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LIMIT DISTRIBUTIONS FOR $\mathrm{SO}(n,1)$ ACTIONS ON k-LATTICES IN $\mathbb{R}^{n+1}$

MICHAEL BERSUDSKY AND NIMISH A. SHAH

Abstract. We study the asymptotic distribution of norm ball averages along orbits of a lattice $\Gamma \subseteq \mathrm{SO}(n,1)$ acting on the moduli space of pairs of orthogonal discrete subgroups of $\mathbb{R}^{n+1}$ up to homothety. Our main result shows that, except for special 2-lattices in $\mathbb{R}^3$ lying in hyperplanes tangent to the light cone, these measures converge to an explicit semi-invariant probability measure supported on the space of homothety classes of pairs of orthogonal lattices tangent to the light cone.

Our main motivation is a conjecture of Sargent and Shapira [SS19], which is resolved as a special case of our general result.

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1. Introduction

The moduli space of k-dimensional lattices of $\mathbb{R}^n$ for $k < n$ has attracted considerable attention in recent years. This subject arises naturally in number theory with its origins tracing back to the works of Roelcke [Roe56], Maass [Maa59], and later Schmidt [Sch98], who studied the distribution of discrete subgroups of $\mathbb{Z}^n$ within large covolume balls — a natural generalization of counting integral vectors in expanding Euclidean balls. More recent interest comes from [EMSS16] and [AES16], which refined the aforementioned works by applying techniques of homogeneous dynamics.

Beyond its number-theoretic motivations, the moduli space of k-lattices is a natural object of study in homogeneous dynamics. This space can be realized as a homogeneous space of $\mathrm{SL}(n,\mathbb{R})$ and admits the structure of a fiber bundle over the Grassmannian of k-planes in $\mathbb{R}^n$, with fibers isomorphic to $\mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z})$. Notably, the action of subgroups of $\mathrm{SL}(n,\mathbb{R})$ on these two spaces exhibits distinct dynamical behaviors. For instance, the action of a one-parameter diagonal subgroup $\{a(t)\}_{t\in\mathbb{R}} \subseteq \mathrm{SL}(n,\mathbb{R})$ is ergodic and mixing on $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$, whereas on the Grassmannian, the same flow is contracting: points either remain fixed or converge to a limit. A concrete example explored in this paper is the action of the diagonal subgroup in $\mathrm{SO}(n,1)$ on the space of k-lattices (see Lemma 4.1). In this case, all k-dimensional subspaces contract toward a specific limit, and the orbit of almost every k-lattice accumulates densely on the collection of k-lattices contained in this limiting subspace.

Our main motivation is the following conjecture stated in [SS19]. Let $X_2 \cong \mathrm{PGL}(2,\mathbb{R})/\mathrm{PGL}(2,\mathbb{Z})$ denote the space of full-rank lattices in $\mathbb{R}^2$ up to scaling. Given a rank-2 discrete subgroup $\Lambda \subseteq \mathbb{R}^3$, its *shape* is the equivalence class $\mathrm{s}(\Lambda) \in \mathrm{PO}(2)\backslash X_2$ of Λ under scaling and orthogonal transformations

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of $\mathbb{R}^3$. Consider $H_{\mathbb{Z}} := \{(x,y,z) \in \mathbb{Z}^3 : 2xz - y^2 = 1\}$, and for each $v \in H_{\mathbb{Z}}$, let

$$\Lambda_v := v^\perp \cap \mathbb{Z}^3,$$

where $v^\perp$ is the orthogonal complement of v. The conjecture is that the collection $\{\mathrm{s}(\Lambda_v) : v \in H_{\mathbb{Z}}\}$ is dense in $\mathrm{PO}(2)\backslash X_2$.

As noted in [SS19], the motivation for this conjecture is related to Furstenberg's approach to resolve a long-standing conjecture on the well-approximability of cubic irrationals. The technique of [SS19] relies on a stationary measure classification for random walks of a Zariski dense subgroup $\Gamma \subseteq \mathrm{SO}(2,1)$ acting on the moduli space of 2-lattices in $\mathbb{R}^3$, building on the techniques of Benoist and Quint, see [BQ16]. However, an incomplete stationary measure classification and certain unexpected stationary measures called k-extensions prevented a full resolution of the conjecture in that work[1]. We also refer the reader to the more recent work [GLS24] studying random walks of lattices in representations of $\mathrm{SL}(2,\mathbb{R})$ acting on general hybrids of projective and lattice type quotients and proving stronger stationary measure classification.

In this paper, we take a direct approach to resolving the conjecture by analyzing the limiting distribution of Γ-orbits in the moduli space of k-lattices, averaged over growing norm balls, where $\Gamma \subseteq \mathrm{SO}(n,1)$ is a lattice. In fact, the conjecture follows as a special case of our general result, Theorem 1.1. Additionally, we describe the limiting distribution of the aforementioned exceptional orbits in Theorem 1.8, and in higher dimensions we obtain the limiting joint distribution of orbits of pairs of orthogonal complementary lattices, see Theorem 1.1 and Theorem 1.2.

The present work also fits into the more general study of limiting distributions of orbits of lattice subgroups in homogeneous spaces via averages over growing balls; see, e.g., [Led99, Oh05, GW07, GN14]. Although this problem has been studied in great generality, the corresponding questions for hybrid projective-lattice-type quotients remain less well understood. In this connection, we also note the recent work [BX24], which studies the action of lattices $\Gamma \subseteq \mathrm{SL}(m+1,\mathbb{R})$ on the moduli space of homothety classes of m-lattices in $\mathbb{R}^{m+1}$. In that setup all orbits are dense, whereas in the present work the relevant orbits accumulate on a lower-dimensional subset; see Proposition 2.1. Moreover, the method of [BX24], based on the duality principle [GW07, Theorem 2.2], does not seem to apply in our setting.

Concrete definitions and results. For natural numbers $r \le n$, a *r-lattice in $\mathbb{R}^{n+1}$* is a discrete subgroup of the form

$$\Lambda = \mathrm{Span}_{\mathbb{Z}}\{v_1, ..., v_r\},$$

where $v_1, ..., v_r \in \mathbb{R}^{n+1}$ are linearly independent. The homothety class $[\Lambda]$ of an r-lattice Λ is defined as the equivalence class with respect to the equivalence relation $\Lambda \sim \Lambda' \iff \Lambda = \alpha\Lambda'$, for $\alpha \in \mathbb{R}^\times$.

We consider the moduli space $X_{r,n+1}$ given by

$$\{([\Lambda_1],[\Lambda_2]) : \Lambda_1 \text{ and } \Lambda_2 \text{ are respectively } r \text{ and } (n+1-r)\text{-lattices in } \mathbb{R}^{n+1} \text{ such that } \Lambda_1 \perp \Lambda_2\},$$

where $\perp$ represents the Euclidean orthogonality. Note that if $r = 1$ or $r = n$, the pair notation is redundant: one of the lattices is one dimensional, and a one-dimensional lattice is unique up to homothety.

Induced by the usual linear representation, the matrix group $\mathrm{SL}(n+1,\mathbb{R})$ acts on $X_{r,n+1}$ as:

$$g \cdot ([\Lambda_1],[\Lambda_2]) := ([g\Lambda_1],[g^*\Lambda_2]), \quad g \in \mathrm{SL}(n+1,\mathbb{R}), \quad ([\Lambda_1],[\Lambda_2]) \in X_{r,n+1}, \tag{1.1}$$

where $g^* := (g^\intercal)^{-1}$.

[1]The conjecture itself was resolved several years ago by Uri Bader and Uri Shapira by a different method. Their work, even though announced publicly in various talks, is still unpublished. As explained to us in private communication by Uri Shapira, their proof goes through establishing a refined classification of stationary measures (similar to [GLS24]) coupled with an instability result which allows to upgrade the classification to understanding random walks. We stress that the Bader-Shapira results hold only for 2-lattices in $\mathbb{R}^3$, while this paper considers joint distributions of pairs of orthogonal lattices in higher dimensions.

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Consider the quadratic form $q(x_1,...,x_{n+1}) := x_1^2+\cdots+x_n^2-x_{n+1}^2$, and denote by $G := \mathrm{SO}(n,1)^\circ$ the identity component of the special orthogonal group of q. Let $\Gamma \subseteq G$ be a lattice, let $\|\cdot\|$ be a norm on $M_{n+1}(\mathbb{R})$, and denote

$$\Gamma_T := \{\gamma \in \Gamma : \|\gamma\| \leq T\}.$$

Our goal is to analyze the limiting distribution of the Γ-norm ball averages on $X_{r,n+1}$:

$$\frac{1}{\#\Gamma_T}\sum_{\gamma\in\Gamma_T} f(\gamma\cdot([\Lambda_1],[\Lambda_2])),\ \forall f\in C_c(X_{r,n+1}),\ \text{as } T\to\infty, \tag{1.2}$$

for any $([\Lambda_1],[\Lambda_2]) \in X_{r,n+1}$. Our main results below show that the above averages converge to an explicit measure which depends on $([\Lambda_1],[\Lambda_2])$ and on the chosen norm. The limiting measures are concentrated on the collection of pairs of lattices lying in subspaces tangent to the light cone $\{\mathbf{x}\in\mathbb{R}^{n+1} : q(\mathbf{x})=0\}$, and we now describe the aforementioned subspaces in more detail.

We say that a r-dimensional vector subspace V of $\mathbb{R}^{n+1}$ is degenerate, if the restricted quadratic form $q|_V$ is degenerate. A subspace V is degenerate if and only if V is tangent to the light cone, and for our particular quadratic form q, its Euclidean orthogonal complement $V^\perp$ is also degenerate. Let $\mathrm{Gr}_r(\mathbb{R}^{n+1})$ denote the Grassmannian of r-dimensional subspaces of $\mathbb{R}^{n+1}$. We define

$$\mathcal{D}_{r,n+1} := \{(P_1,P_2)\in \mathrm{Gr}_r(\mathbb{R}^{n+1})\times \mathrm{Gr}_{n+1-r}(\mathbb{R}^{n+1}) : P_1 \text{ is degenerate and } P_2 = P_1^\perp\}.$$

Let $\mathbf{e}_1,...,\mathbf{e}_{n+1}$ represent the canonical basis vectors of $\mathbb{R}^{n+1}$. For $r\leq n$, we reserve a special notation $\mathbf{P}^\infty = (P_1^\infty, P_2^\infty)$ for the following pair of degenerate subspaces:

$$P_1^\infty := \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1+\mathbf{e}_{n+1},\mathbf{e}_2,...,\mathbf{e}_r\}, \text{ and}$$

$$P_2^\infty := (P_1^\infty)^\perp = \begin{cases}\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1-\mathbf{e}_{n+1},\mathbf{e}_{r+1},...,\mathbf{e}_n\}, & \text{if } r\leq n-1\\ \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1-\mathbf{e}_{n+1}\}, & \text{if } r=n.\end{cases}$$

Consider the maximal compact subgroup of $\mathrm{SO}(n,1)^0$ given by:

$$K := \begin{bmatrix}\mathrm{SO}(n) & \\ & 1\end{bmatrix}, \tag{1.3}$$

where $\mathrm{SO}(n)$ is the special orthogonal group of the form $x_1^2+\cdots+x_n^2$. The subgroup K acts transitively on $\mathcal{D}_{r,n+1}$, so that $\mathcal{D}_{r,n+1}\cong K/K_{\mathbf{P}^\infty}$, where $K_{\mathbf{P}^\infty} = \mathrm{Stab}_K(P_1^\infty,P_2^\infty)$.

We will call a r-lattice degenerate, if it is contained in a degenerate subspace, and we define

$$\mathcal{C}_{r,n+1} := \{([\Lambda_1],[\Lambda_2])\in X_{r,n+1} : \Lambda_1 \text{ is degenerate}\}.$$

The collection $\mathcal{C}_{r,n+1}$ has the following natural fiber-bundle structure over $\mathcal{D}_{r,n+1}$. Consider the natural projection $\pi : X_{r,n+1}\to \mathrm{Gr}_r(\mathbb{R}^{n+1})\times\mathrm{Gr}_{n+1-r}(\mathbb{R}^{n+1})$, defined as:

$$\pi([\Lambda_1],[\Lambda_2]) := (\mathrm{Span}_{\mathbb{R}}(\Lambda_1),\mathrm{Span}_{\mathbb{R}}(\Lambda_2)).$$

For $k\in\mathbb{N}$ we let

$$X_k := \mathrm{PGL}(k,\mathbb{R})/\mathrm{PGL}(k,\mathbb{Z}) \cong \mathrm{SL}(k,\mathbb{R})/\mathrm{SL}(k,\mathbb{Z}),$$

which is identified with the space of (full rank) unimodular lattices of $\mathbb{R}^k$. Then, the fibers of π are isomorphic to $X_r\times X_{n+1-r}$; namely, $\mathcal{C}_{r,n+1}$ is a fiber-bundle over $\mathcal{D}_{r,n+1}$, where the fibers are isomorphic to $X_r\times X_{n+1-r}$. For $\mathbf{P}=(P_1,P_2)\in\mathcal{D}_{r,n+1}$, we denote by $X_{\mathbf{P}}$ the fiber over $\mathbf{P}$, i.e.:

$$X_{\mathbf{P}} = \{([\Lambda_1],[\Lambda_2])\in X_{r,n+1} : \Lambda_1\subseteq P_1,\ \Lambda_2\subseteq P_2\}.$$

We equip $X_{\mathbf{P}}$ with the $\mathrm{PGL}(r,\mathbb{R})\times\mathrm{PGL}(n+1-r,\mathbb{R})$-invariant probability measure $\mu_{\mathbf{P}}$ as follows. Choosing a basis of P_1 and P_2, we get a representation of $\mathrm{PGL}(r,\mathbb{R})\times\mathrm{PGL}(n+1-r,\mathbb{R})$ on $P_1\times P_2$, which in turn identifies $X_{\mathbf{P}}$ with $X_r\times X_{n+1-r}$ as a $\mathrm{PGL}(r,\mathbb{R})\times\mathrm{PGL}(n+1-r,\mathbb{R})$-space, and it admits a unique $\mathrm{PGL}(r,\mathbb{R})\times\mathrm{PGL}(n+1-r,\mathbb{R})$-invariant probability measure. This measure is independent of the choice of the basis for the subspaces P_1 and P_2.

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Orbits of nondegenerate orthogonal lattices. Let (Λ_1, Λ_2) be a nondegenerate pair. Without loss of generality, throughout the paper, we will assume that Λ_1 is such that $q|_{\mathrm{Span}_{\mathbb{R}}(\Lambda_1)}$ is positive definite. This is convenient for stating the results in a uniform manner, and there is no loss of generality in that choice.

The limiting measure of (1.2) for a non-degenerate pair will be obtained by integrating the measures $\mu_{\mathbf{P}}$ with respect to a smooth measure on $\mathcal{D}_{r,n+1} \cong K/K_{\mathbf{P}^\infty}$. We will now describe the density on $\mathcal{D}_{r,n+1}$ of the aforementioned measure, which depends on the starting point and the norm.

Consider the subgroup

$$H = \begin{bmatrix} \mathrm{SO}(r) & \mathbf{0} \\ \mathbf{0} & \mathrm{SO}(n-r,1)^\circ \end{bmatrix} \subseteq G,$$

and let dh be a Haar measure on H. Let dk and $d\kappa$ denote the Haar probability measures on K and $K_{\mathbf{P}^\infty}$, respectively.

For a positive-definite subspace $P_0 \in \mathrm{Gr}_r(\mathbb{R}^{n+1})$, we define the density function $w_{P_0} \in C(\mathcal{D}_{r,n+1})$ as follows. By Witt's theorem, there exists a $g_0 \in G$ such that $P_0 = g_0 \cdot \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, ..., \mathbf{e}_r\}$. We define for all $k_0 \in K$,

$$w_{P_0}(k_0 K_{\mathbf{P}^\infty}) := \begin{cases} \dfrac{\int_{K_{\mathbf{P}^\infty}} \int_H \|k_0\kappa a(\infty) h g_0^{-1}\|^{-(n-1)} dh d\kappa}{\int_K \int_H \|ka(\infty)hg_0^{-1}\|^{-(n-1)} dh dk}, & \text{if } r > 1, \\ \dfrac{\int_{K_{\mathbf{P}^\infty}} \int_H \left(\|k_0\kappa a(\infty) h g_0^{-1}\|^{-(n-1)} + \|k_0\kappa a(-\infty) h g_0^{-1}\|^{-(n-1)}\right) dh d\kappa}{\int_K \int_H \left(\|ka(\infty)hg_0^{-1}\|^{-(n-1)} + \|ka(-\infty)hg_0^{-1}\|^{-(n-1)}\right) dh dk}, & \text{if } r = 1, \end{cases}$$

$$\text{where} \quad a(\infty) := \begin{bmatrix} \frac{1}{2} & \mathbf{0} & \frac{1}{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{1}{2} & \mathbf{0} & \frac{1}{2} \end{bmatrix} \quad \text{and} \quad a(-\infty) := \begin{bmatrix} \frac{1}{2} & \mathbf{0} & -\frac{1}{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\frac{1}{2} & \mathbf{0} & \frac{1}{2} \end{bmatrix}. \tag{1.4}$$

To see that the above expressions are well defined, we refer to Lemma 3.2 proving convergence of the integrals. In addition, the expression defining w_{P_0} is independent of the choice of g_0 because H is the subgroup of G preserving $\mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, ..., \mathbf{e}_r\}$, and dh is right-H invariant.

Observe that if $\|\cdot\|$ is left K-invariant, then w_{P_0} is the constant function 1, for every P_0.

Theorem 1.1. *Let $\Gamma \subseteq G$ be a lattice, and let $([\Lambda_1], [\Lambda_2]) \in X_{r,n+1}$ where $\mathrm{Span}_{\mathbb{R}}(\Lambda_1)$ is positive definite. We denote $P_0 := \mathrm{Span}_{\mathbb{R}}(\Lambda_1)$. Then, for all $f \in C_c(X_{r,n+1})$,*

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma \in \Gamma_T} f(\gamma \cdot ([\Lambda_1], [\Lambda_2])) = \int_{K/K_{\mathbf{P}^\infty}} \mu_{k\cdot\mathbf{P}^\infty}(f) w_{P_0}(kK_{\mathbf{P}^\infty}) d(kK_{\mathbf{P}^\infty}),$$

where $d(kK_{\mathbf{P}^\infty})$ is the K-invariant probability on $K/K_{\mathbf{P}^\infty}$.

We will now verify the conjecture of Sargent–Shapira [SS19] affirmatively using Theorem 1.1.

Proof of the Sargent–Shapira conjecture. Since for a 2-lattice $\Lambda \subseteq \mathbb{R}^3$, the homothety class of any lattice in its orthogonal line is unique, the pair notation for $X_{2,3}$ is redundant; that is, we write

$$X_{2,3} = \{[\Lambda] : \Lambda \text{ is a 2-lattice in } \mathbb{R}^3\}.$$

For a plane $P \in \mathrm{Gr}_2(\mathbb{R}^3)$, let $X_P = \{[\Lambda] \in X_{2,3} : \Lambda \subseteq P\}$, and let μ_P be the $\mathrm{PGL}(2,\mathbb{R})$-invariant probability on X_P.

We note the identification $\mathrm{SO}(3)\backslash X_{2,3} \cong \mathrm{PO}(2)\backslash X_2$ obtained by identifying the class $\mathrm{SO}(3)[\Lambda]$ with the shapes of lattices in the horizontal plane $\mathbb{R}^2 \times 0$. Hence, the map sending a 2-lattice in $\mathbb{R}^3$ to its shape is the natural quotient map $\mathrm{s} : X_{2,3} \to \mathrm{SO}(3)\backslash X_{2,3}$. With this identification, the pushed measure $\mathrm{s}_*\mu_P$ is the natural measure $\mu_{\mathrm{PO}(2)\backslash X_2}$ on $\mathrm{PO}(2)\backslash X_2$, for every $P \in \mathrm{Gr}_2(\mathbb{R}^3)$. Therefore,

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given a lattice $\tilde{\Gamma} \subseteq \mathrm{SO}(2,1)^\circ$ and a non-degenerate 2-lattice $\Lambda \subseteq \mathbb{R}^3$, we obtain from Theorem 1.1

$$\lim_{T\to\infty} \frac{1}{\#\tilde{\Gamma}_T} \sum_{\tilde{\gamma}\in\tilde{\Gamma}_T} f(\mathrm{s}([\tilde{\gamma}\Lambda])) = \mu_{\mathrm{PO}(2)\backslash X_2}(f),\ \forall f \in C_c(\mathrm{PO}(2)\backslash X_2). \tag{1.5}$$

Let $Q(x,y,z) = 2xz - y^2$ and let $\mathrm{SO}(Q)$ be the special orthogonal group preserving Q. Consider $\Gamma := \mathrm{SO}(Q)^\circ \cap \mathrm{SL}(3,\mathbb{Z})$, and note that $\Gamma^* = \Gamma$, where $\Gamma^* := (\Gamma^\intercal)^{-1}$. For $v \in \mathbb{Z}^3 \smallsetminus 0$, we have that

$$\gamma\Lambda_v = \Lambda_{\gamma^* v}, \quad \text{for } \gamma \in \Gamma,$$

where $\Lambda_v := v^\perp \cap \mathbb{Z}^3$. Thus, to prove that the shapes $\mathrm{s}(\Lambda_v)$ for $v \in H_\mathbb{Z} := \{x \in \mathbb{Z}^3 : Q(x) = 1\}$ are dense in $\mathrm{PO}(2)\backslash X_2$, it is sufficient to show that the shapes of the subcollection $\Gamma\Lambda_{v_0}$ are dense, where $v_0 := (1,1,1) \in H_\mathbb{Z}$.

To apply Theorem 1.1, we may choose $M \in \mathrm{O}(3)$ such that $M^{-1}\mathrm{SO}(Q)^\circ M = \mathrm{SO}(2,1)^\circ$, and we observe that the plane $\mathrm{Span}_\mathbb{R}(M^{-1}\Lambda_{v_0})$ is nondegenerate for the form $x^2+y^2-z^2$. Thus, we may apply Theorem 1.1 for $\tilde{\Gamma} = M^{-1}\Gamma M$ and $\Lambda = M^{-1}\Lambda_{v_0}$. Let $x \in \mathrm{PO}(2)\backslash X_2$ and let $\mathcal{U} \subseteq \mathrm{PO}(2)\backslash X_2$ be a compact neighborhood of x. Choose $f \in C_c(\mathrm{PO}(2)\backslash X_2)$ supported in $\overline{\mathcal{U}}$ and positive on $\mathcal{U}$. By (1.5),

$$0 < \mu_{\mathrm{PO}(2)\backslash X_2}(f) = \lim_{T\to\infty} \frac{1}{\#\tilde{\Gamma}_T} \sum_{\tilde{\gamma}\in\tilde{\Gamma}_T} f(\mathrm{s}([\tilde{\gamma}(M^{-1}\Lambda_{v_0})]) = \lim_{T\to\infty} \frac{1}{\#\tilde{\Gamma}_T} \sum_{\gamma\in\Gamma,\ \|M^{-1}\gamma M\|\le T} f(\mathrm{s}([\gamma\Lambda_{v_0}])).$$

In particular, there exists $\gamma \in \Gamma$ such that $\mathrm{s}[\gamma\Lambda_{v_0}] \in \mathcal{U}$. □

Orbits of degenerate lattices. We now present our results on the limits of the norm-ball averages (1.2) for a degenerate pair (Λ_1, Λ_2). We split the discussion into the higher-dimensional case $n \ge 3$ and the low-dimensional case $n = 2$, due to a special arithmetic phenomenon occurring only when $n = 2$.

When $n \ge 3$, the limits of (1.2) for $([\Lambda_1],[\Lambda_2]) \in \mathcal{C}_{r,n+1}$ are described by a limiting measure of the same form as in Theorem 1.1, where the only difference is the density function on $\mathcal{D}_{r,n+1}$.

Orbits of degenerate lattices for $n \ge 3$. Consider the unipotent subgroup $U \subseteq G$ given by

$$U := \left\{ u(\mathbf{x}) := \exp\left(\begin{bmatrix} 0 & \mathbf{x} & 0 \\ -\mathbf{x}^\intercal & \mathbf{0} & \mathbf{x}^\intercal \\ 0 & \mathbf{x} & 0 \end{bmatrix} \right) : \mathbf{x} \in \mathbb{R}^{n-1} \right\}.$$

Given a degenerate r-dimensional subspace $P_0 \in \mathrm{Gr}_r(\mathbb{R}^{n+1})$, there exists $g_0 \in G$ such that

$$P_0 = g_0 \mathrm{Span}_\mathbb{R}\{\mathbf{e}_1 - \mathbf{e}_{n+1}, \mathbf{e}_2, \dots, \mathbf{e}_r\}.$$

We define a density function $w^\infty_{P_0} \in C(\mathcal{D}_{r,n+1})$ by:

$$w^\infty_{P_0}(k_0 K_{\mathbf{P}^\infty}) := \frac{\int_{K_{\mathbf{P}^\infty}} \int_{\mathbb{R}^{n-1}} \|k_0\kappa a(\infty) u(\mathbf{x}) g_0^{-1}\|^{-(n-1)}\, d\mathbf{x} d\kappa}{\int_K \int_{\mathbb{R}^{n-1}} \|k a(\infty) u(\mathbf{x}) g_0^{-1}\|^{-(n-1)}\, d\mathbf{x} dk}, \text{ for } k_0 \in K.$$

Here $a(\infty)$ is given in (1.4). For the convergence of the above integral, see Lemma 3.3. The definition $w^\infty_{P_0}$ is independent of the choice of g_0; indeed, choosing a different g_0' such that $g_0'\mathrm{Span}_\mathbb{R}\{\mathbf{e}_1 - \mathbf{e}_{n+1}, \mathbf{e}_2, \dots, \mathbf{e}_r\} = P_0$ changes the above expressions by the same scalar multiple, which cancels in the quotient.

We note that if $\|\cdot\|$ is left K-invariant, then $w^\infty_{P_0} \equiv 1$, for every $P_0 \in \mathrm{Gr}_r(\mathbb{R}^{n+1})$.

Theorem 1.2. *Let $n \ge 3$. Let $\Gamma \subseteq G$ be a lattice, and let $([\Lambda_1],[\Lambda_2]) \in \mathcal{C}_{r,n+1}$, i.e. $\mathrm{Span}_\mathbb{R}(\Lambda_1)$ is degenerate. We denote $P_0 := \mathrm{Span}_\mathbb{R}(\Lambda_1)$. Then, for all $f \in C_c(X_{r,n+1})$,*

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} f(\gamma\cdot([\Lambda_1],[\Lambda_2])) = \int_{K/K_{\mathbf{P}^\infty}} \mu_{k\cdot\mathbf{P}^\infty}(f) w^\infty_{P_0}(kK_{\mathbf{P}^\infty}) d(kK_{\mathbf{P}^\infty}),$$

where $d(kK_{\mathbf{P}^\infty})$ is the K-invariant probability on $K/K_{\mathbf{P}^\infty}$.

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Special orbits of degenerate 2*-lattices in* $\mathbb{R}^3$. In this case, the pair notation is redundant. In particular, the collection $\mathcal{D}_{2,3} \subseteq \mathrm{Gr}_2(\mathbb{R}^3)$ is identified with the circle of planes tangent to the light cone, and the collection $\mathcal{C}_{2,3}$ of degenerate 2-lattices is a bundle over that circle.

Consider the diagonal subgroup $A \subseteq G$ and its expanding unipotent subgroup $U \subset G$, given by

$$A := \left\{ a(s) := \exp\left[\begin{smallmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ s & 0 & 0 \end{smallmatrix}\right] : s \in \mathbb{R} \right\}, \quad U := \left\{ u(t) := \exp\left(\left[\begin{smallmatrix} 0 & t & 0 \\ -t & 0 & t \\ 0 & t & 0 \end{smallmatrix}\right] \right) : t \in \mathbb{R} \right\}.$$

We denote

$$\delta(s) := \begin{bmatrix} e^{\frac{s}{2}} & 0 \\ 0 & e^{-\frac{s}{2}} \end{bmatrix}, \quad \upsilon(t) := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix},$$

and we let $\Phi : \mathrm{SL}(2,\mathbb{R}) \to G$ be the special isogeny (surjective map with finite kernel) characterized by

$$\Phi(\delta(s)) = a(s), \quad \Phi(\upsilon(t)) = u(t). \tag{1.6}$$

In the following, we define a family of representations of $\mathrm{SL}(2,\mathbb{R})$ acting on each degenerate plane, which align in a certain way with the $\mathrm{SL}(2,\mathbb{R})$ representation on $\mathbb{R}^3$ given by Φ.

First consider the standard degenerate plane $P_0^\infty := \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2\}$. Denote

$$v^+ := \mathbf{e}_1 + \mathbf{e}_3, \quad v^0 := \mathbf{e}_2.$$

By representing linear maps on P_0^∞ with respect to the ordered basis (v^+, v^0), we get an isomorphism

$$\rho^\infty : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(P_0^\infty);$$

that is,

$$\rho^\infty\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] v^+ = av^+ + cv^0, \quad \rho^\infty\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] v^0 = bv^+ + dv^0.$$

Observe that the vectors v^+, v^0 are eigenvectors of $a(s)$, and the parabolic subgroup $AU \subseteq G$ preserves P_0^∞ and acts on $X_{P_0^\infty}$. Explicitly:

$$\begin{aligned} a(s)v^+ &= e^s v^+, \quad a(s)v^0 = v^0, \\ u(t)v^+ &= v^+, \quad u(t)v^0 = tv^+ + v^0. \end{aligned} \tag{1.7}$$

Then, by (1.7), we get the following correspondence:

$$\Phi(\delta(s)) \cdot [\Lambda] = [\rho^\infty(\delta(s))\Lambda], \quad \Phi(\upsilon(t)) \cdot [\Lambda] = [\rho^\infty(\upsilon(t))\Lambda], \quad \forall \Lambda \subseteq P_0^\infty. \tag{1.8}$$

Next, to describe the correspondence for the remaining degenerate planes, set $P_\theta^\infty := k_\theta P_0^\infty$, where

$$k_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \in G.$$

We define the representation $\rho_\theta^\infty : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(P_\theta^\infty)$ by

$$\rho_\theta^\infty(g) := k_\theta \rho^\infty(g) k_\theta^{-1}, \quad g \in \mathrm{SL}(2,\mathbb{R}). \tag{1.9}$$

Notice that the parabolic subgroup $k_\theta AU k_\theta^{-1}$ preserves P_θ^∞, and similarly to (1.8), we have

$$k_\theta \Phi(\delta(s)) k_\theta^{-1} \cdot [\Lambda] = [\rho_\theta^\infty(\delta(s))\Lambda], \quad k_\theta \Phi(\upsilon(t)) k_\theta^{-1} \cdot [\Lambda] = [\rho_\theta^\infty(\upsilon(t))\Lambda], \quad \forall \Lambda \subseteq P_\theta^\infty. \tag{1.10}$$

We will now define the special points.

Definition 1.3. Let $\Gamma \subseteq \mathrm{SO}(2,1)^\circ$ be a lattice, and for $\theta \in [0, 2\pi)$, define

$$\Gamma_\theta^\Phi := \{ g \in \mathrm{SL}(2,\mathbb{R}) : k_\theta \Phi(g) k_\theta^{-1} \in \Gamma \}.$$

We say that a 2-lattice $\Lambda \subseteq P_\theta^\infty$ is Γ-special, if the orbit $\{[\rho_\theta^\infty(g)\Lambda] : g \in \Gamma_\theta^\Phi\}$ is finite, say

$$\{[\rho_\theta^\infty(g)\Lambda] : g \in \Gamma_\theta^\Phi\} =: \{[\Lambda_1], \dots, [\Lambda_m]\}.$$

We refer to this finite set as the (Γ, Λ)-packet.

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Remark 1.4. Let

$$\mathrm{Stab}_\theta([\Lambda]) := \{g \in \mathrm{SL}(2,\mathbb{R}) : [\rho_\theta^\infty(g)\Lambda] = [\Lambda]\}.$$

Note that both $\mathrm{Stab}_\theta([\Lambda])$ and Γ_θ^Φ are lattices in $\mathrm{SL}(2,\mathbb{R})$. Then Λ is Γ-special if and only if $\mathrm{Stab}_\theta([\Lambda])$ is commensurable with Γ_θ^Φ, equivalently if and only if $\mathrm{Stab}_\theta([\Lambda]) \cap \Gamma_\theta^\Phi$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$.

Remark 1.5. If Λ is Γ-special, then a conjugate of Γ is commensurable with $\Phi(\mathrm{SL}(2,\mathbb{Z}))$.

The following lemma shows that the Γ-orbit of a special point is trapped in a multi-section. Denote:

$$\kappa_\vartheta := \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{bmatrix},$$

and note that $\Phi(\kappa_{\theta/2}) = k_\theta$.

Lemma 1.6. *Let $\Gamma \subseteq G$ be a lattice and let $\Lambda \subset P_{\theta_0}^\infty$ be a Γ-special 2-lattice. Then*

$$\Gamma \cdot [\Lambda] \subseteq \left\{ k_\theta \left[\rho_{\theta_0}^\infty(\kappa_{-\theta/2})\Lambda_i\right] : \ 1 \le i \le m, \ \theta \in [0, 2\pi) \right\},$$

where $\{[\Lambda_i] : 1 \le i \le m\}$ is the (Γ,Λ)-packet.

Remark 1.7. Note that $\rho_{\theta_0}^\infty(\kappa_{-\theta/2})$ is not a Euclidean rotation. Namely, the shapes of the lattices $\rho_{\theta_0}^\infty(\kappa_{-\theta/2})\Lambda_i$ vary when varying θ.

Proof. Take $\gamma \in \Gamma$. Since $\gamma P_{\theta_0}^\infty$ is degenerate, there exists $\theta \in [0, 2\pi)$ such that $\gamma P_{\theta_0}^\infty = P_{\theta+\theta_0}^\infty$. Hence

$$b_\gamma := k_{\theta+\theta_0}^{-1} \gamma k_{\theta_0} \in AU.$$

Choose $g_\gamma = \delta(s)\upsilon(t) \in \mathrm{SL}(2,\mathbb{R})$ such that $\Phi(g_\gamma) = b_\gamma$, and note that

$$\gamma = k_\theta \left(k_{\theta_0} \Phi(g_\gamma) k_{\theta_0}^{-1} \right).$$

Then, by (1.10), $\gamma \cdot [\Lambda] = k_\theta \cdot [\rho_{\theta_0}(g_\gamma)\Lambda]$.

Using that $\Phi(\kappa_{\theta/2}) = k_\theta$, we also see that $\gamma = k_{\theta_0} \Phi(\kappa_{\theta/2} g_\gamma) k_{\theta_0}^{-1}$. Therefore, $\kappa_{\theta/2} g_\gamma \in \Gamma_{\theta_0}^\Phi$, and by definition of the packet, there exists i such that

$$[\Lambda_i] = [\rho_{\theta_0}^\infty(\kappa_{\theta/2} g_\gamma)\Lambda].$$

We may now conclude that:

$$\gamma \cdot [\Lambda] = k_\theta \left[\rho_{\theta_0}^\infty(\kappa_{-\theta/2}(\kappa_{\theta/2} g_\gamma))\Lambda\right] = k_\theta \left[\rho_{\theta_0}^\infty(\kappa_{-\theta/2})\Lambda_i\right],$$

which proves the claim. □

Our main theorem below proves that orbits of special points become dense in their multi-section and are distributed according to an explicit measure. This limiting measure is an m-extension, a notion introduced in [SS19, Definition 1.5]. If a lattice is not Γ-special, then the corresponding limiting distribution is as in Theorem 1.2.

Theorem 1.8. *Let $\Gamma \subseteq G$ be a lattice, and let $[\Lambda] \in X_{2,3}$ be a degenerate 2-lattice. Choose $k_{\theta_0} \in K$ such that $\Lambda \subseteq k_{\theta_0} P^\infty$. Let*

$$w_{\theta_0}(\vartheta) := \frac{\int_{\mathbb{R}} \frac{1}{\|k_\vartheta a(\infty) u(x) k_{\theta_0}^{-1}\|} \, dx}{\int_0^{2\pi} \int_{\mathbb{R}} \frac{1}{\|k_\theta a(\infty) u(x) k_{\theta_0}^{-1}\|} \, dx \, d\theta}.$$

Then:

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(1) If Λ is Γ-special, then for all $f \in C_c(X_{2,3})$,

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} f(\gamma\cdot[\Lambda]) = \frac{1}{m}\sum_{i=1}^{m} \int_0^{2\pi} f\big(k_\theta\big[\rho_{\theta_0}^{\infty}(\kappa_{-\theta/2})\Lambda_i\big]\big)\, w_{\theta_0}(\theta+\theta_0)\, d\theta,$$

where $\{[\Lambda_1],\dots,[\Lambda_m]\}$ is the (Γ,Λ)-packet.

(2) If Λ is not Γ-special, then

$$\lim_{T\to\infty} \frac{1}{\#\Gamma_T} \sum_{\gamma\in\Gamma_T} f(\gamma\cdot[\Lambda]) = \int_0^{2\pi} \mu_{k_\theta P^\infty}(f)\, w_{\theta_0}(\theta)\, d\theta.$$

2. Sums to integrals

This section describes a general method used to obtain our main result. It is a slight adaptation of [GW07, Proposition 3.1], and the proof follows the same argument closely.

In the following, $G \le \mathrm{SL}(n,\mathbb{R})$ is a closed subgroup, $\Delta \le G$ is a lattice, and $\|\cdot\|$ is a norm. Let dg be the Haar measure on G such that $\mathrm{Vol}(G/\Delta)=1$. For a subset $L\subseteq G$ and $T>0$, write

$$L_T := \{l \in L : \|l\| \le T\}.$$

We assume:

- For every $\epsilon>0$ there exists $\delta>0$ and $T_0>0$ such that for all $T>T_0$

$$\frac{\mathrm{Vol}(G_{T(1+\delta)})}{\mathrm{Vol}(G_T)} \le 1+\epsilon. \tag{2.1}$$

- It holds that

$$\lim_{\tau\to\infty} \frac{\#\Delta_\tau}{\mathrm{Vol}(G_\tau)} = 1. \tag{2.2}$$

These assumptions hold in considerable generality; in particular, they hold for simple groups G, see for example [GN12].

Suppose that G acts continuously on a manifold X, and fix $x_0\in X$. For $T>0$, define a probability measure μ_T on $G/\Delta\times X$ by

$$\mu_T(\varphi\otimes f) := \frac{1}{\mathrm{Vol}(G_T)} \int_{G_T} \varphi(g\Delta)\, f(gx_0)\, dg,$$

for $\varphi\in C_c(G/\Delta)$ and $f\in C_c(X)$.

Let $\{O_\delta\}_{\delta\in(0,1]}$ be a nested family of symmetric neighborhoods of the identity in G such that $O_\delta \downarrow \{e\}$ as $\delta\to 0$. For each $\delta\in(0,1]$, choose a nonnegative bump function $\chi_\delta\in C_c(G)$ supported in O_δ and normalized by $\int_G \chi_\delta(g)\, dg = 1$. Define

$$\varphi_\delta(g\Delta) := \sum_{\gamma\in\Delta} \chi_\delta(g\gamma^{-1}).$$

Then $\varphi_\delta\in C_c(G/\Delta)$ and $\int_{G/\Delta}\varphi_\delta = 1$.

Proposition 2.1. *Fix $x_0\in X$, and suppose that there exists a measure ν on $G/\Delta\times X$ such that*

$$\mu_T \xrightarrow[T\to\infty]{} \nu \tag{2.3}$$

in the weak- topology. Then, for every $f\in C_c(X)$,*

$$\lim_{T\to\infty} \frac{1}{\#\Delta_T} \sum_{\gamma\in\Delta_T} f(\gamma x_0) = \lim_{\delta\to 0} \nu(\varphi_\delta\otimes f).$$

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Proof. It is enough to treat the case $f \geq 0$, since the general case follows by writing $f = f^+ - f^-$.

Fix $\varepsilon > 0$. Since f is compactly supported, it is left uniformly continuous with respect to the G-action. Therefore, for $\delta > 0$ sufficiently small,

$$|f(ugx_0) - f(gx_0)| \leq \varepsilon \quad \text{for all } u \in O_\delta \text{ and } g \in G. \tag{2.4}$$

We also choose δ so that

$$O_\delta G_T \subseteq G_{(1+\delta)T} \tag{2.5}$$

for all sufficiently large T.

We first prove the upper bound. If $\gamma \in \Delta_T$, then the support of $g \mapsto \chi_\delta(g\gamma^{-1})$ is contained in $O_\delta\gamma \subseteq G_{(1+\delta)T}$, so

$$1 = \int_G \chi_\delta(g\gamma^{-1})\, dg = \int_{G_{(1+\delta)T}} \chi_\delta(g\gamma^{-1})\, dg.$$

Hence

$$\sum_{\gamma\in\Delta_T} f(\gamma x_0) \leq \sum_{\gamma\in\Delta} \int_{G_{(1+\delta)T}} \chi_\delta(g\gamma^{-1})\, f(\gamma x_0)\, dg. \tag{2.6}$$

Moreover, if $\|\gamma\| > (1+\delta)^2 T$, then

$$\int_{G_{(1+\delta)T}} \chi_\delta(g\gamma^{-1})\, f(\gamma x_0)\, dg = 0.$$

Indeed, if the integrand is nonzero for some $g \in G_{(1+\delta)T}$, then $g\gamma^{-1} \in O_\delta$, so $\gamma = ug$ for some $u \in O_\delta^{-1} = O_\delta$. By (2.5), this implies $\gamma \in G_{(1+\delta)^2 T}$, a contradiction.

Next, for every measurable $B \subseteq G$ and every $\gamma \in \Delta$, (2.4) gives

$$\left| \int_B \chi_\delta(g\gamma^{-1}) \big(f(gx_0) - f(\gamma x_0)\big)\, dg \right| \leq \varepsilon \int_B \chi_\delta(g\gamma^{-1})\, dg \leq \varepsilon.$$

Summing over γ, and using the vanishing just proved, we obtain

$$\begin{aligned} \sum_{\gamma\in\Delta} \int_{G_{(1+\delta)T}} \chi_\delta(g\gamma^{-1}) f(\gamma x_0)\, dg &= \sum_{\gamma\in\Delta_{(1+\delta)^2 T}} \int_{G_{(1+\delta)T}} \chi_\delta(g\gamma^{-1}) f(\gamma x_0)\, dg \\ &= \int_{G_{(1+\delta)T}} \left(\sum_{\gamma\in\Delta} \chi_\delta(g\gamma^{-1}) \right) f(gx_0)\, dg + O(\varepsilon)\, \#\Delta_{(1+\delta)^2 T}. \end{aligned} \tag{2.7}$$

Since $\sum_{\gamma\in\Delta} \chi_\delta(g\gamma^{-1}) = \varphi_\delta(g\Delta)$, combining (2.6) with (2.7) and using (2.2) yields

$$\frac{1}{\mathrm{Vol}(G_T)} \sum_{\gamma\in\Delta_T} f(\gamma x_0) \leq \frac{1}{\mathrm{Vol}(G_T)} \int_{G_{(1+\delta)T}} \varphi_\delta(g\Delta)\, f(gx_0)\, dg + O(\varepsilon). \tag{2.8}$$

We now prove the lower bound. Similarly to above, if $\|\gamma\| > T$, then for every $u \in O_\delta = O_\delta^{-1}$ we have $\|u\gamma\| > T/(1+\delta)$, and therefore

$$\int_{G_{T/(1+\delta)}} \chi_\delta(g\gamma^{-1})\, dg = 0.$$

It follows that

$$\sum_{\gamma\in\Delta} \int_{G_{T/(1+\delta)}} \chi_\delta(g\gamma^{-1})\, f(\gamma x_0)\, dg = \sum_{\gamma\in\Delta_T} \int_{G_{T/(1+\delta)}} \chi_\delta(g\gamma^{-1})\, f(\gamma x_0)\, dg \leq \sum_{\gamma\in\Delta_T} f(\gamma x_0). \tag{2.9}$$

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Using (2.4) exactly as in the upper-bound argument, but now integrating over $G_{T/(1+\delta)}$, we obtain

$$\sum_{\gamma\in\Delta}\int_{G_{T/(1+\delta)}}\chi_\delta(g\gamma^{-1})\,f(\gamma x_0)\,dg=\int_{G_{T/(1+\delta)}}\varphi_\delta(g\Delta)f(gx_0)\,dg+O(\varepsilon)\,\#\Delta_T. \tag{2.10}$$

Therefore, by (2.9) and (2.10)

$$\frac{1}{\mathrm{Vol}(G_T)}\int_{G_{T/(1+\delta)}}\varphi_\delta(g\Delta)\,f(gx_0)\,dg+O(\varepsilon)\le\frac{1}{\mathrm{Vol}(G_T)}\sum_{\gamma\in\Delta_T}f(\gamma x_0). \tag{2.11}$$

Define

$$L^-(\delta):=\liminf_{T\to\infty}\frac{\mathrm{Vol}(G_{T/(1+\delta)})}{\mathrm{Vol}(G_T)},\quad L^+(\delta):=\limsup_{T\to\infty}\frac{\mathrm{Vol}(G_{(1+\delta)T})}{\mathrm{Vol}(G_T)}.$$

Using (2.8), (2.11), and the weak-* convergence in (2.3), we get

$$\begin{aligned}L^-(\delta)\,\nu(\varphi_\delta\otimes f)+O(\varepsilon)&\le\liminf_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\sum_{\gamma\in\Delta_T}f(\gamma x_0)\\&\le\limsup_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\sum_{\gamma\in\Delta_T}f(\gamma x_0)\\&\le L^+(\delta)\,\nu(\varphi_\delta\otimes f)+O(\varepsilon).\end{aligned} \tag{2.12}$$

By (2.1), we have $L^\pm(\delta)\to 1$ as $\delta\to 0$. Since $\varepsilon>0$ is arbitrary, (2.12) implies

$$\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\sum_{\gamma\in\Delta_T}f(\gamma x_0)=\lim_{\delta\to 0}\nu(\varphi_\delta\otimes f).$$

Since $\lim_{T\to\infty}\frac{\#\Delta_T}{\mathrm{Vol}(G_T)}=1$, this completes the proof. □

3. Volume computations

We let $n\ge 2$ and denote $G:=\mathrm{SO}(n,1)^\circ$. In this section, we compute the growth of the norm balls $G_T:=\{g\in G:\|g\|\le T\}$ and isolate the part of the Haar integral that contributes to the limiting densities appearing in Theorems 1.1, 1.2, and 1.8. To obtain the volume estimates, we use decompositions of G of the form KAH, where A is a fixed Cartan subgroup of G, K is the corresponding maximal compact subgroup, and H is either a symmetric subgroup or the expanding horospherical subgroup. We begin by introducing these subgroups and recording the relevant Haar-measure decompositions.

For $i,j\in\mathbb{N}\cup\{0\}$ with $i+j=n-1$, let

$$H_{i,j}=\begin{bmatrix}\mathrm{SO}(i+1)&\mathbf{0}\\\mathbf{0}&\mathrm{SO}(j,1)^\circ\end{bmatrix}. \tag{3.1}$$

Here $\mathrm{SO}(0,1)^\circ$ and $\mathrm{SO}(1)$ are understood to be the trivial group $\{\mathrm{id}\}$.

Our diagonalizable subgroup $A\subseteq G$ is given by

$$A=\left\{a(s):=\exp\left(\left[\begin{smallmatrix}\mathbf{0}&1\\1&\mathbf{0}\end{smallmatrix}\right]s\right):s\in\mathbb{R}\right\},$$

where

$$\exp\left(\left[\begin{smallmatrix}\mathbf{0}&1\\1&\mathbf{0}\end{smallmatrix}\right]s\right)=\begin{bmatrix}\cosh(s)&\mathbf{0}&\sinh(s)\\\mathbf{0}&I_{n-1}&\mathbf{0}\\\sinh(s)&\mathbf{0}&\cosh(s)\end{bmatrix}, \tag{3.2}$$

and its expanding horosphere is

$$U=\left\{u(\mathbf{x}):=\exp\left(\begin{bmatrix}0&\mathbf{x}&0\\-\mathbf{x}^\intercal&\mathbf{0}&\mathbf{x}^\intercal\\0&\mathbf{x}&0\end{bmatrix}\right):\mathbf{x}\in\mathbb{R}^{n-1}\right\},$$

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where

$$\exp\left(\begin{bmatrix} 0 & \mathbf{x} & 0 \\ -\mathbf{x}^\intercal & \mathbf{0} & \mathbf{x}^\intercal \\ 0 & \mathbf{x} & 0 \end{bmatrix}\right) = \begin{bmatrix} 1-\frac{\|\mathbf{x}\|^2}{2} & \mathbf{x} & \frac{\|\mathbf{x}\|^2}{2} \\ -\mathbf{x}^\intercal & I_{n-1} & \mathbf{x}^\intercal \\ -\frac{\|\mathbf{x}\|^2}{2} & \mathbf{x} & 1+\frac{\|\mathbf{x}\|^2}{2} \end{bmatrix}. \tag{3.3}$$

The Haar measure on G admits the following decompositions; see [Sch84, Chapter 8] and [GOS09, Section 3] for further remarks. Let K be the maximal compact subgroup given by (1.3). When $i = 0$, we have

$$\begin{aligned} \int_G f(g)\,dg &= \int_K \int_{H_{0,n-1}} \int_0^\infty \left(f(ka(s)h) + f\left(ka(s)\begin{bmatrix} -1 & \\ & I_{n-1} \end{bmatrix} h\right)\right) \cosh(s)^{n-1}\,ds\,dh\,dk \\ &= \int_K \int_{H_{0,n-1}} \int_0^\infty \left(f(ka(s)h) + f\left(ka(s)h\begin{bmatrix} -1 & \\ & I_{n-1} \end{bmatrix}\right)\right) \cosh(s)^{n-1}\,ds\,dh\,dk. \end{aligned} \tag{3.4}$$

For $i \neq 0$ and $i + j = n - 1$,

$$\int_G f(g)\,dg = \int_K \int_{H_{i,j}} \int_0^\infty f(ka(s)h) \sinh(s)^i \cosh(s)^j\,ds\,dh\,dk. \tag{3.5}$$

Moreover,

$$\int_G f(g)\,dg = \int_K \int_{\mathbb{R}^{n-1}} \int_{-\infty}^\infty f(ka(s)u(\mathbf{x}))e^{(n-1)s}\,ds\,d\mathbf{x}\,dk. \tag{3.6}$$

Consider

$$a(\infty) := \begin{bmatrix} 1/2 & \mathbf{0} & 1/2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 1/2 & \mathbf{0} & 1/2 \end{bmatrix}.$$

The matrix $a(\infty)$ will be useful below, since $a(s) = e^s a(\infty) + O(1)$ as $s \to +\infty$.

Lemma 3.1. *The following estimates hold:*

(1) For $h \in H_{i,j}$, $s \in \mathbb{R}$, we have $e^{|s|}\|h\| \ll \|ka(s)h\|$ uniformly in $k \in K$.

(2) Suppose that $h \in H_{i,j}$, where $j \neq 0$. Write h in the following Cartan form:

$$h = \begin{bmatrix} c & \\ & c_1 a_j(t) c_2 \end{bmatrix}, \quad c \in \mathrm{SO}(i+1), \quad c_1, c_2 \in \begin{bmatrix} \mathrm{SO}(j) & \\ & 1 \end{bmatrix},$$

where

$$a_j(t) = \begin{bmatrix} \cosh(t) & \mathbf{0} & \sinh(t) \\ \mathbf{0} & I_{j-1} & \mathbf{0} \\ \sinh(t) & \mathbf{0} & \cosh(t) \end{bmatrix}.$$

Then

$$\|h\| \asymp \|a_j(t)\| \asymp e^{|t|}.$$

(3) For $u(\mathbf{x}) \in U$, $s \in \mathbb{R}$, we have $e^s\|\mathbf{x}\|^2 + e^{|s|} \ll \|a(s)u(\mathbf{x})\|$. Moreover, if $s \geq 0$, then $e^s\|u(\mathbf{x})\| \ll \|a(s)u(\mathbf{x})\|$.

(4) Let $u(\mathbf{x}) \in U$. Then $\|u(\mathbf{x})\| \asymp 1 + \|\mathbf{x}\|^2$.

(5) The ratio $\|h\|/\|a(\infty)h\|$ is uniformly bounded for $h \in H_{i,j}$, and the ratio $\|u(\mathbf{x})\|/\|a(\infty)u(\mathbf{x})\|$ is uniformly bounded for $u(\mathbf{x}) \in U$.

Proof. By equivalence of norms, there is no loss of generality in assuming that $\|\cdot\|$ is the Hilbert–Schmidt norm, namely $\|g\| = \sqrt{\mathrm{Trace}(g^\intercal g)}$, which is bi-$K$-invariant. In particular,

$$\|ka(s)h\| = \|a(s)h\|, \ \forall k \in K, \forall a(s) \in A, \forall h \in H.$$

We now prove (1) and (2). First consider $H = H_{n-1,0}$, namely $H = K$. By bi-K-invariance, and since $\|h\| = \|I_{n+1}\| = \sqrt{n+1}$

$$\|ka(s)h\| = \|a(s)\| \asymp e^{|s|} \asymp \|h\|e^{|s|},$$

which gives (1) in this case.

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Now assume $H = H_{i,j}$ with $j \neq 0$. Writing h in Cartan form as in (2), and using again bi-K-invariance,

$$\|h\| = \left\| \begin{bmatrix} I_{i+1} & \\ & a_j(t) \end{bmatrix} \right\|.$$

Since

$$a_j(t) = \exp\left(\begin{bmatrix} \mathbf{0} & 1 \\ 1 & \mathbf{0} \end{bmatrix} t \right) = \begin{bmatrix} \cosh(t) & 0 & \sinh(t) \\ 0 & I_{j-1} & 0 \\ \sinh(t) & 0 & \cosh(t) \end{bmatrix},$$

we get

$$\|h\|^2 = (i+1) + (j-1) + 2\cosh(2t) = n - 1 + 2\cosh(2t) \asymp \cosh^2(t).$$

Hence $\|h\| \asymp \cosh(t) \asymp e^{|t|}$, which proves (2).

To prove (1), note that the bottom right entry of $a(s)h$ is $\cosh(s)\cosh(t)$. Therefore

$$\|a(s)h\| \geq \cosh(s)\cosh(t) \gg e^{|s|}\cosh(t) \asymp e^{|s|}\|h\|,$$

and (1) follows.

We now verify (3). Note that the bottom right entry of $a(s)u(\mathbf{x})$ equals $\frac{e^s}{2}\|\mathbf{x}\|^2 + \cosh(s)$. Thus

$$\|a(s)u(\mathbf{x})\| \geq \frac{e^s}{2}\|\mathbf{x}\|^2 + \cosh(s) \gg e^s\|\mathbf{x}\|^2 + e^{|s|},$$

which is exactly (3). If $s \geq 0$, then by (4),

$$\|a(s)u(\mathbf{x})\| \gg e^s\big(1 + \|\mathbf{x}\|^2\big) \asymp e^s\|u(\mathbf{x})\|,$$

which proves the second assertion in (3).

To prove (4), observe first that the bottom right entry of $u(\mathbf{x})$ is $1 + \frac{1}{2}\|\mathbf{x}\|^2$, so

$$\|u(\mathbf{x})\| \gg 1 + \|\mathbf{x}\|^2.$$

On the other hand, from the explicit expression (3.3) for $u(\mathbf{x})$, each matrix entry is $O(1 + \|\mathbf{x}\|^2)$, and therefore

$$\|u(\mathbf{x})\| \ll 1 + \|\mathbf{x}\|^2.$$

Hence

$$\|u(\mathbf{x})\| \asymp 1 + \|\mathbf{x}\|^2,$$

proving (4).

Finally, we verify (5). If $H = H_{n-1,0}$, then H is compact and, again by orthogonal invariance,

$$\|h\|/\|a(\infty)h\| = \sqrt{n+1}/\|a(\infty)\|.$$

If $H = H_{i,j}$ with $j \neq 0$, then the bottom right entry of $a(\infty)h$ is $\frac{1}{2}\cosh(t)$, and therefore

$$\|a(\infty)h\| \gg \cosh(t) \asymp \|h\|.$$

Likewise, the bottom right entry of $a(\infty)u(\mathbf{x})$ is $\frac{1}{2}\big(1 + \|\mathbf{x}\|^2\big)$, so

$$\|a(\infty)u(\mathbf{x})\| \gg 1 + \|\mathbf{x}\|^2 \asymp \|u(\mathbf{x})\|.$$

This proves (5). □

Lemma 3.2. *Let J denote either I_{n+1} or $a(\infty)$. Let $H = H_{i,j}$, and denote by dh a Haar measure on H. If $n - 2 < \alpha$, then $\int_H \frac{1}{\|Jh\|^\alpha} dh$ converges with the following tail estimate:*

$$\int_{T \leq \|h\|} \frac{1}{\|Jh\|^\alpha} dh \ll T^{(n-2)-\alpha}.$$

Otherwise, for $n - 2 \geq \alpha$:

$$\int_{\|h\| \leq T} \frac{1}{\|Jh\|^\alpha} dh \ll \begin{cases} \log(T) & \alpha = n-2, \\ T^{n-2-\alpha} & \alpha < n-2, \end{cases}$$

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Proof. If $j=0$, then $H_{n-1,0}=K$ is compact, so the claims are immediate. Thus assume $j>0$.

In the decomposition of $h\in H_{i,j}$ as

$$h=\begin{bmatrix} c & \\ & c_1a_j(t)c_2\end{bmatrix},\ t\geq 0,\quad c\in \mathrm{SO}(i+1),\quad c_1,c_2\in \begin{bmatrix}\mathrm{SO}(j) & \\ & 1\end{bmatrix},$$

the integration element of the $H_{i,j}$-Haar measure is given by $dh=dc\,dc_1\,dc_2\sinh(t)^{j-1}dt$; cf. (3.4). Then, by compactness, and by Lemma 3.1,

$$\int_{\|h\|\leq T}\frac{1}{\|Jh\|^\alpha}\,dh\ll\int_{e^t\leq cT}e^{-\alpha t}\sinh(t)^{j-1}\,dt\ll\int_{e^t\leq cT}e^{(j-1-\alpha)t}\,dt.$$

Since $j-1\leq n-2$, this gives

$$\int_{\|h\|\leq T}\frac{1}{\|Jh\|^\alpha}\,dh\ll\int_{e^t\leq cT}e^{((n-2)-\alpha)t}\,dt\ll\begin{cases}1, & \alpha>n-2,\\ \log T, & \alpha=n-2,\\ T^{n-2-\alpha}, & \alpha<n-2.\end{cases}$$

This proves the stated bound when $\alpha\leq n-2$, and also shows convergence of $\int_H\|Jh\|^{-\alpha}\,dh$ when $\alpha>n-2$.

For the tail, again using $\|Jh\|\gg e^t$ and $\|h\|\asymp e^t$,

$$\int_{T\leq\|h\|}\frac{1}{\|Jh\|^\alpha}\,dh\ll\int_{e^t\geq c^{-1}T}e^{-\alpha t}\sinh(t)^{j-1}\,dt\ll\int_{e^t\geq c^{-1}T}e^{((n-2)-\alpha)t}\,dt\ll T^{(n-2)-\alpha},$$

provided $\alpha>n-2$. This proves the lemma. □

We get the following analogous estimates for U.

Lemma 3.3. *Let J denote either I_{n+1} or $a(\infty)$, and let $d\mathbf{x}$ be a Haar measure on $U\cong\mathbb{R}^{n-1}$. If $\frac{n-1}{2}<\alpha$, then:*

$$\int_{T\leq\|u(\mathbf{x})\|}\frac{1}{\|Ju(\mathbf{x})\|^\alpha}d\mathbf{x}\ll T^{\frac{n-1}{2}-\alpha},$$

and otherwise:

$$\int_{\|u(\mathbf{x})\|\leq T}\frac{1}{\|Ju(\mathbf{x})\|^\alpha}d\mathbf{x}\ll\begin{cases}\log(T) & \alpha=\frac{n-1}{2},\\ T^{\frac{n-1}{2}-\alpha} & \alpha<\frac{n-1}{2},\end{cases}$$

for $J=I_{n+1}$ or $J=a(\infty)$.

Proof. By Lemma 3.1, there exists some $c>0$ such that:

$$\begin{aligned}\int_{\|u(\mathbf{x})\|\leq T}\frac{1}{\|Ju(\mathbf{x})\|^\alpha}d\mathbf{x}&\ll\int_{\|\mathbf{x}\|\leq\sqrt{cT-1}}\frac{1}{(\|\mathbf{x}\|^2+1)^\alpha}d\mathbf{x}\\ &\ll 1+\int_1^{\sqrt{T}}r^{(n-2)-2\alpha}dr.\end{aligned}$$

The tail estimate follows similarly. □

For $T>0,\ k\in K,\ h\in H$ consider

$$b_T(k,h)=\{0\leq s:\|ka(s)h\|\leq T\}. \tag{3.7}$$

The following asymptotic estimate will be key in describing the density in the limiting measure appearing in the main theorems.

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Lemma 3.4. *Fix $k \in K$. Then*

$$\lim_{T\to\infty} \frac{1}{T^{n-1}} \int_H \int_{b_T(k,h)} \omega(s)\, ds\, dh = \begin{cases} \frac{1}{(n-1)2^{n-1}} \int_H \frac{1}{\|ka(\infty)h\|^{n-1}}\, dh, & H = H_{i,j}, \\ \frac{1}{n-1} \int_H \frac{1}{\|ka(\infty)h\|^{n-1}}\, dh, & H = U, \end{cases}$$

where $\omega(s) = \sinh(s)^i \cosh(s)^j$ when $H = H_{i,j}$, and $\omega(s) = e^{(n-1)s}$ when $H = U$.

Proof. By Lemma 3.1(1) and (3), there exists $c_0 > 0$ such that $b_T(k,h) = \emptyset$ whenever $\|h\| \geq c_0 T$. Hence

$$\frac{1}{T^{n-1}} \int_H \int_{b_T(k,h)} \omega(s)\, ds\, dh = \frac{1}{T^{n-1}} \int_{\|h\|\leq c_0 T} \int_{b_T(k,h)} \omega(s)\, ds\, dh.$$

Also, we observe that

$$\big|\|ka(s)h\| - e^s \|ka(\infty)h\|\big| \leq \|k(a(s) - e^s a(\infty))h\| \ll \|h\|,$$

which implies, together with Lemma 3.1(5),

$$\|ka(s)h\| = e^s \|ka(\infty)h\| \left(1 + O\left(\frac{\|h\|}{e^s \|ka(\infty)h\|}\right)\right) = e^s \|ka(\infty)h\| \big(1 + O(e^{-s})\big).$$

Fix $\epsilon \in (0,1)$, and choose $s_0 > 0$ so that for all $s \geq s_0$,

$$\|ka(s)h\| = e^s \|ka(\infty)h\| \big(1 + O(\epsilon)\big).$$

Therefore, for some absolute $C > 0$,

$$\left\{ s \geq s_0 :\ e^s \leq \frac{T}{(1 + C\epsilon)\|ka(\infty)h\|} \right\} \subseteq b_T(k,h) \subseteq [0, s_0] \cup \left\{ s \geq s_0 :\ e^s \leq \frac{T}{(1 - C\epsilon)\|ka(\infty)h\|} \right\}. \tag{3.8}$$

The contribution of $[0, s_0]$ is negligible. Indeed,

$$\int_{\|h\|\leq c_0 T} \int_0^{s_0} \omega(s)\, ds\, dh \ll_{s_0} \int_{\|h\|\leq c_0 T} dh.$$

If $H = H_{i,j}$, then by Lemma 3.2 with $\alpha = 0$,

$$\int_{\|h\|\leq c_0 T} dh \ll \begin{cases} \log T, & n = 2, \\ T^{n-2}, & n \geq 3, \end{cases}$$

while if $H = U$, Lemma 3.3 with $\alpha = 0$ gives

$$\int_{\|h\|\leq c_0 T} dh \ll T^{\frac{n-1}{2}}.$$

Hence in both cases

$$\frac{1}{T^{n-1}} \int_{\|h\|\leq c_0 T} \int_0^{s_0} \omega(s)\, ds\, dh = o(1).$$

We proceed to establish the estimate in the two cases $H = H_{i,j}$ and U separately.
Case 1: $H = H_{i,j}$. Since $i + j = n - 1$,

$$\omega(s) = \sinh(s)^i \cosh(s)^j = \frac{1}{2^{n-1}} e^{(n-1)s} + r_n(s),$$

where $r_n(s) \ll e^{(n-3)s}$.

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Using the previous sandwich (3.8) for $b_T(k,h)$ and the negligible contribution of $[0,s_0]$, we get

$$\frac{1}{T^{n-1}}\int_H\int_{b_T(k,h)}\omega(s)\,ds\,dh=\frac{(1+O(\epsilon))^{n-1}}{(n-1)2^{n-1}}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh + E_T+o(1),$$

where the error E_T comes from $r_n(s)$.

We proceed to estimate E_T using the inclusion (3.8) and Lemmata 3.2–3.3. For $n=2$,

$$E_T\ll\frac{1}{T}\int_{\|h\|\le c_0T}dh\ll\frac{\log T}{T}=o(1).$$

For $n=3$,

$$E_T\ll\frac{\log T}{T^2}\int_{\|h\|\le c_0T}dh\ll\frac{\log T}{T}=o(1).$$

For $n\ge 4$,

$$E_T\ll\frac{1}{T^2}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-3}}\,dh=O(T^{-1})=o(1),$$

by Lemma 3.2, since $n-3<n-2$.

Thus

$$\frac{1}{T^{n-1}}\int_H\int_{b_T(k,h)}\omega(s)\,ds\,dh=\frac{(1+O(\epsilon))^{n-1}}{(n-1)2^{n-1}}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh+o(1).$$

Since $n-1>n-2$, Lemma 3.2 gives

$$\lim_{T\to\infty}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh=\int_H\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh.$$

Letting $T\to\infty$ and then $\epsilon\to 0$ yields

$$\lim_{T\to\infty}\frac{1}{T^{n-1}}\int_H\int_{b_T(k,h)}\omega(s)\,ds\,dh=\frac{1}{(n-1)2^{n-1}}\int_H\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh.$$

Case 2: $H=U$. Here $\omega(s)=e^{(n-1)s}$ exactly, so the same argument gives

$$\frac{1}{T^{n-1}}\int_H\int_{b_T(k,h)}\omega(s)\,ds\,dh=\frac{(1+O(\epsilon))^{n-1}}{n-1}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh+o(1).$$

Since $n-1>\frac{n-1}{2}$, Lemma 3.3 implies

$$\lim_{T\to\infty}\int_{\|h\|\le c_0T}\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh=\int_H\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh.$$

Letting $T\to\infty$ and then $\epsilon\to 0$ finishes the proof. □

Corollary 3.5. *Let*

$$G_T:=\{g\in G:\|g\|\le T\},\quad \mathrm{J}:=\begin{bmatrix}-1 & \\ & I_n\end{bmatrix},\quad a(-\infty):=Ja(\infty)J.$$

If $H=H_{0,n-1}$, then

$$\lim_{T\to\infty}\frac{\mathrm{Vol}(G_T)}{T^{n-1}}=\frac{1}{(n-1)2^{n-1}}\int_K\int_H\left(\frac{1}{\|ka(\infty)h\|^{n-1}}+\frac{1}{\|ka(-\infty)h\|^{n-1}}\right)dh\,dk. \tag{3.9}$$

If $H=H_{i,j}$ with $i\ne 0$, then

$$\lim_{T\to\infty}\frac{\mathrm{Vol}(G_T)}{T^{n-1}}=\frac{1}{(n-1)2^{n-1}}\int_K\int_H\frac{1}{\|ka(\infty)h\|^{n-1}}\,dh\,dk. \tag{3.10}$$

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If $H = U$, then

$$\lim_{T\to\infty} \frac{\mathrm{Vol}(G_T)}{T^{n-1}} = \frac{1}{n-1}\int_K \int_{\mathbb{R}^{n-1}} \frac{1}{\|ka(\infty)u(\mathbf{x})\|^{n-1}}\, d\mathbf{x}\, dk. \tag{3.11}$$

Proof. We prove (3.9) and (3.11). The proof of (3.10) is similar.

Case 1: $H = H_{0,n-1}$. For $k \in K$ and $h \in H$, define

$$b_T^+(k,h) := \{s \geq 0 : \|ka(s)h\| \leq T\}, \quad b_T^-(k,h) := \{s < 0 : \|ka(s)h\| \leq T\}.$$

Since $a(-s) = \mathrm{J}a(s)\mathrm{J}$, and since $\mathrm{J}h = h\mathrm{J}$, we have

$$b_T^-(k,h) := \{s < 0 : \|ka(s)h\| \leq T\} = \{s < 0 : \|Jk^{\mathrm{J}}a(-s)h\mathrm{J}\| \leq T\},$$

where $k^{\mathrm{J}} = \mathrm{J}k\mathrm{J}$. Then, after the change of variable $t = -s$ we have

$$\int_{b_T^-(k,h)} \cosh(s)^{n-1}\, ds = \int_{\widetilde{b}_T(k^{\mathrm{J}},h)} \cosh(t)^{n-1}\, dt,$$

where $\widetilde{b}_T(k^{\mathrm{J}},h) := \{t \geq 0 : \|k^{\mathrm{J}}a(t)h\|_{\mathrm{J}} \leq T\}$, with $\|g\|_{\mathrm{J}} := \|\mathrm{J}g\mathrm{J}\|$.

Hence, by the Haar measure formula (3.4) for $H_{0,n-1}$,

$$\frac{\mathrm{Vol}(G_T)}{T^{n-1}} = \frac{1}{T^{n-1}}\int_K \int_H \left(\int_{b_T^+(k,h)} + \int_{\widetilde{b}_T(k^{\mathrm{J}},h)}\right) \cosh(s)^{n-1}\, ds\, dh\, dk.$$

Apply Lemma 3.4 to the first inner integral with respect to the norm $\|\cdot\|$, and to the second inner integral with respect to the norm $\|\cdot\|_{\mathrm{J}}$, and then dominated convergence in k, we obtain

$$\lim_{T\to\infty} \frac{\mathrm{Vol}(G_T)}{T^{n-1}} = \frac{1}{(n-1)2^{n-1}}\int_K \int_H \left(\frac{1}{\|ka(\infty)h\|^{n-1}} + \frac{1}{\|k^{\mathrm{J}}a(\infty)h\|_{\mathrm{J}}^{n-1}}\right) dh\, dk.$$

For the second term above, note that

$$\frac{1}{\|k^{\mathrm{J}}a(\infty)h\|_{\mathrm{J}}^{n-1}} = \frac{1}{\|\mathrm{J}k^{\mathrm{J}}a(\infty)h\mathrm{J}\|^{n-1}} = \frac{1}{\|ka(-\infty)h\|^{n-1}},$$

which proves (3.9).

Case 2: $H = U$. By the Haar measure formula for U,

$$\frac{\mathrm{Vol}(G_T)}{T^{n-1}} = \frac{1}{T^{n-1}}\int_K \int_{\mathbb{R}^{n-1}} \left(\int_{b_T^+(k,u(\mathbf{x}))} + \int_{b_T^-(k,u(\mathbf{x}))}\right) e^{(n-1)s}\, ds\, d\mathbf{x}\, dk.$$

We claim that

$$\frac{1}{T^{n-1}}\int_K \int_{\mathbb{R}^{n-1}} \int_{b_T^-(k,u(\mathbf{x}))} e^{(n-1)s}\, ds\, d\mathbf{x}\, dk = O\left(\frac{\log T}{T^{\frac{n-1}{2}}}\right). \tag{3.12}$$

Indeed, by Lemma 3.1, (3), there exists $c_0 > 0$ such that

$$b_T^-(k,u(\mathbf{x})) \subseteq \{s < 0 : e^s\|\mathbf{x}\|^2 + e^{-s} \leq c_0 T\}.$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} \int_{b_T^-(k,u(\mathbf{x}))} e^{(n-1)s}\, ds\, d\mathbf{x} &\ll \int_{-\log(c_0T)}^0 \left(\frac{c_0T - e^{-s}}{e^s}\right)^{\frac{n-1}{2}} e^{(n-1)s}\, ds \\ &= \int_{-\log(c_0T)}^0 (c_0Te^s - 1)^{\frac{n-1}{2}}\, ds \\ &\ll T^{\frac{n-1}{2}}\log T,\end{aligned}$$

which proves (3.12).

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Hence the negative-s contribution is negligible after division by T^{n-1}, and Lemma 3.4 gives

$$\lim_{T\to\infty}\frac{\mathrm{Vol}(G_T)}{T^{n-1}} = \lim_{T\to\infty}\frac{1}{T^{n-1}}\int_K\int_{\mathbb{R}^{n-1}}\int_{b_T^+(k,u(\mathbf{x}))} e^{(n-1)s}\,ds\,d\mathbf{x}\,dk$$
$$= \frac{1}{n-1}\int_K\int_{\mathbb{R}^{n-1}}\frac{1}{\|ka(\infty)u(\mathbf{x})\|^{n-1}}\,d\mathbf{x}\,dk.$$

This proves (3.11). □

Next, for $L>0$, $k\in K$, and $h\in H$, define

$$d_{T,L}(k,h) := \{0\le s\le \log T - L:\ \|ka(s)h\|\le T\}.$$

Lemma 3.6. *Let H be either $H_{i,j}$ or U. For every $\epsilon>0$ there exist $L_0=L_0(\epsilon)$ and $T_0=T_0(\epsilon)$ such that, for all $L\ge L_0$, $T\ge T_0$, and all $k\in K$,*

$$\frac{1}{T^{n-1}}\int_H\int_{d_{T,L}(k,h)}\omega(s)\,ds\,dh\le\epsilon,$$

where $\omega(s)=\sinh(s)^i\cosh(s)^j$ if $H=H_{i,j}$, and $\omega(s)=e^{(n-1)s}$ if $H=U$.

Proof. By Lemma 3.1, (1), and (3), there exists $c_0>0$ such that

$$d_{T,L}(k,h)\subseteq\left\{0\le s\le\log T-L:\ e^s\le\frac{c_0T}{\|h\|}\right\}.$$

Since $\omega(s)\ll e^{(n-1)s}$, we get

$$\frac{1}{T^{n-1}}\int_H\int_{d_{T,L}(k,h)}\omega(s)\,ds\,dh \ll \frac{1}{T^{n-1}}\int_{\|h\|\le c_0e^L}\int_0^{\log T-L} e^{(n-1)s}\,ds\,dh$$
$$+\frac{1}{T^{n-1}}\int_{c_0e^L\le\|h\|\le c_0T}\int_0^{\log(c_0T/\|h\|)} e^{(n-1)s}\,ds\,dh.$$

For the first term,

$$\frac{1}{T^{n-1}}\int_{\|h\|\le c_0e^L}\int_0^{\log T-L} e^{(n-1)s}\,ds\,dh \ll e^{-(n-1)L}\int_{\|h\|\le c_0e^L}dh = o_L(1),$$

by Lemma 3.2 or Lemma 3.3 with $\alpha=0$.

For the second term,

$$\frac{1}{T^{n-1}}\int_{c_0e^L\le\|h\|\le c_0T}\int_0^{\log(c_0T/\|h\|)} e^{(n-1)s}\,ds\,dh \ll \int_{\|h\|\ge c_0e^L}\frac{1}{\|h\|^{n-1}}\,dh+\frac{1}{T^{n-1}}\int_{\|h\|\le c_0T}dh$$
$$= o_L(1)+o_T(1),$$

by Lemma 3.2 or Lemma 3.3 with $\alpha=n-1$ for the first term and $\alpha=0$ for the second. Now choose L_0 so that the $o_L(1)$ terms are $<\epsilon/2$ for $L\ge L_0$, and then choose T_0 so that the $o_T(1)$ term is $<\epsilon/2$ for $T\ge T_0$. □

We denote

$$b_{T,L}(k,h) := b_T(k,h)\setminus d_{T,L}(k,h) = \{s\ge\log T-L:\ \|ka(s)h\|\le T\}. \tag{3.13}$$

Note that there exists $c_0>0$ such that if $\|h\|\ge c_0e^L$, then $b_{T,L}(k,h)=\emptyset$ by Lemma 3.1. Thus, after restricting to the late range $s\ge\log T-L$, the H-integration is automatically confined to the bounded region $\|h\|\ll e^L$, uniformly in T. This is used later in Section 6.1 in the proof deriving the limiting distribution, via Corollary 3.7 below, to truncate the h-integration to a compact range before applying the equidistribution argument.

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Corollary 3.7. *Let $f : G \to \mathbb{R}$ be bounded. For every $\epsilon > 0$ there exist $L_0 = L_0(\epsilon, f)$ and $T_0 = T_0(\epsilon, f)$ such that, for all $L \geq L_0$, $T \geq T_0$, and all $k \in K$,*

$$\left| \frac{1}{\mathrm{Vol}(G_T)} \int_H \int_{b_T(k,h)} f(ka(s)h)\omega(s)\, ds\, dh - \frac{1}{\mathrm{Vol}(G_T)} \int_H \int_{b_{T,L}(k,h)} f(ka(s)h)\omega(s)\, ds\, dh \right| \leq \epsilon.$$

Moreover, there exists $c_0 > 0$ such that $b_{T,L}(k,h) = \emptyset$ whenever $\|h\| \geq c_0 e^L$.

Proof. Since $b_T(k,h) \setminus b_{T,L}(k,h) = d_{T,L}(k,h)$,

$$\left| \frac{1}{\mathrm{Vol}(G_T)} \int_H \int_{b_T(k,h)} f(ka(s)h)\omega(s)\, ds\, dh - \frac{1}{\mathrm{Vol}(G_T)} \int_H \int_{b_{T,L}(k,h)} f(ka(s)h)\omega(s)\, ds\, dh \right|$$
$$\leq \frac{\|f\|_\infty}{\mathrm{Vol}(G_T)} \int_H \int_{d_{T,L}(k,h)} \omega(s)\, ds\, dh.$$

By Corollary 3.5, $\mathrm{Vol}(G_T) \asymp T^{n-1}$, and by Lemma 3.6, $\frac{1}{\mathrm{Vol}(G_T)} \int_H \int_{d_{T,L}(k,h)} \omega(s)\, ds\, dh$ can be made arbitrarily small by taking first L and then T sufficiently large. This proves the first claim. The second is exactly the observation above. □

4. Action on orthogonal lattices

Let $n \geq 2$ and $r < n+1$ be natural numbers and denote $X := X_{r,n+1}$. A challenge in analyzing the G-action (1.1) on X is that the lattices not only change shape, but also move between different ambient subspaces. In our setting, a convenient way to separate these two effects is to use a KAH decomposition. The subgroup H is chosen to preserve the ambient subspaces of the lattices, and therefore acts on the lattices inside these subspaces by explicit transformations. The essential point is then to understand the action of the diagonal part A. We show that asymptotically this action can be written in the form $R\delta$, where δ acts on the lattices inside their subspaces by explicit diagonal matrices in $\mathrm{PGL}(k, \mathbb{R})$, while R moves the ambient subspaces to the distinguished degenerate pair $\mathbf{P}^\infty$; see Lemma 4.2.

Combined with Corollary 3.7, this reduces the original equidistribution problem to the well-studied problem of equidistribution of expanding translates of bounded manifolds treated in Section 5. Those results yield equidistribution for the lattices contained in $\mathbf{P}^\infty$, and the K-component then moves this limiting distribution to the lattices contained in the other degenerate pairs.

4.1. **The case of $H_{i,j}$.** Fix nonnegative integers i, j such that $i + j = n - 1$. Consider the pair of nondegenerate orthogonal complementary subspaces $P_1^{(0)}, P_2^{(0)} \subseteq \mathbb{R}^{n+1}$ defined by

$$\begin{aligned} P_1^{(0)} &:= \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \dots, \mathbf{e}_{i+1}\}, \\ P_2^{(0)} &:= \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_{i+2}, \dots, \mathbf{e}_{n+1}\}, \\ \mathbf{P}^{(0)} &:= (P_1^{(0)}, P_2^{(0)}). \end{aligned} \tag{4.1}$$

Consider $X_{\mathbf{P}^{(0)}} := \{([\Lambda_1], [\Lambda_2]) \in X : \Lambda_1 \subseteq P_1^{(0)}\}$, and observe that the subgroup $H_{i,j} \subseteq \mathrm{SO}(n,1)^\circ$ acts on $X_{\mathbf{P}^{(0)}}$ with respect to the action induced by the G-action (1.1). More specifically, for $h \in H_{i,j}$, write

$$h = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix},\ h_1 \in \mathrm{SO}(i+1),\ h_2 \in \mathrm{SO}(j,1)^\circ.$$

Then,

$$h \cdot ([\Lambda_1], [\Lambda_2]) = ([h_1\Lambda_1], [h_2^*\Lambda_2]), \quad h \in H_{i,j}.$$

It will be convenient to work with the following matrix representation. Let

$$\rho^{(0)} : \mathrm{GL}(i+1, \mathbb{R}) \times \mathrm{GL}(j+1, \mathbb{R}) \to \mathrm{GL}(P_1^{(0)}) \times \mathrm{GL}(P_2^{(0)})$$

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be the representation obtained by expressing linear maps $P_1^{(0)} \to P_1^{(0)}$ and $P_2^{(0)} \to P_2^{(0)}$ in the ordered bases

$$B_1 := (\mathbf{e}_1, \dots, \mathbf{e}_{i+1}), \quad B_2 := (\mathbf{e}_{i+2}, \dots, \mathbf{e}_{n+1}).$$

For $k \in \mathbb{N}$, denote

$$X_k := \mathrm{PGL}(k, \mathbb{R})/\mathrm{PGL}(k, \mathbb{Z}).$$

The representation $\rho^{(0)}$ induces an action of $\mathrm{PGL}(i+1,\mathbb{R}) \times \mathrm{PGL}(j+1,\mathbb{R})$ and hence an identification

$$X_{\mathbf{P}^{(0)}} \cong X_{i+1} \times X_{j+1},$$

under which the identity coset corresponds to $\big([\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_1, \dots, \mathbf{e}_{i+1}\}], [\mathrm{Span}_{\mathbb{Z}}\{\mathbf{e}_{i+2}, \dots, \mathbf{e}_{n+1}\}]\big)$.

For $([\Lambda_1], [\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$, if $(y_1, y_2) \in X_{i+1} \times X_{j+1}$ is such that $\rho^{(0)}(y_1, y_2) = ([\Lambda_1], [\Lambda_2])$, then

$$\left[\begin{smallmatrix} h_1 & 0 \\ 0 & h_2 \end{smallmatrix}\right] \cdot ([\Lambda_1], [\Lambda_2]) = \rho^{(0)}\big(h_1 y_1, h_2^* y_2\big).$$

We now define matrices $g_1(s)$ and $g_2(s)$, for $s > 0$, that describe the asymptotic action of $a(s)$ on $X_{\mathbf{P}^{(0)}}$.

Let $v^+, v^- \in \mathbb{R}^{n+1}$ be given by

$$v^+ := \left(\begin{smallmatrix} 1 \\ \mathbf{0} \\ 1 \end{smallmatrix}\right), \; v^- := \left(\begin{smallmatrix} 1 \\ \mathbf{0} \\ -1 \end{smallmatrix}\right).$$

Then

$$a(s)v^+ = e^s v^+, \quad a(s)v^- = e^{-s}v^-, \quad a(s)\mathbf{e}_q = \mathbf{e}_q \quad (2 \le q \le n). \tag{4.2}$$

Since $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, v^-)$ and $(v^+, \mathbf{e}_2, \dots, \mathbf{e}_n, \mathbf{e}_{n+1})$ are bases of $\mathbb{R}^{n+1}$, we define $g_1(s), g_2(s) \in \mathrm{GL}(n+1, \mathbb{R})$ by

$$g_1(s)\mathbf{e}_1 := \frac{e^s}{2} v^+, \quad g_1(s)\mathbf{e}_q := \mathbf{e}_q \; (2 \le q \le n), \quad g_1(s)v^- := e^{-s}v^-,$$

and

$$g_2(s)v^+ := e^{-s}v^+, \quad g_2(s)\mathbf{e}_q := \mathbf{e}_q \; (2 \le q \le n), \quad g_2(s)\mathbf{e}_{n+1} := -\frac{e^s}{2} v^-.$$

Lemma 4.1. *It holds that*

$$\lim_{s\to\infty} a(s)(g_1(s))^{-1} = \mathrm{id}, \quad \lim_{s\to\infty} a(s)^*(g_2(s))^{-1} = \mathrm{id}.$$

Proof. It is enough to check the convergence on the basis $(v^+, \mathbf{e}_2, \dots, \mathbf{e}_n, v^-)$.

We first prove that $a(s)(g_1(s))^{-1} \to \mathrm{id}$. For $2 \le q \le n$,

$$a(s)(g_1(s))^{-1}\mathbf{e}_q = \mathbf{e}_q.$$

Also,

$$a(s)(g_1(s))^{-1}v^- = a(s)(e^s v^-) = v^-.$$

Finally, since $2\mathbf{e}_1 = v^+ + v^-$, we get

$$a(s)(g_1(s))^{-1}v^+ = a(s)(2e^{-s}\mathbf{e}_1) = e^{-s}a(s)(v^+ + v^-) = v^+ + e^{-2s}v^- \longrightarrow v^+.$$

Hence $a(s)(g_1(s))^{-1} \to \mathrm{id}$.

Similarly, for the second limit, for $2 \le q \le n$,

$$a(s)^*(g_2(s))^{-1}\mathbf{e}_q = \mathbf{e}_q,$$

and

$$a(s)^*(g_2(s))^{-1}v^+ = a(s)^*(e^s v^+) = v^+.$$

Since $2\mathbf{e}_{n+1} = v^+ - v^-$, we have

$$(g_2(s))^{-1}v^- = -2e^{-s}\mathbf{e}_{n+1} = e^{-s}(v^- - v^+).$$

Therefore,

$$a(s)^*(g_2(s))^{-1}v^- = e^{-s}\big(a(s)^*v^- - a(s)^*v^+\big) = v^- - e^{-2s}v^+ \longrightarrow v^-.$$

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This proves $a(s)^*(g_2(s))^{-1} \to \mathrm{id}$. □

Choose $R_1 \in \mathrm{SL}(n+1,\mathbb{R})$ such that

$$R_1\mathbf{e}_1 = \frac{1}{2}v^+,\ R_1\mathbf{e}_q = \mathbf{e}_q \text{ for } 2 \le q \le n,$$

and choose $R_2 \in \mathrm{SL}(n+1,\mathbb{R})$ such that

$$R_2\mathbf{e}_q = \mathbf{e}_q \text{ for } 2 \le q \le n,\ R_2\mathbf{e}_{n+1} = -\frac{1}{2}v^-.$$

We denote

$$R([\Lambda_1],[\Lambda_2]) := ([R_1\Lambda_1],[R_2\Lambda_2]),\ ([\Lambda_1],[\Lambda_2]) \in X_{r,n+1}.$$

Consider the complementary orthogonal degenerate subspaces

$$\begin{aligned} P_1^\infty &:= \mathrm{Span}_{\mathbb{R}}\{v^+, \mathbf{e}_2, \dots, \mathbf{e}_{i+1}\},\\ P_2^\infty &:= \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_{i+2}, \dots, \mathbf{e}_n, v^-\},\\ \mathbf{P}^\infty &:= (P_1^\infty, P_2^\infty). \end{aligned} \tag{4.3}$$

We notice that both R_i and $g_i(s)$ map $P_i^{(0)}$ into P_i^∞ for $i = 1, 2$. In particular, R maps $X_{\mathbf{P}^{(0)}}$ bijectively onto $X_{\mathbf{P}^\infty}$.

Denote

$$d_1(s) := \begin{bmatrix} e^{\frac{is}{i+1}} & 0 \\ 0 & e^{-\frac{s}{i+1}} I_i \end{bmatrix} \in \mathrm{SL}(i+1,\mathbb{R}), \quad d_2(s) := \begin{bmatrix} e^{-\frac{s}{j+1}} I_j & 0 \\ 0 & e^{\frac{js}{j+1}} \end{bmatrix} \in \mathrm{SL}(j+1,\mathbb{R}). \tag{4.4}$$

Lemma 4.2. *Let $([\Lambda_1],[\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$ and let $(y_1, y_2) \in X_{i+1} \times X_{j+1}$ such that $\rho^{(0)}(y_1, y_2) = ([\Lambda_1],[\Lambda_2])$. Then*

$$([g_1(s)\Lambda_1],[g_2(s)\Lambda_2]) = R\Big(\rho^{(0)}\big(d_1(s)y_1, d_2(s)y_2\big)\Big). \tag{4.5}$$

Proof. Since $g_i(s)$ and R_i map $P_i^{(0)}$ to P_i^∞ for $i = 1, 2$, we obtain that

$$A_1(s) := R_1^{-1} g_1(s)\big|_{P_1^{(0)}} \in \mathrm{GL}(P_1^{(0)}), \quad A_2(s) := R_2^{-1} g_2(s)\big|_{P_2^{(0)}} \in \mathrm{GL}(P_2^{(0)}).$$

To prove the claim, we compute the matrices representing $A_1(s)$ and $A_2(s)$ in the ordered bases $(\mathbf{e}_1, \dots, \mathbf{e}_{i+1})$ and $(\mathbf{e}_{i+2}, \dots, \mathbf{e}_{n+1})$ respectively.

For $A_1(s)$, we have

$$A_1(s)\mathbf{e}_1 = R_1^{-1}\left(\frac{e^s}{2}v^+\right) = e^s\mathbf{e}_1, \quad A_1(s)\mathbf{e}_q = \mathbf{e}_q \quad (2 \le q \le i+1).$$

Hence

$$[A_1(s)]_{B_1} = \mathrm{diag}(e^s, 1, \dots, 1) = e^{\frac{s}{i+1}} d_1(s).$$

For $A_2(s)$, we have

$$A_2(s)\mathbf{e}_q = \mathbf{e}_q \quad (i+2 \le q \le n), \quad A_2(s)\mathbf{e}_{n+1} = R_2^{-1}\left(-\frac{e^s}{2}v^-\right) = e^s\mathbf{e}_{n+1}.$$

Hence

$$[A_2(s)]_{B_2} = \mathrm{diag}(1, \dots, 1, e^s) = e^{\frac{s}{j+1}} d_2(s).$$

Therefore, for $([\Lambda_1],[\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$ and $(y_1, y_2) \in X_{i+1} \times X_{j+1}$ such that $\rho^{(0)}(y_1, y_2) = ([\Lambda_1],[\Lambda_2])$, we obtain that

$$([A_1(s)\Lambda_1],[A_2(s)\Lambda_2]) = \rho^{(0)}(e^{\frac{s}{i+1}} d_1(s)y_1, e^{\frac{s}{j+1}} d_2(s)y_2) = \rho^{(0)}(d_1(s)y_1, d_2(s)y_2).$$

Applying R, we obtain (4.5). □

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For $x \in G/\Gamma$, $y = (y_1, y_2) \in X_{i+1} \times X_{j+1}$, $k \in K$, and $T > 0$, we define the following measure on $G/\Gamma \times X_{i+1} \times X_{j+1}$ by

$$\mu_{x,y,k,T}(F) := \frac{1}{\mathrm{Vol}(G_T)} \int_{H_{i,j}} \int_{b_T(k,h)} F\big(a(s)hx, d_1(s)h_1y_1, d_2(s)h_2^*y_2\big)\omega(s)\, ds\, dh, \tag{4.6}$$

where $b_T(k,h)$ is given in (3.7), and $\omega(s) = \sinh(s)^i \cosh(s)^j$.

Define $\Psi : G/\Gamma \times X_{i+1} \times X_{j+1} \to G/\Gamma \times X_{\mathbf{P}^\infty}$ by:

$$\Psi(x,y) := \big(x, R\rho^{(0)}(y)\big), \quad x \in G/\Gamma,\ y \in X_{i+1} \times X_{j+1}, \tag{4.7}$$

and, for $k \in K$, define

$$k\Psi(x,y) := \big(kx, k \cdot (R\rho^{(0)}(y))\big), \tag{4.8}$$

where in the second coordinate k acts as in (1.1).

Lemma 4.3. *Fix $x_0 \in G/\Gamma$, let $([\Lambda_1],[\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$, and choose $y_0 \in X_{i+1} \times X_{j+1}$ with $\rho^{(0)}(y_0) = ([\Lambda_1],[\Lambda_2])$.*

Suppose that for every $k \in K$,

$$\operatorname*{weak}^*\lim_{T\to\infty} \mu_{x_0,y_0,k,T} = \mu_{k,\infty}. \tag{4.9}$$

Then, for every $F \in C_c(G/\Gamma \times X)$,

$$\lim_{T\to\infty} \frac{1}{\mathrm{Vol}(G_T)} \int_{H_{i,j}} \int_{b_T(k,h)} F\big(ka(s)hx_0, ka(s)h \cdot ([\Lambda_1],[\Lambda_2])\big)\omega(s)\, ds\, dh = (k\Psi)_*\mu_{k,\infty}(F).$$

The above lemma follows as a direct corollary of Lemmata 4.1–4.2 as follows. By applying Lemma 4.2 to the point $h \cdot ([\Lambda_1],[\Lambda_2]) = ([h_1\Lambda_1],[h_2^*\Lambda_2])$, and then using Lemma 4.1 to replace $([g_1(s)h_1\Lambda_1],[g_2(s)h_2^*\Lambda_2])$ by $a(s)h\cdot([\Lambda_1],[\Lambda_2])$ inside the test function. The contribution of bounded values of s is negligible after normalization by $\mathrm{Vol}(G_T)$, exactly as in the proof of Lemma 3.4.

4.2. **The case of U.** The discussion here is analogous to the previous subsection treating the case $H = H_{i,j}$. Recall the pair of orthogonal degenerate subspaces $\mathbf{P}^\infty = (P_1^\infty, P_2^\infty)$ from (4.3). We consider the representation

$$\rho^\infty : \mathrm{GL}(i+1,\mathbb{R}) \times \mathrm{GL}(j+1,\mathbb{R}) \to \mathrm{GL}(P_1^\infty) \times \mathrm{GL}(P_2^\infty),$$

defined by expressing linear transformations $P_1^\infty \to P_1^\infty$ and $P_2^\infty \to P_2^\infty$ in the ordered bases

$$B_1^\infty := (v^+, \mathbf{e}_2, \dots, \mathbf{e}_{i+1}), \quad B_2^\infty := (v^-, \mathbf{e}_{i+2}, \dots, \mathbf{e}_n),$$

respectively.

For $\mathbf{x} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{n-1} = \mathbb{R}^i \times \mathbb{R}^j$, we define

$$\upsilon(\mathbf{x}) = \upsilon(\boldsymbol{a},\boldsymbol{b}) := \left(\begin{bmatrix} 1 & \boldsymbol{a} \\ \mathbf{0} & I_i \end{bmatrix}, \begin{bmatrix} 1 & \boldsymbol{b} \\ \mathbf{0} & I_j \end{bmatrix}\right) \in \mathrm{GL}(i+1,\mathbb{R}) \times \mathrm{GL}(j+1,\mathbb{R}). \tag{4.10}$$

We also define

$$\delta(s) := \left(\begin{bmatrix} e^{\frac{is}{i+1}} & 0 \\ 0 & e^{-\frac{s}{i+1}} I_i \end{bmatrix}, \begin{bmatrix} e^{\frac{js}{j+1}} & 0 \\ 0 & e^{-\frac{s}{j+1}} I_j \end{bmatrix}\right), \quad s > 0. \tag{4.11}$$

Before proving the next lemma, recall that

$$u(\mathbf{x})v^+ = v^+, \quad u(\mathbf{x})\mathbf{e}_q = x_q v^+ + \mathbf{e}_q \quad (2 \le q \le n). \tag{4.12}$$

Also, using the explicit formula for $u(\mathbf{x})$, one checks that

$$u(\mathbf{x})^* v^- = v^-, \quad u(\mathbf{x})^*\mathbf{e}_q = x_q v^- + \mathbf{e}_q \quad (2 \le q \le n). \tag{4.13}$$

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Lemma 4.4. *Let $([\Lambda_1],[\Lambda_2]) \in X_{\mathbf{P}^\infty}$ and let $y \in X_{i+1} \times X_{j+1}$ such that $\rho^\infty(y) = ([\Lambda_1],[\Lambda_2])$. Then, for every $\mathbf{x} = (\boldsymbol{a},\boldsymbol{b}) \in \mathbb{R}^i \times \mathbb{R}^j$ and every $s > 0$,*

$$a(s)u(\mathbf{x}) \cdot ([\Lambda_1],[\Lambda_2]) = \rho^\infty\big((\delta(s)\upsilon(\mathbf{x})) \cdot y\big). \tag{4.14}$$

Proof. Since $a(s)u(\mathbf{x})$ preserves P_1^∞, and $(a(s)u(\mathbf{x}))^*$ preserves P_2^∞, it is enough to compute the representing matrices of

$$a(s)u(\mathbf{x})\big|_{P_1^\infty} \quad \text{and} \quad (a(s)u(\mathbf{x}))^*\big|_{P_2^\infty}$$

in the bases B_1^∞ and B_2^∞.

Using (4.2) and (4.12), we get $a(s)u(\mathbf{x})v^+ = e^s v^+$, and,

$$a(s)u(\mathbf{x})\mathbf{e}_q = a(s)(x_q v^+ + \mathbf{e}_q) = e^s x_q v^+ + \mathbf{e}_q, \quad \text{for } 2 \le q \le i+1.$$

Therefore the representing matrix of $a(s)u(\mathbf{x})\big|_{P_1^\infty}$ in the basis B_1^∞ is

$$\begin{bmatrix} e^s & e^s\boldsymbol{a} \\ 0 & I_i \end{bmatrix} = \begin{bmatrix} e^s & 0 \\ 0 & I_i \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{a} \\ 0 & I_i \end{bmatrix} = e^{\frac{s}{i+1}} \begin{bmatrix} e^{\frac{is}{i+1}} & 0 \\ 0 & e^{-\frac{s}{i+1}} I_i \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{a} \\ 0 & I_i \end{bmatrix}.$$

Similarly, using (4.13) and the fact that $a(s)^* v^- = e^s v^-$ and $a(s)^* \mathbf{e}_q = \mathbf{e}_q$ for $2 \le q \le n$, we get

$$(a(s)u(\mathbf{x}))^* v^- = e^s v^-,$$

and, for $i+2 \le q \le n$,

$$(a(s)u(\mathbf{x}))^* \mathbf{e}_q = a(s)^*(x_q v^- + \mathbf{e}_q) = e^s x_q v^- + \mathbf{e}_q.$$

Therefore the representing matrix of $(a(s)u(\mathbf{x}))^*\big|_{P_2^\infty}$ in the basis B_2^∞ is

$$\begin{bmatrix} e^s & e^s\boldsymbol{b} \\ 0 & I_j \end{bmatrix} = \begin{bmatrix} e^s & 0 \\ 0 & I_j \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{b} \\ 0 & I_j \end{bmatrix} = e^{\frac{s}{j+1}} \begin{bmatrix} e^{\frac{js}{j+1}} & 0 \\ 0 & e^{-\frac{s}{j+1}} I_j \end{bmatrix} \begin{bmatrix} 1 & \boldsymbol{b} \\ 0 & I_j \end{bmatrix}.$$

Since scalar homotheties are trivial in the projective actions on X_{i+1} and X_{j+1}, the extra scalar factors $e^{\frac{s}{i+1}}$ and $e^{\frac{s}{j+1}}$ are absorbed. This proves (4.14). □

For $x \in G/\Gamma$, $y = (y_1, y_2) \in X_{i+1} \times X_{j+1}$, $k \in K$, and $T > 0$, define a measure on $G/\Gamma \times X_{i+1} \times X_{j+1}$ by

$$\nu_{x,y,k,T}(F) := \frac{1}{\mathrm{Vol}(G_T)} \int_{\mathbb{R}^{n-1}} \int_{b_T(k,u(\mathbf{x}))} F\big(a(s)u(\mathbf{x})x, \delta(s)\upsilon(\mathbf{x})y\big) e^{(n-1)s}\, ds\, d\mathbf{x}, \tag{4.15}$$

where $b_T(k,u(\mathbf{x}))$ is given in (3.7).

If $n = 2$, then either $i = 0$ or $j = 0$, so one of X_{i+1}, X_{j+1} is a singleton. In that case, we identify $X_{i+1} \times X_{j+1}$ with X_2, and (4.15) becomes

$$\nu_{x,y,k,T}(F) := \frac{1}{\mathrm{Vol}(G_T)} \int_{\mathbb{R}} \int_{b_T(k,u(t))} F\big(a(s)u(t)x, \delta(s)\upsilon(t)y\big) e^s\, ds\, dt,$$

where $\upsilon(t) = \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$.

We consider the map $\Psi^\infty : G/\Gamma \times X_{i+1} \times X_{j+1} \to G/\Gamma \times X_{\mathbf{P}^\infty}$,

$$\Psi^\infty(x,y) := (x, \rho^\infty(y)),\ x \in G/\Gamma,\ y \in X_{i+1} \times X_{j+1},$$

and, for $k \in K$, we define

$$k\Psi^\infty(x,y) := (kx, k \cdot \rho^\infty(y)),\ x \in G/\Gamma,\ y \in X_{i+1} \times X_{j+1},$$

where in the second coordinate k acts as in (1.1).

We have the following statement, analogous to Lemma 4.3. It is an immediate consequence of Lemma 4.4.

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Lemma 4.5. *Fix $x_0 \in G/\Gamma$, let $([\Lambda_1],[\Lambda_2]) \in X_{\mathbf{P}^\infty}$, and let $y_0 \in X_{i+1} \times X_{j+1}$ such that $\rho^\infty(y_0) = ([\Lambda_1],[\Lambda_2])$. Suppose that for every $k \in K$,*

$$\operatorname*{weak}^*\lim_{T\to\infty} \nu_{x_0,y_0,k,T} = \nu_{k,\infty}. \tag{4.16}$$

Then, for all $F \in C_c(G/\Gamma \times X)$,

$$\lim_{T\to\infty} \frac{1}{\mathrm{Vol}(G_T)} \int_{\mathbb{R}^{n-1}} \int_{b_T(k,u(\mathbf{x}))} F\big(ka(s)u(\mathbf{x})x_0, ka(s)u(\mathbf{x})\cdot([\Lambda_1],[\Lambda_2])\big) e^{(n-1)s}\, ds\, d\mathbf{x}$$
$$= (k\Psi^\infty)_* \nu_{k,\infty}(F).$$

5. Equidistribution statements

The goal of this section is to prove the equidistribution statements needed to establish the convergence of (4.9) and (4.16).

We divide the argument into two cases, $n \geq 3$ and the $n = 2$ case.

For $n \geq 3$, we will show that every weak-* limit has Haar marginals in each factor and is invariant under a nontrivial unipotent element whose projection to each factor acts ergodically. We will then use the fact that, in this setting, there are no nontrivial algebraic joinings. It will follow that the limiting measure is the full product measure.

The case $n = 2$ is more delicate because of the exceptional isogeny $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}(2,1)^\circ$, which allows additional algebraic joinings and therefore requires a separate analysis.

5.1. **The $n \geq 3$-case.** Let $n \geq 3$, and fix nonnegative integers i, j such that $i + j = n - 1$. Set $\mathcal{G} := \mathrm{SO}(n,1)^\circ \times \mathrm{SL}(i+1,\mathbb{R}) \times \mathrm{SL}(j+1,\mathbb{R})$, and consider the subgroup

$$\mathcal{H} := \left\{(h, h_1, h_2^*) : h = \begin{bmatrix} h_1 & \\ & h_2 \end{bmatrix} \in H_{i,j}\right\} \subseteq \mathcal{G},$$

where $H_{i,j} \cong \mathrm{SO}(i+1) \times \mathrm{SO}(j,1)^\circ$ is as in (3.1). We also define

$$\mathcal{U} := \{(u(\mathbf{x}), \upsilon(\mathbf{x})) : \mathbf{x} \in \mathbb{R}^{n-1}\},$$

where $u(\mathbf{x})$ is given by (3.3) and $\upsilon(\mathbf{x}) \in \mathrm{SL}(i+1,\mathbb{R}) \times \mathrm{SL}(j+1,\mathbb{R})$ is defined in (4.10).

Next, we consider the diagonalizable one-parameter subgroups of $\mathcal{G}$ given by

$$\alpha_{\mathcal{H}}(t) := (a(t), d_1(t), d_2(t)),\ t \in \mathbb{R},$$

where $a(t), d_1(t), d_2(t)$ are as in (3.2), (4.4), and

$$\alpha_{\mathcal{U}}(t) := (a(t), \delta(t)),\ t \in \mathbb{R},$$

where $\delta(t) \in \mathrm{SL}(i+1,\mathbb{R}) \times \mathrm{SL}(j+1,\mathbb{R})$ is as in (4.11).

Proposition 5.1. *Let $\mathcal{L}$ denote either $\mathcal{H}$ or $\mathcal{U}$, and let $\alpha_{\mathcal{L}}$ be the corresponding diagonalizable subgroup defined above. Let $\tilde{\Gamma} := \Gamma \times \mathrm{SL}(i+1,\mathbb{Z}) \times \mathrm{SL}(j+1,\mathbb{Z})$, where $\Gamma \subseteq \mathrm{SO}(n,1)^\circ$ is a lattice, and fix $x \in \mathcal{G}/\tilde{\Gamma}$. Let $w \in C_c(\mathcal{L})$ be nonnegative, and define*

$$\mu_s^{\mathcal{L}}(F) := \int_{\mathcal{L}} F(\alpha_{\mathcal{L}}(s) l x)\, w(l)\, dl,\ F \in C_c(\mathcal{G}/\tilde{\Gamma}),$$

where dl denotes a Haar measure on $\mathcal{L}$. Then

$$\lim_{s\to\infty} \mu_s^{\mathcal{L}}(F) = \mu_{\mathcal{G}/\tilde{\Gamma}}(F) \int_{\mathcal{L}} w(l)\, dl,$$

for all $F \in C_c(\mathcal{G}/\tilde{\Gamma})$.

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5.1.1. *Generalities.* To prove Proposition 5.1, we will use the following general facts about equidistribution on homogeneous spaces.

Throughout this subsection, let $\mathcal{G} \subseteq \mathrm{SL}(m, \mathbb{R})$ be a connected semisimple Lie group, let $\Gamma \subseteq \mathcal{G}$ be a lattice, and let $\mathcal{H} \subseteq \mathcal{G}$ be a closed connected subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras.

Let $\{\alpha(t)\}_{t\in\mathbb{R}} \subseteq \mathcal{G}$ be a nontrivial one-parameter subgroup consisting of diagonalizable elements. Then $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+$, where $\mathfrak{g}^0$ is the fixed-point subspace of $\mathrm{Ad}(\alpha(t))$, and $\mathfrak{g}^+$ and $\mathfrak{g}^-$ are the sums of the eigenspaces corresponding, respectively, to the positive and negative weights of the one-parameter group $\mathrm{Ad}(\alpha(t))$. We write $\pi^+ : \mathfrak{g} \to \mathfrak{g}^+$ for the canonical projection associated with this direct sum decomposition.

Theorem 5.2. *Assume further that $\mathcal{G}$ is simple and that $\pi^+(\mathfrak{h}) = \mathfrak{g}^+$. Let $x_0 \in \mathcal{G}/\Gamma$, and let $w \in C_c(\mathcal{H})$ be nonnegative. Then*

$$\lim_{s\to\infty} \int_{\mathcal{H}} f(\alpha(s)hx_0)\, w(h)\, dh = \mu_{\mathcal{G}/\Gamma}(f)\mu_{\mathcal{H}}(w)$$

for all $f \in C_c(\mathcal{G}/\Gamma)$. Here, $\mu_{\mathcal{G}/\Gamma}$ denotes the $\mathcal{G}$-invariant probability measure on $\mathcal{G}/\Gamma$, and $\mu_{\mathcal{H}}$ denotes a Haar measure on $\mathcal{H}$.

Proof. The result follows from mixing of the $\{\alpha(t)\}$-action with respect to $\mu_{\mathcal{G}/\Gamma}$, together with the classical "banana trick" due to Margulis; see [EM93]. □

To prove Proposition 5.1, we will use the above theorem to obtain equidistribution in each simple factor. We will supplement this with the following lemma, which ensures that every limiting measure is invariant under a suitable unipotent element.

Lemma 5.3. *Let $w \in C_c(\mathcal{H})$ be nonnegative, and consider the measures*

$$\mu_s(f) := \int_{\mathcal{H}} f(\alpha(s)hx_0)\, w(h)\, dh, \quad f \in C_c(\mathcal{G}/\Gamma).$$

Suppose that $\boldsymbol{v} \in \mathfrak{g}^+$ is an eigenvector for $\{\mathrm{Ad}(\alpha(t))\}_{t\in\mathbb{R}}$ and that $\boldsymbol{v} \in \pi^+(\mathfrak{h})$. Then every weak- limit of μ_s as $s \to \infty$ is invariant under left translation by $\exp(\boldsymbol{v})$.*

Proof. Suppose that $\mathrm{Ad}(\alpha(t))\boldsymbol{v} = e^{\chi(t)}\boldsymbol{v}$, where $\chi(t) \to \infty$ as $t \to \infty$, and assume that there exists $\boldsymbol{h} \in \mathfrak{h}$ such that $\boldsymbol{v} = \pi^+(\boldsymbol{h})$. Since $\ker(\pi^+) = \mathfrak{g}^0 \oplus \mathfrak{g}^-$, there exists $\boldsymbol{u} \in \mathfrak{g}^0 \oplus \mathfrak{g}^-$ such that $\boldsymbol{h} = \boldsymbol{v} + \boldsymbol{u}$.

Denote $\boldsymbol{h}_t := e^{-\chi(t)}\boldsymbol{h}$. Then

$$\mathrm{Ad}(\alpha(t))\boldsymbol{h}_t = \boldsymbol{v} + O(e^{-\chi(t)}).$$

Now suppose that $\mu_{s_i} \to \nu$ in the weak-* topology. For $f \in C_c(\mathcal{G}/\Gamma)$, using the left invariance of Haar measure on $\mathcal{H}$, we obtain

$$\begin{aligned}
\nu(f) &= \lim_{i\to\infty} \int_{\mathcal{H}} f(\alpha(s_i)hx_0)\, w(h)\, dh \\
&= \lim_{i\to\infty} \int_{\mathcal{H}} f(\alpha(s_i)\exp(\boldsymbol{h}_{s_i})hx_0)\, w(\exp(\boldsymbol{h}_{s_i})h)\, dh \\
&= \lim_{i\to\infty} \int_{\mathcal{H}} f(\exp(\mathrm{Ad}(\alpha(s_i))\boldsymbol{h}_{s_i})\alpha(s_i)hx_0)\, w(\exp(\boldsymbol{h}_{s_i})h)\, dh \\
&= \lim_{i\to\infty} \int_{\mathcal{H}} f(\exp(\boldsymbol{v} + O(e^{-\chi(s_i)}))\alpha(s_i)hx_0)\, w(\exp(O(e^{-\chi(s_i)}))h)\, dh \\
&= \lim_{i\to\infty} \int_{\mathcal{H}} f(\exp(\boldsymbol{v})\alpha(s_i)hx_0)\, w(h)\, dh \\
&= (\exp(\boldsymbol{v})_*\nu)(f),
\end{aligned}$$

where in the second-to-last line we use the uniform continuity of f and w. This proves the claimed invariance. □

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The following proposition shows that, once the limiting measure has Haar marginals and is invariant under a suitable unipotent element, full product equidistribution follows provided there are no proper algebraic joinings.

Proposition 5.4. *Suppose that $\mathcal{G} = \mathcal{G}_1 \times \cdots \times \mathcal{G}_m$ and $\Gamma = \Gamma_1 \times \cdots \times \Gamma_m$, where each $\mathcal{G}_l$ is a connected Lie group and each Γ_l is a lattice in $\mathcal{G}_l$. For each l, let $\pi_l : \mathcal{G} \to \mathcal{G}_l$ and $\bar{\pi}_l : \mathcal{G}/\Gamma \to \mathcal{G}_l/\Gamma_l$ denote the natural projections. Suppose that $u \in \mathcal{G}$ is an* Ad*-unipotent element such that $\pi_l(u)$ acts ergodically with respect to the $\mathcal{G}_l$-invariant probability measure $\mu_{\mathcal{G}_l/\Gamma_l}$ on $\mathcal{G}_l/\Gamma_l$.*

Let μ be a u-invariant Borel probability measure on $\mathcal{G}/\Gamma$ such that

$$(\bar{\pi}_l)_*(\mu) = \mu_{\mathcal{G}_l/\Gamma_l}$$

for each l. Assume further that there is no proper subgroup $L \subseteq \mathcal{G}$ such that $L\Gamma/\Gamma$ is closed and $\pi_l(L) = \mathcal{G}_l$ for every l. Then μ is $\mathcal{G}$-invariant.

Proof. Consider the ergodic decomposition of μ with respect to the u-action:

$$\mu = \int_Y \mu_y \, d\bar{\mu}(y),$$

where each μ_y is a u-ergodic u-invariant Borel probability measure on $\mathcal{G}/\Gamma$, and $(Y, \bar{\mu})$ is a Borel probability space on which u acts trivially.

Fix $1 \le l \le m$. Since $\bar{\pi}_l$ is equivariant with respect to the action of u on $\mathcal{G}/\Gamma$ and the action of $\pi_l(u)$ on $\mathcal{G}_l/\Gamma_l$, the measure $(\bar{\pi}_l)_*(\mu_y)$ is $\pi_l(u)$-ergodic and $\pi_l(u)$-invariant for $\bar{\mu}$-almost every $y \in Y$. Hence

$$(\bar{\pi}_l)_*\mu = \int_Y (\bar{\pi}_l)_*(\mu_y) \, d\bar{\mu}(y)$$

is an ergodic decomposition of $(\bar{\pi}_l)_*\mu$.

By hypothesis, $(\bar{\pi}_l)_*\mu = \mu_{\mathcal{G}_l/\Gamma_l}$, and this measure is $\pi_l(u)$-ergodic. Therefore, for $\bar{\mu}$-almost every $y \in Y$,

$$(\bar{\pi}_l)_*(\mu_y) = \mu_{\mathcal{G}_l/\Gamma_l}. \tag{5.1}$$

Now let $y \in Y$ be such that μ_y is u-ergodic and (5.1) holds for every $1 \le l \le m$. By Ratner's measure-classification theorem, there exists a closed connected subgroup $L \subseteq \mathcal{G}$ such that μ_y is L-invariant and supported on a closed orbit $Lg\Gamma/\Gamma$ for some $g \in \mathcal{G}$. For each l, the projection

$$\pi_l(Lg\Gamma/\Gamma) = \pi_l(L)\pi_l(g)\Gamma_l/\Gamma_l$$

has full $\mu_{\mathcal{G}_l/\Gamma_l}$-measure by (5.1). This is only possible if $\pi_l(L) = \mathcal{G}_l$, since otherwise $\pi_l(L)$ is a proper lower-dimensional subgroup and the projected orbit has zero Haar measure.

Finally, let $L_1 = g^{-1}Lg$. Then $L_1\Gamma/\Gamma$ is closed, and $\pi_l(L_1) = \mathcal{G}_l$ for every l. By assumption, this forces $L_1 = \mathcal{G}$, and hence $L = \mathcal{G}$. Therefore μ_y is the $\mathcal{G}$-invariant probability measure $\mu_{\mathcal{G}/\Gamma}$.

Thus $\mu_y = \mu_{\mathcal{G}/\Gamma}$ for $\bar{\mu}$-almost every y, and consequently $\mu = \mu_{\mathcal{G}/\Gamma}$. □

5.1.2. *Proving Proposition 5.1.* Let $n \ge 3$ and let i, j be nonnegative integers such that $i+j = n-1$. Set

$$\mathcal{G}_1 := \mathrm{SO}(n,1)^\circ, \quad \mathcal{G}_2 := \mathrm{SL}(i+1, \mathbb{R}), \quad \mathcal{G}_3 := \mathrm{SL}(j+1, \mathbb{R}),$$

and let $\mathfrak{g}_l$ denote the corresponding Lie algebras.

The subgroup $\mathcal{H}$. Let

$$\alpha_1(t) := a(t), \quad \alpha_2(t) := d_1(t), \quad \alpha_3(t) := d_2(t),$$

where $a(t), d_1(t), d_2(t)$ are as in (3.2) and (4.4). Let $\mathfrak{g}_l = \mathfrak{g}_l^- \oplus \mathfrak{g}_l^0 \oplus \mathfrak{g}_l^+$ be the corresponding weight-space decomposition for $\mathrm{Ad}(\alpha_l(t))$, and let $\pi_l^+ : \mathfrak{g}_l \to \mathfrak{g}_l^+$ be the associated projection. Then

$$\mathfrak{g}_1^+ = \mathfrak{u}, \quad \mathfrak{g}_2^+ = \left\{ \left[\begin{smallmatrix} 0 & \boldsymbol{x} \\ 0 & 0 \end{smallmatrix}\right] : \boldsymbol{x} \in \mathbb{R}^i \right\}, \quad \mathfrak{g}_3^+ = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ \boldsymbol{y} & 0 \end{smallmatrix}\right] : \boldsymbol{y} \in \mathbb{R}^j \right\},$$

where $\mathfrak{u} = \mathrm{Lie}(U)$.

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Now denote

$$\mathcal{H}_1 := H_{i,j}, \quad \mathcal{H}_2 := \mathrm{SO}(i+1), \quad \mathcal{H}_3 := \mathrm{SO}(j,1)^\circ,$$

and let $\mathfrak{h}_l := \mathrm{Lie}(\mathcal{H}_l)$.

Lemma 5.5. *For each $l = 1, 2, 3$ we have $\pi_l^+(\mathfrak{h}_l) = \mathfrak{g}_l^+$.*

Proof. For $\boldsymbol{a} \in \mathbb{R}^i$ and $\boldsymbol{b} \in \mathbb{R}^j$, consider

$$\boldsymbol{v}_1 := \begin{bmatrix} 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \\ -(\boldsymbol{a},\boldsymbol{b})^\intercal & \mathbf{0} & (\boldsymbol{a},\boldsymbol{b})^\intercal \\ 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \end{bmatrix}, \quad \boldsymbol{v}_2 := \begin{bmatrix} 0 & 2\boldsymbol{a} \\ 0 & 0 \end{bmatrix} \quad \boldsymbol{v}_3 := \begin{bmatrix} 0 & 0 \\ -2\boldsymbol{b} & 0 \end{bmatrix}. \tag{5.2}$$

Let

$$\boldsymbol{h}_1 := 2\begin{bmatrix} 0 & (\boldsymbol{a},\mathbf{0}) & 0 \\ -(\boldsymbol{a},\mathbf{0})^\intercal & \mathbf{0} & (\mathbf{0},\boldsymbol{b})^\intercal \\ 0 & (\mathbf{0},\boldsymbol{b}) & 0 \end{bmatrix}, \quad \boldsymbol{h}_2 := 2\begin{bmatrix} 0 & \boldsymbol{a} \\ -\boldsymbol{a}^\intercal & 0 \end{bmatrix}, \quad \boldsymbol{h}_3 := 2\begin{bmatrix} 0 & -\boldsymbol{b}^\intercal \\ -\boldsymbol{b} & 0 \end{bmatrix}. \tag{5.3}$$

Then $\boldsymbol{h}_l \in \mathfrak{h}_l$ for $l = 1, 2, 3$, and

$$\mathrm{Ad}(\alpha_l(t))(\boldsymbol{h}_l - \boldsymbol{v}_l) = e^{-t}(\boldsymbol{h}_l - \boldsymbol{v}_l), \quad l = 1, 2, 3.$$

Hence $\boldsymbol{h}_l - \boldsymbol{v}_l \in \ker(\pi_l^+)$, so $\pi_l^+(\boldsymbol{h}_l) = \boldsymbol{v}_l,\ l = 1, 2, 3$.

As $\boldsymbol{a}$ and $\boldsymbol{b}$ vary, the vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ run through all of $\mathfrak{g}_1^+, \mathfrak{g}_2^+, \mathfrak{g}_3^+$. This proves the lemma. □

Let

$$\mathcal{H} := \left\{(h, h_1, h_2^*) : h = \begin{bmatrix} h_1 & \\ & h_2 \end{bmatrix} \in H_{i,j}\right\} \subseteq \mathcal{H}_1 \times \mathcal{H}_2 \times \mathcal{H}_3,$$

and let $\mathfrak{h} := \mathrm{Lie}(\mathcal{H})$. Set

$$\alpha(t) := (\alpha_1(t), \alpha_2(t), \alpha_3(t)),$$

and let $\pi^+ : \mathfrak{g} \to \mathfrak{g}^+$ be the positive-weight projection for $\mathrm{Ad}(\alpha(t))$.

Lemma 5.6. *For $\boldsymbol{a} \in \mathbb{R}^i$ and $\boldsymbol{b} \in \mathbb{R}^j$, set*

$$\boldsymbol{v}^+_{\boldsymbol{a},\boldsymbol{b}} = \left(\begin{bmatrix} 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \\ -(\boldsymbol{a},\boldsymbol{b})^\intercal & \mathbf{0} & (\boldsymbol{a},\boldsymbol{b})^\intercal \\ 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2\boldsymbol{a} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2\boldsymbol{b} & 0 \end{bmatrix}\right).$$

Then

$$\mathrm{Ad}(\alpha(s))\boldsymbol{v}^+_{\boldsymbol{a},\boldsymbol{b}} = e^s \boldsymbol{v}^+_{\boldsymbol{a},\boldsymbol{b}}, \quad \boldsymbol{v}^+_{\boldsymbol{a},\boldsymbol{b}} \in \pi^+(\mathfrak{h}).$$

Proof. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ be as in (5.2) and (5.3), and set

$$\boldsymbol{h} := (\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3) \in \mathfrak{h}.$$

By Lemma 5.5, $\pi_l^+(\boldsymbol{h}_l) = \boldsymbol{v}_l,\ l = 1, 2, 3$.

Hence

$$\pi^+(\boldsymbol{h}) = \big(\pi_1^+(\boldsymbol{h}_1), \pi_2^+(\boldsymbol{h}_2), \pi_3^+(\boldsymbol{h}_3)\big) = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = \boldsymbol{v}^+_{\boldsymbol{a},\boldsymbol{b}}.$$

The eigenvalue relation is immediate from the definitions. □

The subgroup $\mathcal{U}$. Write

$$\delta(t) = \big(\delta_1(t), \delta_2(t)\big) \in \mathrm{SL}(i+1, \mathbb{R}) \times \mathrm{SL}(j+1, \mathbb{R}),$$

where $\delta(t)$ is as in (4.11), and set

$$\beta_1(t) := a(t), \quad \beta_2(t) := \delta_1(t), \quad \beta_3(t) := \delta_2(t).$$

Let $\mathfrak{g}_l = \mathfrak{g}_l^- \oplus \mathfrak{g}_l^0 \oplus \mathfrak{g}_l^+$ be the weight decomposition for $\mathrm{Ad}(\beta_l(t))$, and let $\varpi_l^+ : \mathfrak{g}_l \to \mathfrak{g}_l^+$ be the associated projection. Then

$$\mathfrak{g}_1^+ = \mathfrak{u}, \quad \mathfrak{g}_2^+ = \left\{\begin{bmatrix} 0 & \boldsymbol{a} \\ 0 & 0 \end{bmatrix} : \boldsymbol{a} \in \mathbb{R}^i\right\}, \quad \mathfrak{g}_3^+ = \left\{\begin{bmatrix} 0 & \boldsymbol{b} \\ 0 & 0 \end{bmatrix} : \boldsymbol{b} \in \mathbb{R}^j\right\}. \tag{5.4}$$

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Let

$$\mathcal{U}_1 := U \subseteq \mathcal{G}_1, \quad \mathcal{U}_2 := \left\{ \begin{bmatrix} 1 & \boldsymbol{a} \\ 0 & I_i \end{bmatrix} : \boldsymbol{a} \in \mathbb{R}^i \right\} \subseteq \mathcal{G}_2, \quad \mathcal{U}_3 := \left\{ \begin{bmatrix} 1 & \boldsymbol{b} \\ 0 & I_j \end{bmatrix} : \boldsymbol{b} \in \mathbb{R}^j \right\} \subseteq \mathcal{G}_3,$$

and let $\mathfrak{u}_l := \mathrm{Lie}(\mathcal{U}_l)$.

Lemma 5.7. *For each $l = 1, 2, 3$ we have $\varpi_l^+(\mathfrak{u}_l) = \mathfrak{g}_l^+$.*

Proof. For $l = 1$, we have $\mathfrak{u}_1 = \mathfrak{u} = \mathfrak{g}_1^+$. For $l = 2, 3$, the groups $\mathcal{U}_l$ are precisely the expanding horospherical subgroups of $\beta_l(t)$, so $\mathfrak{u}_l = \mathfrak{g}_l^+$. The claim follows. □

Let $\mathfrak{u}_\Delta := \mathrm{Lie}(\mathcal{U}) \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, and write $\boldsymbol{x} = (\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^i \times \mathbb{R}^j$. Define

$$\boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}} := \left(\begin{bmatrix} 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \\ -(\boldsymbol{a},\boldsymbol{b})^\intercal & \mathbf{0} & (\boldsymbol{a},\boldsymbol{b})^\intercal \\ 0 & (\boldsymbol{a},\boldsymbol{b}) & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{a} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boldsymbol{b} \\ 0 & 0 \end{bmatrix} \right).$$

Lemma 5.8. *For every $\boldsymbol{a} \in \mathbb{R}^i$ and $\boldsymbol{b} \in \mathbb{R}^j$,*

$$\mathrm{Ad}\big((a(s), \delta(s))\big)\boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}} = e^s \boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}}, \quad \boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}} \in \pi^+(\mathfrak{u}_\Delta),$$

where π^+ denotes the positive-weight projection for $\mathrm{Ad}((a(s), \delta(s)))$.

Proof. The vector $\boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}}$ is tangent at the identity to the one-parameter subgroup

$$t \mapsto \big(u(t(\boldsymbol{a},\boldsymbol{b})), \upsilon(t(\boldsymbol{a},\boldsymbol{b}))\big) \subseteq \mathcal{U},$$

hence $\boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}} \in \mathfrak{u}_\Delta$. By (5.4), $\mathfrak{u}_\Delta \subseteq \mathfrak{g}^+$, and therefore $\pi^+(\boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}}) = \boldsymbol{w}^+_{\boldsymbol{a},\boldsymbol{b}}$.

The eigenvalue relation follows from the definitions of $a(s)$, $\delta_1(s)$, and $\delta_2(s)$. □

Proof of Proposition 5.1 for $n \geq 3$. Let $\Gamma \subseteq \mathrm{SO}(n,1)^\circ$ be a lattice, and set

$$\Gamma_1 := \Gamma, \quad \Gamma_2 := \mathrm{SL}(i+1, \mathbb{Z}), \quad \Gamma_3 := \mathrm{SL}(j+1, \mathbb{Z}),$$

so that $\tilde{\Gamma} = \Gamma_1 \times \Gamma_2 \times \Gamma_3$.

We first verify that there are no proper algebraic joinings. Set

$$R := \mathrm{PGL}(i+1, \mathbb{R}) \times \mathrm{PGL}(j+1, \mathbb{R}),$$

and let $\pi_{\mathrm{left}} : \mathcal{G}_1 \times R \to \mathcal{G}_1$ and $\pi_{\mathrm{right}} : \mathcal{G}_1 \times R \to R$ be the natural projections.

We claim that if $L \subseteq \mathcal{G}_1 \times R$ is a closed connected subgroup such that both $\pi_{\mathrm{left}}|_L$ and $\pi_{\mathrm{right}}|_L$ are onto, then $L = \mathcal{G}_1 \times R$.

Indeed, let $\mathfrak{l} \subseteq \mathfrak{g}_1 \oplus \mathfrak{r}$ be the Lie algebra of L, where

$$\mathfrak{g}_1 = \mathfrak{so}(n,1), \quad \mathfrak{r} = \mathfrak{pgl}(i+1, \mathbb{R}) \oplus \mathfrak{pgl}(j+1, \mathbb{R}) \cong \mathfrak{sl}_{i+1}(\mathbb{R}) \oplus \mathfrak{sl}_{j+1}(\mathbb{R}).$$

Since both projections are surjective, the differentials $d\pi_{\mathrm{left}}|_{\mathfrak{l}} : \mathfrak{l} \to \mathfrak{g}_1$ and $d\pi_{\mathrm{right}}|_{\mathfrak{l}} : \mathfrak{l} \to \mathfrak{r}$ are surjective. Consider

$$\mathfrak{n} := \ker(d\pi_{\mathrm{right}}|_{\mathfrak{l}}) \subseteq \mathfrak{g}_1 \oplus \{0\}.$$

Identifying $\mathfrak{n}$ with its projection to $\mathfrak{g}_1$, we see that $\mathfrak{n}$ is an ideal in $\mathfrak{g}_1$. Since $\mathfrak{g}_1 = \mathfrak{so}(n,1)$ is simple for $n \geq 3$, either $\mathfrak{n} = 0$ or $\mathfrak{n} = \mathfrak{g}_1 \oplus \{0\}$.

If $\mathfrak{n} = \mathfrak{g}_1 \oplus \{0\}$, then $\mathfrak{g}_1 \oplus \{0\} \subseteq \mathfrak{l}$, and surjectivity of $d\pi_{\mathrm{right}}$ gives $\mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{r}$, hence $L = \mathcal{G}_1 \times R$.

Assume now that $\mathfrak{n} = 0$. Then $d\pi_{\mathrm{right}}|_{\mathfrak{l}}$ is an isomorphism, so $\mathfrak{l}$ is the graph of a surjective Lie algebra homomorphism $\varphi : \mathfrak{r} \to \mathfrak{g}_1$. Writing

$$\mathfrak{r} = \mathfrak{r}_1 \oplus \mathfrak{r}_2, \quad \mathfrak{r}_1 \cong \mathfrak{sl}_{i+1}(\mathbb{R}), \quad \mathfrak{r}_2 \cong \mathfrak{sl}_{j+1}(\mathbb{R}),$$

we have $[\mathfrak{r}_1, \mathfrak{r}_2] = 0$, so $\varphi(\mathfrak{r}_1)$ and $\varphi(\mathfrak{r}_2)$ are commuting ideals in $\mathfrak{g}_1$. Since $\mathfrak{g}_1$ is simple, one of them is 0 and the other is all of $\mathfrak{g}_1$. Thus for some $m \in \{i+1, j+1\}$ there exists a surjective Lie algebra homomorphism $\mathfrak{sl}_m(\mathbb{R}) \twoheadrightarrow \mathfrak{so}(n,1)$.

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As both Lie algebras are simple, this must be an isomorphism. But $\operatorname{rank}_{\mathbb{R}}(\mathfrak{sl}_m(\mathbb{R})) = m-1$ and $\operatorname{rank}_{\mathbb{R}}(\mathfrak{so}(n,1)) = 1$, so this is possible only when $m = 2$, which would force $\mathfrak{sl}_2(\mathbb{R}) \cong \mathfrak{so}(2,1)$, contrary to $n \geq 3$. This proves the claim.

Now fix $\mathcal{L} \in \{\mathcal{H}, \mathcal{U}\}$, and let ν be a weak-* limit of $\mu_s^{\mathcal{L}}$. We verify the hypotheses of Proposition 5.4 uniformly in $\mathcal{L}$.

For $k = 1, 2, 3$, define

$$\mathcal{L}_k := \begin{cases} \mathcal{H}_k, & \mathcal{L} = \mathcal{H}, \\ \mathcal{U}_k, & \mathcal{L} = \mathcal{U}, \end{cases} \qquad \alpha_k^{\mathcal{L}}(t) := \begin{cases} \alpha_k(t), & \mathcal{L} = \mathcal{H}, \\ \beta_k(t), & \mathcal{L} = \mathcal{U}. \end{cases}$$

Then the k-th marginal of $\mu_s^{\mathcal{L}}$ is of the form

$$f \longmapsto \int_{\mathcal{L}_k} f(\alpha_k^{\mathcal{L}}(s)\ell x_k)\, w_k(\ell)\, d\ell,$$

for some nonnegative $w_k \in C_c(\mathcal{L}_i)$. By Theorem 5.2, together with Lemma 5.5 when $\mathcal{L} = \mathcal{H}$ and Lemma 5.7 when $\mathcal{L} = \mathcal{U}$, it follows that $(\bar{\pi}_k)_*\nu = \mu_{\mathcal{G}_k/\Gamma_l}$ for $(\bar{\pi}_k)_*\nu = \mu_{\mathcal{G}_k/\Gamma_l}, k = 1, 2, 3$, with the obvious convention in the trivial-factor cases $i = 0$ or $j = 0$.

Next, by Lemma 5.3, together with Lemma 5.6 when $\mathcal{L} = \mathcal{H}$ and Lemma 5.8 when $\mathcal{L} = \mathcal{U}$, the measure ν is invariant under a nontrivial unipotent element whose projection to each nontrivial factor is nontrivial. By [Moo66], these projections act ergodically on the corresponding factor spaces.

Thus, all the hypotheses of Proposition 5.4 are now satisfied, so $\nu = \mu_{\mathcal{G}/\tilde{\Gamma}}$. □

5.2. **The case of $n = 2$.** We first treat the unipotent case, and then the two possibilities $H_{1,0}$ and $H_{0,1}$.

The unipotent case. Let

$$\delta(s) := \begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix}, \quad \upsilon(t) := \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}. \tag{5.5}$$

Let $G := \mathrm{SO}(2,1)^\circ$. Then, there exists an isogeny $\Phi : \mathrm{SL}(2,\mathbb{R}) \to G$ such that:

$$a(s) = \Phi(\delta(s)) = \exp\begin{bmatrix} 0 & 0 & s \\ 0 & 0 & 0 \\ s & 0 & 0 \end{bmatrix}, \quad u(t) = \Phi(\upsilon(t)) = \exp\begin{bmatrix} 0 & t & 0 \\ -t & 0 & t \\ 0 & t & 0 \end{bmatrix}.$$

Let $X_2 := \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$, let $\Gamma \subseteq G$ be a lattice, and let $w \in C_c(\mathbb{R})$ be nonnegative. Given $(x_0, y_0) \in G/\Gamma \times X_2$, for each $s \in \mathbb{R}$, let μ_s^U be the measure defined by

$$\mu_s^U(F) := \int_{\mathbb{R}} F(a(s)u(t)x_0, \delta(s)\upsilon(t)y_0)\, w(t)\, dt, \quad \forall\, F \in C_c(G/\Gamma \times X_2).$$

Set $\widetilde{\Gamma} := \Phi^{-1}(\Gamma)$, and let $\bar{\Phi} : \mathrm{SL}(2,\mathbb{R})/\widetilde{\Gamma} \to G/\Gamma$ be the map induced by Φ. If $\tilde{x}_0 \in \mathrm{SL}(2,\mathbb{R})/\widetilde{\Gamma}$ projects to x_0, then

$$\mu_s^U(F) = \int_{\mathbb{R}} F\big(\bar{\Phi}(\delta(s)\upsilon(t)\tilde{x}_0), \delta(s)\upsilon(t)y_0\big)\, w(t)\, dt.$$

Definition 5.9. Let

$$\mathrm{SL}^{\Delta}(2,\mathbb{R}) := \{(\Phi(g), g) : g \in \mathrm{SL}(2,\mathbb{R})\} \subseteq G \times \mathrm{SL}(2,\mathbb{R}).$$

We say that $(x_0, y_0) \in G/\Gamma \times X_2$ is *commensurable* if the $\mathrm{SL}^{\Delta}(2,\mathbb{R})$-orbit of (x_0, y_0) is closed.

Proposition 5.10. *Let $w \in C_c(\mathbb{R})$ be nonnegative, and let $(x_0, y_0) \in G/\Gamma \times X_2$. Then, for every $F \in C_c(G/\Gamma \times X_2)$, the following hold:*

(1) If (x_0, y_0) is not commensurable, then

$$\lim_{s\to\infty} \mu_s^U(F) = \mu_{G/\Gamma} \otimes \mu_{X_2}(F) \int_{\mathbb{R}} w(t)\, dt. \tag{5.6}$$

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(2) If (x_0, y_0) is commensurable, then

$$\lim_{s\to\infty} \mu_s^U(F) = \mu_{\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)}(F) \int_{\mathbb{R}} w(t)\,dt, \tag{5.7}$$

where $\mu_{\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)}$ denotes the $\mathrm{SL}^\Delta(2,\mathbb{R})$-invariant probability measure on the closed orbit $\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)$.

Proof. Let

$$M := G \times \mathrm{SL}(2,\mathbb{R}), \quad H := \mathrm{SL}^\Delta(2,\mathbb{R}) \subseteq M.$$

The subgroup $\{(\Phi(\upsilon(t)), \upsilon(t)) : t \in \mathbb{R}\} \subseteq H$ is the expanding horospherical subgroup of $(a(s), \delta(s)) \in H$. Let $L \subseteq M$ be the smallest closed connected subgroup containing H such that $L(x_0, y_0)$ is closed. Since $H \cong \mathrm{SL}(2,\mathbb{R})$ is generated by one-parameter unipotent subgroups, $L(x_0, y_0)$ carries a unique L-invariant probability measure $\mu_{L(x_0,y_0)}$; see [Mar88, Property-D], [Sha91, (2.3) Theorem]. By [Sha96, Theorem 1.4],

$$\lim_{s\to\infty} \mu_s^U(F) = \mu_{L(x_0,y_0)}(F) \int_{\mathbb{R}} w(t)\,dt.$$

Let $p_1 : M \to G$ and $p_2 : M \to \mathrm{SL}(2,\mathbb{R})$ be the coordinate projections. Since $p_1(H) = G$, we have $p_1(L) = G$. Set

$$N := L \cap (G \times \{e\}).$$

Then $p_1(N)$ is a connected normal subgroup of the simple group G, so either $N = \{e\}$ or $N = G \times \{e\}$.

If $N = G \times \{e\}$, then L contains $G \times \{e\}$ and H, hence $L = (G \times \{e\})H = M$. Therefore, $\mu_{L(x_0,y_0)} = \mu_{G/\Gamma} \otimes \mu_{X_2}$, and (5.6) follows.

If $N = \{e\}$, then $p_2|_L : L \to \mathrm{SL}(2,\mathbb{R})$ is injective. Since $H \subseteq L$ and $p_2(H) = \mathrm{SL}(2,\mathbb{R})$, it is also surjective. Hence $p_2|_L$ is an isomorphism. For each $g \in \mathrm{SL}(2,\mathbb{R})$, the unique point of L with second coordinate g must be $(\Phi(g), g) \in H$, since $H \subseteq L$. Thus $L = H = \mathrm{SL}^\Delta(2,\mathbb{R})$. Therefore (x_0, y_0) is commensurable and $\mu_{L(x_0,y_0)} = \mu_{\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)}$, which gives (5.7). □

The cases $H_{1,0}$ and $H_{0,1}$. Recall

$$H_{1,0} = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}, \quad H_{0,1} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{bmatrix} : t \in \mathbb{R} \right\},$$

and let

$$h_{1,0}(t) := \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}, \quad h_{0,1}(t) := \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.$$

Let $w \in C_c(\mathbb{R})$ be nonnegative, let $(x_0, y_0) \in G/\Gamma \times X_2$, and for each $s \in \mathbb{R}$, define

$$\mu_s^{1,0}(F) := \int_{\mathbb{R}} F\left(a(s) \begin{bmatrix} h_{1,0}(t) & 0 \\ 0 & 1 \end{bmatrix} x_0,\ \delta(s) h_{1,0}(t) y_0\right) w(t)\,dt,$$

$$\mu_s^{0,1}(F) := \int_{\mathbb{R}} F\left(a(s) \begin{bmatrix} 1 & 0 \\ 0 & h_{0,1}(t) \end{bmatrix} x_0,\ (\delta(s) h_{0,1}(t))^* y_0\right) w(t)\,dt,$$

for all $F \in C_c(G/\Gamma \times X_2)$.

Proposition 5.11. *For $(i,j) = (1,0), (0,1)$ and every $F \in C_c(G/\Gamma \times X_2)$,*

$$\lim_{s\to\infty} \mu_s^{i,j}(F) = \mu_{G/\Gamma} \otimes \mu_{X_2}(F) \int_{\mathbb{R}} w(t)\,dt.$$

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We observe that there exists an isogeny $\Phi : \mathrm{SL}(2,\mathbb{R}) \to G$ such that

$$\begin{aligned} \Phi\begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix} &= a(s), \\ \Phi(h_{1,0}(\theta/2)) &= \begin{bmatrix} h_{1,0}(\theta) & 0 \\ 0 & 1 \end{bmatrix}, \\ \Phi(h_{0,1}(t/2)) &= \begin{bmatrix} 1 & 0 \\ 0 & h_{0,1}(t) \end{bmatrix}. \end{aligned} \tag{5.8}$$

Using this isogeny, we reduce the proposition to an equidistribution statement in $\mathrm{SL}(2,\mathbb{R})/\widetilde{\Gamma} \times \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. Let $\widetilde{\Gamma} := \Phi^{-1}(\Gamma)$ and let $\bar{\Phi} : \mathrm{SL}(2,\mathbb{R})/\widetilde{\Gamma} \to G/\Gamma$ be the induced map. Also let $\Psi(g) := (g^{\top})^{-1}$; since $\Psi(\mathrm{SL}(2,\mathbb{Z})) = \mathrm{SL}(2,\mathbb{Z})$, it induces an involution of X_2, still denoted by Ψ.

If $\tilde{x}_0 \in \mathrm{SL}(2,\mathbb{R})/\widetilde{\Gamma}$ projects to x_0, then by (5.8),

$$\mu_s^{1,0}(F) = \int_{\mathbb{R}} F\big(\bar{\Phi}(\delta(s)h_{1,0}(t/2)\tilde{x}_0), \delta(s)h_{1,0}(t)y_0\big)\, w(t)\, dt.$$

Similarly,

$$\mu_s^{0,1}(F) = \int_{\mathbb{R}} F\big(\bar{\Phi}(\delta(s)h_{0,1}(t/2)\tilde{x}_0), \Psi(\delta(s)h_{0,1}(t)\tilde{y}_0)\big)\, w(t)\, dt.$$

Here $\tilde{y}_0 = \Psi(y_0)$.

The required equidistribution is therefore a consequence of the following product statement.

Proposition 5.12. *Let $\Gamma = \Gamma_1 \times \Gamma_2 \subseteq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) =: M$ be a lattice, and let $z_0 \in M/\Gamma$. Let $I = [a,b]$, and let*

$$\psi(t) := (h(t/2), h(t)),$$

where $h(t)$ is either $h_{1,0}(t)$ or $h_{0,1}(t)$.

Then, for every continuous nonnegative nonzero function $w : I \to \mathbb{R}$, the probability measures

$$\eta_s(f) := \frac{\int_I f\big((\delta(s),\delta(s))\psi(t)z_0\big)\, w(t)\, dt}{\int_I w(t)\, dt}, \quad f \in C_c(M/\Gamma),$$

converge to the M-invariant probability measure on M/Γ.

We now prove Proposition 5.12. Our main tool is [Yan22, Theorem 1.6, Remark 1.7], which is stated as follows in a form suitable for our needs.

Theorem 5.13. *Let $\Gamma = \Gamma_1 \times \Gamma_2 \subseteq \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) =: M$ be a lattice, and let $z_0 \in M/\Gamma$. Let $\varphi = (\varphi_1, \varphi_2) : [a,b] \to \mathbb{R}^2$ be analytic. Assume that there exists $s_0 \in [a,b]$ such that*

$$(\varphi(t) - \varphi(s_0))^{-1} := \left(\frac{1}{\varphi_1(t) - \varphi_1(s_0)}, \frac{1}{\varphi_2(t) - \varphi_2(s_0)}\right)$$

is defined for almost every $t \in [a,b]$, and assume that

$$\Big\{(\varphi(s_1) - \varphi(s_0))^{-1} - (\varphi(s_2) - \varphi(s_0))^{-1} : s_1, s_2 \in [a,b]\Big\} \tag{5.9}$$

is not contained in any line in $\mathbb{R}^2$. Then, for every subinterval $J \subseteq [a,b]$, every $f \in C_c(M/\Gamma)$, and every continuous function $w : J \to \mathbb{R}$,

$$\lim_{s\to\infty} \int_J f\big((\delta(s)\upsilon(\varphi_1(t)), \delta(s)\upsilon(\varphi_2(t)))z_0\big)\, w(t)\, dt = \mu_{M/\Gamma}(f) \int_J w(t)\, dt,$$

where $\upsilon(t)$ and $\delta(s)$ are given by (5.5).

We need the following lemma to verify the geometric condition (5.9) in our proof.

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Lemma 5.14. *For $\alpha_0, \alpha_1 \in \mathbb{R}$, consider the curves*

$$\gamma_{\tan}(t) := \left(\frac{1}{\tan(t/2) - \alpha_1}, \frac{1}{\tan t - \alpha_0} \right), \ \gamma_{\tanh}(t) := \left(\frac{1}{\tanh(t/2) - \alpha_1}, \frac{1}{\tanh t - \alpha_0} \right)$$

defined wherever the denominators are nonzero. Then neither image is contained in an affine line in $\mathbb{R}^2$.

Proof. We show the proof for $\gamma_{\tan}$; the proof for $\gamma_{\tanh}$ is the same as $\tanh(t) = -i\tan(it)$.

Suppose, toward a contradiction, that its image is contained in an affine line. Then there exist $A, B, C \in \mathbb{R}$, not all zero, such that

$$\frac{A}{\tan(t/2) - \alpha_1} + \frac{B}{\tan t - \alpha_0} = C$$

for all t in some interval on which the curve is defined.

Set $x = \tan(t/2)$. Since $\tan t = \frac{2x}{1-x^2}$, this becomes

$$\frac{A}{x - \alpha_1} + B\,\frac{1 - x^2}{\alpha_0 x^2 + 2x - \alpha_0} = C.$$

After clearing denominators,

$$A(\alpha_0 x^2 + 2x - \alpha_0) + B(1 - x^2)(x - \alpha_1) = C(x - \alpha_1)(\alpha_0 x^2 + 2x - \alpha_0).$$

Comparing the coefficients of x^3 and of the constant term gives

$$-B = C\alpha_0, \quad -B\alpha_1 - A\alpha_0 = C\alpha_0\alpha_1.$$

If $\alpha_0 = 0$, then the first identity gives $B = 0$, and the original relation reduces to $\frac{A}{x-\alpha_1} = C$, which is impossible on an interval unless $A = 0$, contradiction. Thus $\alpha_0 \neq 0$. Substituting $C = -B/\alpha_0$ into the second identity yields $A\alpha_0 = 0$, hence $A = 0$. Comparing the coefficient of x^2 now gives

$$B\alpha_1 = \frac{-B}{\alpha_0}(2 - \alpha_0\alpha_1),$$

that is, $\alpha_0\alpha_1 = -2 + \alpha_0\alpha_1$, a contradiction. □

Proof of Proposition 5.12. We first treat $h(t) = h_{1,0}(t)$. Let

$$u_-(x) := \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}, \quad b(r) := \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix}.$$

In view of $u_-(\mathbb{R})b(\mathbb{R}^\times)v(\mathbb{R})$ being Zariski open and dense in $\mathrm{SL}(2,\mathbb{R})$, for $t \notin \frac{\pi}{2} + \pi\mathbb{Z}$, we find that

$$h(t) = u_-(\tan t)\, b(\cos t)\, v(-\tan t),$$

and therefore

$$\delta(s)h(t) = u_-(e^{-s}\tan t)\, b(\cos t)\, \delta(s) v(-\tan t). \tag{5.10}$$

Let $K := \mathrm{supp}(w) \subseteq I$. The singular set

$$\Sigma := K \cap \left(\left(\frac{\pi}{2} + \pi\mathbb{Z} \right) \cup (\pi + 2\pi\mathbb{Z}) \right)$$

is finite; these are precisely the poles of $\tan t$ and $\tan(t/2)$ on K. Fix $\epsilon > 0$. Choose an open neighborhood $\mathcal{U}$ of Σ in I such that $\int_{\mathcal{U}} w(t)\,dt < \epsilon$. Since f is bounded,

$$\left| \int_{\mathcal{U}} f\big((\delta(s)h(t/2), \delta(s)h(t))z_0\big)\, w(t)\, dt \right| \leq \|f\|_\infty\, \epsilon.$$

Write $K \smallsetminus \mathcal{U} = \bigsqcup_{m=1}^{N} J_m$ as a finite disjoint union of compact intervals. On each J_m, the functions $\tan t$, $\tan(t/2)$, $\cos t$, and $\cos(t/2)$ are continuous and bounded. Fix such an interval $J = [c, d]$.

By uniform continuity of f, for every $\epsilon > 0$ there exists an identity neighborhood $O_\epsilon \subseteq M$ such that $|f(gz) - f(z)| \leq \epsilon$ for all $g \in O_\epsilon,\ z \in M/\Gamma$.

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Since $\tan t$ and $\tan(t/2)$ are bounded on J, (5.10) implies that for all sufficiently large s,

$$\big(u_-(e^{-s}\tan(t/2)), u_-(e^{-s}\tan t)\big) \in O_\epsilon \quad \forall\, t \in J.$$

Next, partition J into finitely many closed intervals $J'_1, \dots, J'_r$ and choose $t_\ell \in J'_\ell$ so that

$$\big(b(\cos(t/2)), b(\cos t)\big)\big(b(\cos(t_\ell/2)), b(\cos t_\ell)\big)^{-1} \in O_\epsilon \quad \forall\, t \in J'_\ell.$$

Then, for $t \in J'_\ell$ and all sufficiently large s,

$$f\big((\delta(s)h(t/2), \delta(s)h(t))z_0\big) = f_\ell\big((\delta(s)\upsilon(-\tan(t/2)), \delta(s)\upsilon(-\tan t))z_0\big) + O(\epsilon),$$

where

$$f_\ell(z) := f\big((b(\cos(t_\ell/2)), b(\cos t_\ell))z\big).$$

Hence

$$\begin{aligned}&\int_{J'_\ell} f\big((\delta(s)h(t/2), \delta(s)h(t))z_0\big)\, w(t)\, dt \\ &= \int_{J'_\ell} f_\ell\big((\delta(s)\upsilon(-\tan(t/2)), \delta(s)\upsilon(-\tan t))z_0\big)\, w(t)\, dt + O\left(\epsilon \int_{J'_\ell} w(t)\, dt\right).\end{aligned}$$

We now apply Theorem 5.13 on each J'_ℓ with $\varphi(t) = (-\tan(t/2), -\tan t)$. Fix $s_0 \in J'_\ell$. Then $(\varphi(t) - \varphi(s_0))^{-1}$ is defined for all $t \in J'_\ell \smallsetminus \{s_0\}$. To verify (5.9), suppose the set in (5.9) were contained in a line in $\mathbb{R}^2$. Fixing $s_2 \in J'_\ell$, the curve

$$t \mapsto (\varphi(t) - \varphi(s_0))^{-1} - (\varphi(s_2) - \varphi(s_0))^{-1}$$

would then be contained in an affine line. But this curve is of the form covered by Lemma 5.14, with $\alpha_1 = -\tan(s_0/2)$ and $\alpha_0 = -\tan(s_0)$. Hence (5.9) holds. Therefore, by Theorem 5.13,

$$\lim_{s\to\infty} \int_{J'_\ell} f_\ell\big((\delta(s)\upsilon(-\tan(t/2)), \delta(s)\upsilon(-\tan t))z_0\big)\, w(t)\, dt = \mu_{M/\Gamma}(f) \int_{J'_\ell} w(t)\, dt,$$

where we used the M-invariance of $\mu_{M/\Gamma}$.

Summing over ℓ and then over m, and using the bound on $\mathcal{U}$, we obtain

$$\limsup_{s\to\infty} \left| \int_I f\big((\delta(s)h(t/2), \delta(s)h(t))z_0\big)\, w(t)\, dt - \mu_{M/\Gamma}(f) \int_I w(t)\, dt \right| \ll \epsilon\big(\|f\|_\infty + \int_I w(t)\, dt\big).$$

Letting $\epsilon \to 0$ proves the proposition in the trigonometric case.

The hyperbolic case $h(t) = h_{0,1}(t)$ is identical, using

$$h(t) = u_-(\tanh t)\, b(\cosh t)\, \upsilon(\tanh t), \quad \delta(s)h(t) = u_-(e^{-s}\tanh t)\, b(\cosh t)\, \delta(s)\upsilon(\tanh t),$$

and applying Theorem 5.13 to $\varphi(t) = (\tanh(t/2), \tanh t)$. The required geometric condition follows from Lemma 5.14 in exactly the same way. □

6. Proving the main results

In this section we combine the volume estimates from Section 3 with the equidistribution results from Section 5.

The results from the previous sections apply most naturally to integrals of the form

$$\frac{1}{\mathrm{Vol}(G_T)} \int_{G_T} \varphi(g\Gamma)\, f\big(g \cdot ([\Lambda_1], [\Lambda_2])\big)\, dg$$

when $([\Lambda_1], [\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$ or $([\Lambda_1], [\Lambda_2]) \in X_{\mathbf{P}^\infty}$; see (4.1) and (4.3) to recall $\mathbf{P}^{(0)}$ and $\mathbf{P}^\infty$. For a general point of $X_{r,n+1}$, we reduce to one of these situations by translation as follows.

Let $([\tilde\Lambda_1], [\tilde\Lambda_2]) \in X_{r,n+1}$. According to our conventions, either $\tilde\Lambda_1$ is positive definite or degenerate. If $\tilde\Lambda_1$ is positive definite, choose $g_0 \in G$ so that $g_0^{-1}\tilde\Lambda_1 \subseteq P_1^{(0)} = \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$. If $\tilde\Lambda_1$

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is degenerate, choose $g_0 \in G$ so that $g_0^{-1}\tilde{\Lambda}_1 \subseteq P_1^\infty = \mathrm{Span}_{\mathbb{R}}\{\mathbf{e}_1 + \mathbf{e}_{n+1}, \mathbf{e}_2, \dots, \mathbf{e}_r\}$. Consider the skewed norm $\|g\|_{g_0} := \|gg_0^{-1}\|$ and denote

$$G_T^{g_0} := \{g \in G : \|g\|_{g_0} \le T\}.$$

Then $G_T g_0 = G_T^{g_0}$, and by right-invariance of Haar measure,

$$\frac{1}{\mathrm{Vol}(G_T)} \int_{G_T} \varphi(g\Gamma) f\big(g \cdot ([\tilde{\Lambda}_1], [\tilde{\Lambda}_2])\big)\, dg = \frac{1}{\mathrm{Vol}(G_T^{g_0})} \int_{G_T^{g_0}} \varphi(gg_0^{-1}\Gamma)\, f\big(g \cdot ([\Lambda_1], [\Lambda_2])\big)\, dg, \tag{6.1}$$

where

$$([\Lambda_1], [\Lambda_2]) := g_0^{-1} \cdot ([\tilde{\Lambda}_1], [\tilde{\Lambda}_2]),$$

$([\Lambda_1], [\Lambda_2]) \in X_{\mathbf{P}^{(0)}}$ in the positive-definite case, and $([\Lambda_1], [\Lambda_2]) \in X_{\mathbf{P}^\infty}$ in the degenerate case.

6.1. **Partition of unity.** Our next goal is to identify the weak-$*$ limits of the measures $\mu_{x,y,k,T}$ from (4.6) and $\nu_{x,y,k,T}$ from (4.15). As in [GW07, Proof of Theorem 10.1], the key point is a partition-of-unity argument.

Lemma 6.1. *Let H denote either $H_{i,j}$ or U. For every $0 < \epsilon < 1/2$, there exists a symmetric identity neighborhood $O \subseteq H$ such that for all $h \in H$, $k \in K$, $L > 0$, and $u_1, u_2, u_3 \in O$,*

$$b_{T(1-\epsilon),L-1}(k, hu_1) \subseteq b_{T,L}(k, hu_2) \subseteq b_{T(1+\epsilon),L+1}(k, hu_3),$$

where $b_{T,L}(k,h) \subset \mathbb{R}$ are defined by (3.13).

Proof. Choose $0 < \delta < \epsilon$ such that $(1+\delta)(1-\epsilon) \le 1$. Let $C \ge 1$ be such that $\|AB\| \le C\|A\|\,\|B\|$ for all matrices A, B. Choose a symmetric identity neighborhood $O \subseteq H$ such that

$$\|I - u\| \le \delta/C, \quad \forall\, u \in O^2.$$

If $h_1, h_2 \in hO$, then $h_2^{-1}h_1 \in O^2$, hence for any $s \in \mathbb{R}$,

$$\|ka(s)h_1\| \le \|ka(s)h_2\| + \|ka(s)h_2(h_2^{-1}h_1 - I)\| \le (1+\delta)\|ka(s)h_2\|.$$

Interchanging h_1 and h_2 gives

$$(1-\delta)\|ka(s)h_2\| \le \|ka(s)h_1\| \le (1+\delta)\|ka(s)h_2\|.$$

Now let $s \in b_{T(1-\epsilon),L-1}(k, hu_1)$. Then

$$s \ge \log(T(1-\epsilon)) - (L-1) \ge \log T - L,$$

and

$$\|ka(s)hu_2\| \le (1+\delta)\|ka(s)hu_1\| \le (1+\delta)T(1-\epsilon) \le T.$$

Hence $s \in b_{T,L}(k, hu_2)$. The second inclusion is proved in the same way. □

For $H = H_{i,j}$ with $i \ne 0$, or for $H = U$, define $c : K \to \mathbb{R}$ by

$$c(k_0) := \frac{\int_H \|k_0 a(\infty) h\|^{-(n-1)}\, dh}{\int_K \int_H \|ka(\infty)h\|^{-(n-1)}\, dh\, dk}, \text{ for } k_0 \in K.$$

For $H = H_{0,n-1}$ define

$$c(k_0) := \frac{\int_H \Big(\|k_0 a(\infty) h\|^{-(n-1)} + \|k_0 a(-\infty) h\|^{-(n-1)}\Big)\, dh}{\int_K \int_H \Big(\|ka(\infty)h\|^{-(n-1)} + \|ka(-\infty)h\|^{-(n-1)}\Big)\, dh\, dk}, \text{ for } k_0 \in K.$$

Proposition 6.2. *Let $n \ge 2$, let $i, j \ge 0$ with $i + j = n - 1$, let $x_0 \in G/\Gamma$ and $y_0 \in X_{i+1} \times X_{j+1}$, and let $k \in K$. Then the following hold.*

(1) For every $F \in C_c(G/\Gamma \times X_{i+1} \times X_{j+1})$,

$$\lim_{T\to\infty} \mu_{x_0,y_0,k,T}(F) = c(k)\, \mu_{G/\Gamma} \otimes \mu_{X_{i+1}} \otimes \mu_{X_{j+1}}(F).$$

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(2) *Suppose that $n \geq 3$, or that $n = 2$ and (x_0, y_0) is not commensurable in the sense of Definition 5.9. Then for every $F \in C_c(G/\Gamma \times X_{i+1} \times X_{j+1})$,*

$$\lim_{T\to\infty} \nu_{x_0,y_0,k,T}(F) = c(k)\, \mu_{G/\Gamma} \otimes \mu_{X_{i+1}} \otimes \mu_{X_{j+1}}(F).$$

(3) *Suppose that $n = 2$ and that (x_0, y_0) is commensurable. Then for every $F \in C_c(G/\Gamma \times X_2)$,*

$$\lim_{T\to\infty} \nu_{x_0,y_0,k,T}(F) = c(k)\, \mu_{\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)}(F),$$

where $\mu_{\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)}$ is the $\mathrm{SL}^\Delta(2,\mathbb{R})$-invariant probability measure on $\mathrm{SL}^\Delta(2,\mathbb{R})(x_0, y_0)$.

Proof. We give the details for (1); the proofs of (2) and (3) are identical, with the same partition-of-unity argument and the corresponding U-equidistribution statement from Section 5.

By linearity it is enough to treat $F \geq 0$. Fix $\epsilon > 0$. By Corollary 3.7, there exists $L > 0$ such that

$$\begin{aligned}\mu_{x_0,y_0,k,T}(F) &= \frac{1}{\mathrm{Vol}(G_T)} \int_{H_{i,j}} \int_{b_T(k,h)} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\, ds\, dh \\ &= \frac{1}{\mathrm{Vol}(G_T)} \int_{H_{i,j}} \int_{b_{T,L}(k,h)} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\, ds\, dh + O(\epsilon),\end{aligned}$$

where $\omega(s) = \sinh(s)^i \cosh(s)^j$.

Again by Corollary 3.7, there exists $c_0 > 0$ such that $b_{T,L}(k, h) = \emptyset$ whenever $\|h\| \geq c_0 e^L$.

Choose a symmetric identity neighborhood $O \subseteq H_{i,j}$ as in Lemma 6.1. Since

$$H_{c_0e^L} := \{h \in H_{i,j} : \|h\| \leq c_0 e^L\}$$

is compact, we may cover it by finitely many translates

$$H_{c_0e^L} \subseteq \bigcup_{q=1}^{N} h_q O.$$

Let $w_1, \ldots, w_M \in C_c(H_{i,j})$ be a partition of unity subordinate to this cover. Thus

$$\sum_{l=1}^{M} w_l(h) = 1 \quad \forall\, h \in H_{c_0e^L},$$

and for each l the support of w_l is contained in some $h_{q(l)}O$.

Using this partition of unity, we write

$$\begin{aligned}&\int_{H_{i,j}} \int_{b_{T,L}(k,h)} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\, ds\, dh \\ &= \sum_{l=1}^{M} \int_{h_{q(l)}O} w_l(h) \int_{b_{T,L}(k,h)} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\, ds\, dh.\end{aligned}$$

For fixed l, Lemma 6.1 gives

$$b_{T,L}(k, h) \subseteq b_{T(1+\epsilon),L+1}(k, h_{q(l)}) \quad \forall\, h \in h_{q(l)}O,$$

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and therefore

$$\begin{aligned}
&\int_{h_{q(l)}O} w_l(h) \int_{b_{T,L}(k,h)} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\,ds\,dh \\
&\quad\le \int_{h_{q(l)}O} w_l(h) \int_{b_{T(1+\epsilon),L+1}(k,h_{q(l)})} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)\omega(s)\,ds\,dh \\
&\quad= \int_{b_{T(1+\epsilon),L+1}(k,h_{q(l)})} \omega(s)\,ds \int_{h_{q(l)}O} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)w_l(h)\,dh.
\end{aligned}$$

We may now apply the expanding-$H_{i,j}$ equidistribution statements Proposition 5.1 and Proposition 5.11 to the h-integral. For T sufficiently large,

$$\begin{aligned}
&\int_{b_{T(1+\epsilon),L+1}(k,h_{q(l)})} \omega(s)\,ds \int_{h_{q(l)}O} F\big(a(s)hx_0, d_1(s)h_1y_{0,1}, d_2(s)h_2^*y_{0,2}\big)w_l(h)\,dh \\
&\quad= (1+O(\epsilon))\mu_{G/\Gamma}\otimes\mu_{X_{i+1}}\otimes\mu_{X_{j+1}}(F) \int_{b_{T(1+\epsilon),L+1}(k,h_{q(l)})} \omega(s)\,ds \int_{H_{i,j}} w_l(h)\,dh.
\end{aligned}$$

Using Lemma 6.1 once more,

$$\int_{b_{T(1+\epsilon),L+1}(k,h_{q(l)})} \omega(s)\,ds \int_{H_{i,j}} w_l(h)\,dh \le \int_{H_{i,j}} w_l(h) \int_{b_{T(1+\epsilon)^2,L+2}(k,h)} \omega(s)\,ds\,dh.$$

Summing over l yields

$$\mu_{x_0,y_0,k,T}(F) \le (1+O(\epsilon))\mu_{G/\Gamma}\otimes\mu_{X_{i+1}}\otimes\mu_{X_{j+1}}(F)\frac{\int_{H_{i,j}}\int_{b_{T(1+\epsilon)^2,L+2}(k,h)}\omega(s)\,ds\,dh}{\mathrm{Vol}(G_T)}.$$

The corresponding lower bound is obtained in the same way:

$$\mu_{x_0,y_0,k,T}(F) \ge (1+O(\epsilon))\mu_{G/\Gamma}\otimes\mu_{X_{i+1}}\otimes\mu_{X_{j+1}}(F)\frac{\int_{H_{i,j}}\int_{b_{T(1-\epsilon)^2,L-2}(k,h)}\omega(s)\,ds\,dh}{\mathrm{Vol}(G_T)}.$$

By Lemma 3.4, Corollary 3.5, and Corollary 3.7, both outer ratios converge to $c(k)+O(\epsilon)$. Since $\epsilon>0$ is arbitrary, this proves (1). □

6.2. **Proof of the main results.** We split the argument into two parts. The proofs of Theorems 1.1 and 1.2 and of the non-special case of Theorem 1.8 follow the same pattern. The special case in Theorem 1.8 requires an additional argument to relate the limiting $\mathrm{SL}(2,\mathbb{R})$-diagonal embedded measure to our to the limit of the lattice norm ball averages.

6.2.1. *The generic cases.* We give the details for Theorem 1.1 when $r\ge 2$. The degenerate non-special case is entirely analogous, and the case $r=1$ is handled in the same way after keeping the extra $a(-\infty)$-term coming from the $H_{0,n-1}$-decomposition.

Let $([\tilde\Lambda_1],[\tilde\Lambda_2])\in X$ and assume that $\tilde\Lambda_1$ is positive definite. Let $g_0\in G$ such that $g_0^{-1}\tilde\Lambda_1\subseteq P_1^{(0)}$. Denote $([\Lambda_1],[\Lambda_2]) := g_0^{-1}\cdot([\tilde\Lambda_1],[\tilde\Lambda_2])$ and let $x_0 := g_0\Gamma$. Choose $y_0\in X_{i+1}\times X_{j+1}$ with $\rho^{(0)}(y_0) = ([\Lambda_1],[\Lambda_2])$.

Assume $i>0$. Using (6.1) and the Haar decomposition (3.5), we obtain

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f\big(g\cdot([\tilde\Lambda_1],[\tilde\Lambda_2])\big)\,dg \\
&\quad= \int_K \frac{1}{\mathrm{Vol}(G_T^{g_0})}\int_{H_{i,j}}\int_{b_T^{g_0}(k,h)} \varphi\big(ka(s)hx_0\big)\,f\big(ka(s)h\cdot([\Lambda_1],[\Lambda_2])\big)\,\omega(s)\,ds\,dh\,dk,
\end{aligned}$$

where

$$b_T^{g_0}(k,h) := \{s\ge 0 : \|ka(s)hg_0^{-1}\|\le T\},\quad \omega(s)=\sinh(s)^i\cosh(s)^j.$$

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Apply Lemma 4.3 and Proposition 6.2 to the skewed norm $\|\cdot\|_{g_0}$. The dominated convergence theorem then gives

$$\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f\big(g\cdot([\tilde\Lambda_1],[\tilde\Lambda_2])\big)\,dg=\int_K c_{g_0}(k)\,(k\Psi)_*\big(\mu_{G/\Gamma}\otimes\mu_{X_{i+1}}\otimes\mu_{X_{j+1}}\big)(\varphi\otimes f)\,dk,$$

where

$$c_{g_0}(k_0):=\frac{\int_{H_{i,j}}\|k_0a(\infty)hg_0^{-1}\|^{-(n-1)}\,dh}{\int_K\int_{H_{i,j}}\|ka(\infty)hg_0^{-1}\|^{-(n-1)}\,dh\,dk},\ \text{for } k_0\in K.$$

Since

$$(k\Psi)_*\big(\mu_{G/\Gamma}\otimes\mu_{X_{i+1}}\otimes\mu_{X_{j+1}}\big)(\varphi\otimes f)=\mu_{G/\Gamma}(\varphi)\,\mu_{X_{k\cdot\mathbf{P}^\infty}}(f),$$

we obtain

$$\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f\big(g\cdot([\tilde\Lambda_1],[\tilde\Lambda_2])\big)\,dg=\mu_{G/\Gamma}(\varphi)\int_K\mu_{X_{k\cdot\mathbf{P}^\infty}}(f)\,c_{g_0}(k)\,dk.$$

Finally, $\mu_{X_{k\kappa\cdot\mathbf{P}^\infty}}=\mu_{X_{k\cdot\mathbf{P}^\infty}}$ for every $\kappa\in K_{\mathbf{P}^\infty}$, so

$$\int_K\mu_{X_{k\cdot\mathbf{P}^\infty}}(f)\,c_{g_0}(k)\,dk=\int_{K/K_{\mathbf{P}^\infty}}\mu_{X_{k\cdot\mathbf{P}^\infty}}(f)\left(\int_{K_{\mathbf{P}^\infty}}c_{g_0}(k\kappa)\,d\kappa\right)d(kK_{\mathbf{P}^\infty}).$$

This is exactly the density appearing in Theorem 1.1. Applying Proposition 2.1 finishes the proof.

6.2.2. *The special case in Theorem 1.8.* Let $\Gamma\subseteq G$ be a lattice, and let Λ be a degenerate 2-lattice which is Γ-special. Choose $k_{\theta_0}\in K$ such that $\Lambda\subseteq k_{\theta_0}P^\infty$.

Using (6.1) and the Haar decomposition (3.6), we obtain

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f(g\cdot[\Lambda])\,dg\\&=\int_0^{2\pi}\frac{1}{\mathrm{Vol}(G_T^{k_{\theta_0}})}\int_{\mathbb{R}}\int_{b_T^{\theta_0}(\theta,u(t))}\varphi\big(k_\theta a(s)u(t)k_{\theta_0}^{-1}\Gamma\big)\,f\big(k_\theta a(s)u(t)k_{\theta_0}^{-1}\cdot[\Lambda]\big)\,e^s\,ds\,dt\,d\theta\\&=\int_0^{2\pi}\frac{1}{\mathrm{Vol}(G_T^{k_{\theta_0}})}\int_{\mathbb{R}}\int_{b_T^{\theta_0}(\theta,u(t))}\varphi\big(k_\theta\Phi(\delta(s)\upsilon(t))k_{\theta_0}^{-1}\Gamma\big)\,f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\delta(s)\upsilon(t))\Lambda]\big)\,e^s\,ds\,dt\,d\theta,\end{aligned}$$

where $\Phi:\mathrm{SL}(2,\mathbb{R})\to G$ is the isogeny characterized by (1.6), where recalling (1.9),

$$\rho_{\theta_0}^\infty(g):=k_{\theta_0}\rho^\infty(g)k_{\theta_0}^{-1},\quad g\in\mathrm{SL}(2,\mathbb{R}),$$

and where $b_T^{\theta_0}(\theta,u(t)):=\{s\ge0:\|k_\theta a(s)u(t)k_{\theta_0}^{-1}\|\le T\}$.

Since Λ is Γ-special, the subgroup $S:=\Gamma_{\theta_0}^\Phi\cap\mathrm{Stab}_{\theta_0}([\Lambda])$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$; see Remark 1.4. Set $x_0:=k_{\theta_0}^{-1}\Gamma$, and choose $y_0\in\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ such that $\rho_{\theta_0}^\infty(y_0)=[\Lambda]$.

Then (x_0,y_0) is commensurable in the sense of Definition 5.9. Hence, Lemma 4.5 and Proposition 6.2, applied with respect to the skewed norm $\|\cdot\|_{k_{\theta_0}}$, yield

$$\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f(g\cdot[\Lambda])\,dg=\int_0^{2\pi}w(\theta)\,(k_\theta\Psi^\infty)_*\mu_Y(\varphi\otimes f)\,d\theta,\tag{6.2}$$

where

$$Y:=\mathrm{SL}^\Delta(2,\mathbb{R})(x_0,y_0)=\big\{(k_{\theta_0}\Phi(g)k_{\theta_0}^{-1}\Gamma,[\rho_{\theta_0}^\infty(g)\Lambda]):g\in\mathrm{SL}(2,\mathbb{R})\big\},$$

where μ_Y is the $\mathrm{SL}(2,\mathbb{R})$-invariant probability on Y, and

$$w(\vartheta):=\frac{\int_{\mathbb{R}}\|k_\vartheta a(\infty)u(t)k_{\theta_0}^{-1}\|^{-1}\,dt}{\int_0^{2\pi}\int_{\mathbb{R}}\|k_\theta a(\infty)u(t)k_{\theta_0}^{-1}\|^{-1}\,dt\,d\theta}.$$

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Identifying Y with $\mathrm{SL}(2,\mathbb{R})/S$, and writing $d\bar{\eta}$ for the $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure on $\mathrm{SL}(2,\mathbb{R})/S$, we get

$$\begin{aligned}(k_\theta\Psi^\infty)_*\mu_Y(\varphi\otimes f) &= \int_{\mathrm{SL}(2,\mathbb{R})/S}\varphi\big(k_\theta\Phi(\bar{\eta})k_{\theta_0}^{-1}\Gamma\big)\, f\big(k_\theta\cdot[\rho^\infty(\bar{\eta})k_{\theta_0}^{-1}\Lambda]\big)\,d\bar{\eta}\\ &= \int_{\mathrm{SL}(2,\mathbb{R})/S}\varphi\big(k_{\theta_0}\Phi(\bar{\eta})k_{\theta_0}^{-1}\Gamma\big)\, f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-(\theta-\theta_0)/2}\bar{\eta})\Lambda]\big)\,d\bar{\eta},\end{aligned}$$

where we used the identity $k_\vartheta=\Phi(\kappa_{\vartheta/2})$ and the left-invariance of $d\bar{\eta}$.

Let $S_1:=\Gamma_{\theta_0}^\Phi$, choose representatives $\sigma_1,\dots,\sigma_m\in S_1$ for the finite quotient S_1/S, and define

$$[\Lambda_i]:=[\rho_{\theta_0}^\infty(\sigma_i)\Lambda],\quad i=1,\dots,m.$$

Then $\{[\Lambda_1],\dots,[\Lambda_m]\}$ is precisely the (Γ,Λ)-packet.

Consider the first-coordinate projection $p:Y\to G/\Gamma$. Under the identifications $Y\simeq\mathrm{SL}(2,\mathbb{R})/S$ and $G/\Gamma\simeq\mathrm{SL}(2,\mathbb{R})/S_1$, the map p is the natural quotient map $\mathrm{SL}(2,\mathbb{R})/S\to\mathrm{SL}(2,\mathbb{R})/S_1$. Since $[S_1:S]=m$, it is an m-sheeted finite covering.

Choose a sufficiently small identity neighborhood $U\subseteq G$ and a bump function $\varphi\in C_c(G/\Gamma)$ supported on $U\Gamma/\Gamma$ with $\mu_{G/\Gamma}(\varphi)=1$. Shrinking U if necessary,

$$p^{-1}(U\Gamma/\Gamma)=Y_1\sqcup\cdots\sqcup Y_m,$$

where each restriction $p|_{Y_i}:Y_i\to U\Gamma/\Gamma$ is a homeomorphism, and we may arrange that $\sigma_iS\in Y_i$.

Moreover, each sheet carries mass $1/m$ with respect to $\varphi\circ p$:

$$\int_{Y_i}\varphi\circ p\,d\mu_Y=\frac{1}{m},\quad i=1,\dots,m. \tag{6.3}$$

Indeed, after lifting to a sufficiently small neighborhood $\widetilde{U}\subseteq\mathrm{SL}(2,\mathbb{R})$, the sheets have the form

$$\widetilde{U}\sigma_iS/S,\quad i=1,\dots,m,$$

and each $\widetilde{U}\sigma_i$ projects injectively to both $\mathrm{SL}(2,\mathbb{R})/S$ and $\mathrm{SL}(2,\mathbb{R})/S_1$. Since Haar measure on $\mathrm{SL}(2,\mathbb{R})$ is right-invariant, the corresponding sheet integrals are equal. Summing over i gives

$$\sum_{i=1}^m\int_{Y_i}\varphi\circ p\,d\mu_Y=\int_Y\varphi\circ p\,d\mu_Y=\mu_{G/\Gamma}(\varphi)=1,$$

which proves (6.3).

Fix $\epsilon>0$. By uniform continuity of f and compactness of $\theta\in[0,2\pi]$, after shrinking U further if necessary, we may assume that uniformly in θ and uniformly for $\bar{\eta}\in Y_i$,

$$f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-(\theta-\theta_0)/2}\bar{\eta})\Lambda]\big)=f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-(\theta-\theta_0)/2}\sigma_i)\Lambda]\big)+O(\epsilon).$$

Therefore,

$$\begin{aligned}(k_\theta\Psi^\infty)_*\mu_Y(\varphi\otimes f) &= \sum_{i=1}^m\int_{Y_i}\varphi\big(k_{\theta_0}\Phi(\bar{\eta})k_{\theta_0}^{-1}\Gamma\big)\, f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-\theta-\theta_0)/2}\bar{\eta})\Lambda]\big)\,d\bar{\eta}\\ &= \sum_{i=1}^m f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-\theta-\theta_0)/2}\sigma_i)\Lambda]\big)\int_{Y_i}\varphi\big(k_{\theta_0}\Phi(\bar{\eta})k_{\theta_0}^{-1}\Gamma\big)\,d\bar{\eta}+O(\epsilon)\\ &= \frac{1}{m}\sum_{i=1}^m f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-(\theta-\theta_0)/2}\sigma_i)\Lambda]\big)+O(\epsilon).\end{aligned}$$

Substituting this into (6.2), we obtain

$$\lim_{T\to\infty}\frac{1}{\mathrm{Vol}(G_T)}\int_{G_T}\varphi(g\Gamma)\,f(g\cdot[\Lambda])\,dg=\frac{1}{m}\sum_{i=1}^m\int_0^{2\pi}f\big(k_{\theta-\theta_0}\cdot[\rho_{\theta_0}^\infty(\kappa_{-(\theta-\theta_0)/2}\sigma_i)\Lambda]\big)\,w(\theta)\,d\theta+O(\epsilon).$$

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After the change of variables $\phi = \theta - \theta_0$ and by 2π-periodicity, this is exactly the formula in Theorem 1.8(1), since $w(\phi + \theta_0) = w_{\theta_0}(\phi + \theta_0)$ by definition.

Since $\epsilon > 0$ is arbitrary, Proposition 2.1 now implies Theorem 1.8 in the special case.

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