On the space of ergodic invariant measures of unipotent flows

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Abstract

Let G be a Lie group and Γ be a discrete subgroup. We show that if $\{\mu_n\}$ is a convergent sequence of probability measures on G/Γ which are invariant and ergodic under actions of unipotent one-parameter subgroups, then the limit μ of such a sequence is supported on a closed orbit of the subgroup preserving it, and is invariant and ergodic for the action of a unipotent one-parameter subgroup of G.

1 Introduction

Let G be a connected Lie group, Γ be a discrete subgroup of G, and let $\pi : G \to G/\Gamma$ be the natural quotient map. Let X denote the homogeneous space G/Γ on which G acts by left translations.

Let $\mathcal{P}(X)$ denote the set of borel probability measures on X equipped with the weak* topology. The group G acts on $\mathcal{P}(X)$ such that for every $g \in G$ and $\mu \in \mathcal{P}(X)$, we have $g\mu(A) = \mu(g^{-1}A)$ for all borel measurable subsets $A \subset X$. The action $(g, \mu) \mapsto g\mu$ is continuous.

For $\mu \in \mathcal{P}(X)$, define

$$\operatorname{supp}(\mu) = \{x \in X : \mu(\Omega) > 0 \text{ for every neighbourhood } \Omega \text{ of } x \text{ in } X\}$$

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Then $\operatorname{supp}(\mu)$ is a closed subset of X. Also define the invariance group

$$\Lambda(\mu) = \{g \in G : g\mu = \mu\}.$$

Then $\Lambda(\mu)$ is a closed (and hence a Lie) subgroup of G.

A curve $c : [0, \infty) \to X$ is said to be *uniformly distributed* with respect to a measure $\mu \in \mathcal{P}(X)$ if for every bounded continuous function f on X,

$$\lim_{T\to\infty} \int_0^T f(c(t))\, dt = \int_X f\, d\mu.$$

We recall that a subgroup U of G is said to be *unipotent*, if the linear automorphism $\operatorname{Ad} u$ of the Lie algebra of G is unipotent for all $u \in U$.

Now we state the main result of this paper.

Theorem 1.1 *Let $\{\{u_i(t)\}_{t\in\mathbf{R}}\}$ be a sequence of unipotent one-parameter subgroups of G, and let $\{\mu_i\}$ be a sequence in $\mathcal{P}(X)$ such that for each $i \in \mathbf{N}$, μ_i is an ergodic $\{u_i(t)\}_{t\in\mathbf{R}}$-invariant measure. Suppose that $\mu_i \to \mu$ in $\mathcal{P}(X)$, and let $x \in \operatorname{supp}(\mu)$. Then the following holds:*

1. $\operatorname{supp}(\mu) = \Lambda(\mu)x$.
2. *Let $g_i \to e$ be a sequence in G such that for every $i \in \mathbf{N}$, $g_i x \in \operatorname{supp}(\mu_i)$ and the trajectory $\{u_i(t)g_i x : t > 0\}$ is uniformly distributed with respect to μ_i. Then there exists $i_0 \in \mathbf{N}$ such that for all $i \geq i_0$,*

$$\operatorname{supp}(\mu_i) \subset g_i \cdot \operatorname{supp}(\mu).$$

3. *Let L be the subgroup generated by all the (unipotent one-parameter) subgroups $g_i^{-1}\{u_i(t)\}g_i$, $i \geq i_0$. Then μ is invariant and ergodic for the action of L on X.*

A measure $\mu \in \mathcal{P}(X)$ is said to be *algebraic* if $\operatorname{supp}\mu$ is a (closed) orbit of the invariance group $\Lambda(\mu)$.

Let $\mathcal{Q}(X) = \{\mu \in \mathcal{P}(X)$: the group generated by all unipotent one-parameter subgroups of G contained in $\Lambda(\mu)$ acts ergodically on X with respect to $\mu\}$. In fact, every $\mu \in \mathcal{Q}(X)$ is ergodic for the action of a single unipotent one-parameter subgroup of G contained in $\Lambda(\mu)$ (see Lemma 2.3).

The following fundamental result concerning the rigidity of unipotent actions on homogeneous spaces is one of the essential ingredients in our proof.

The reader is referred to the survey article [Ra93] for some related developments.

Theorem (Ratner [Ra91a]). *Every measure in $\mathcal{Q}(X)$ is algebraic.* $\square$

In addition to this basic result, our proof of Theorem 1.1 involves the study of unipotent trajectories in 'thin' neighbourhoods of certain 'singular' subsets of X. We use the ideas and methods developed in [DS84], [DM90], [Sh91], and [DM93] for studying such trajectories via suitable linear representations of G. The method allows one to 'linearize' the thin neighbourhoods and study unipotent trajectories in the representation space rather than in G/Γ. This facilitates the use of the polynomial behaviour of unipotent trajectories in the representation space for studying the corresponding trajectories in X.

In the above terminology Theorem 1.1 implies the following.

Corollary 1.1 *$\mathcal{Q}(X)$ is a closed subset of $\mathcal{P}(X)$.*

For $x \in X$, define $\mathcal{Q}(x) = \{\mu \in \mathcal{Q}(X) : x \in \operatorname{supp}(\mu)\}$.

Corollary 1.2 *For every $x \in X$, $\mathcal{Q}(x)$ is a closed subset of $\mathcal{P}(X)$.*

For the rest of the results stated in the introduction, we assume that X admits a finite G-invariant measure or, in other words, Γ is a lattice in G.

The following result is crucial in studying the action of a unipotent one-parameter subgroup on a noncompact homogeneous space of finite volume.

Theorem (Dani and Margulis [DM93, Theorem 6.1]). *Given a compact set $C \subset X$ and an $\epsilon > 0$, there exists a compact set $K \subset X$ such that the following holds: For any $x \in C$, any unipotent one-parameter subgroup $\{u(t) : t \in \mathbf{R}\}$ of G, and any $T > 0$,*

$$\frac{1}{T}\ell\left(\{t \in [0,T] : u(t)x \in K\}\right) > 1 - \epsilon,$$

where ℓ denotes the Lebesgue measure on $\mathbf{R}$. $\square$

This fact enables us to strengthen our main result to get the following corollaries.

Let $X \cup \{\infty\}$ denote the one-point compactification of X. Then $\mathcal{P}(X \cup \{\infty\})$ is compact with respect to the weak* topology.

Corollary 1.3 *Let $\{\mu_i\} \subset \mathcal{Q}(X)$ be a sequence of measures converging weakly to a measure $\mu \in \mathcal{P}(X \cup \{\infty\})$. Then either $\mu \in \mathcal{Q}(X)$ or $\mu(\{\infty\}) = 1$.*

Corollary 1.4 *For every $x \in X$, the set $\mathcal{Q}(x)$ is compact with respect to the weak* topology.*

Let $\mathcal{W} = \{U_i = \{u_i(t)\}\}$ be a sequence of unipotent one-parameter subgroups of G. We say that a point $x \in X$ is *regular* for $\mathcal{W}$ if there does not exist any proper closed subgroup F of G such that the orbit Fx is closed and $F \supset U_i$ for infinitely many $i \in \mathbf{N}$.

We say that a point $x \in X$ is *generic* for $\mathcal{W}$ if for every bounded continuous function f of X the following holds: There exists a sequence $S_i \to \infty$ in $\mathbf{R}$ such that for any sequence $\{T_i\}$ with each $T_i \geq S_i$, we have

$$\lim_{i\to\infty} \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x)\,dt = \int_X f\,d\mu_G,$$

where μ_G is the G-invariant probability measure on X.

Corollary 1.5 *A point $x \in X$ is generic for $\mathcal{W}$ if and only if it is regular for $\mathcal{W}$.*

2 Description of finite invariant measures of a unipotent flow

We first note the following.

Lemma 2.1 *Let F be a closed subgroup of G and let $x \in X$ be such that the orbit Fx is closed. Let $\Delta = \{\delta \in F : \delta x = x\}$. Then the map $\phi : F/\Delta \to X$, defined by $\phi(g\Delta) = gx$ for all $g \in F$, is injective and proper.* $\square$

The following simple observation enables us to apply Ratner's theorem in our proof of the main theorem.

Lemma 2.2 *Let the notations be as in the statement of Theorem 1.1. Suppose that $\{u_i(t)\} \neq \{e\}$ for all large $i \in \mathbf{N}$. Then $\Lambda(\mu)$ contains a non-trivial unipotent one-parameter subgroup of G.*

Proof. For each $i \in \mathbf{N}$ there exists $w_i \in \underline{\mathbf{g}}$ such that $\|w_i\| = 1$ and $\{u_i(t) : t \in \mathbf{R}\} = \{\exp(tw_i) : t \in \mathbf{R}\}$, where $\underline{\mathbf{g}}$ is the Lie algebra of G and $\|\cdot\|$ denotes a Euclidean norm on it. By passing to a subsequence, we may assume that $w_i \to w$ for some $w \in \underline{\mathbf{g}}$, $\|w\| = 1$. For any $t \in \mathbf{R}$, we have $\mathrm{Ad}(\exp(tw_i)) \to \mathrm{Ad}(\exp(tw))$ as $i \to \infty$. Therefore $U = \{\exp(tw) : t \in \mathbf{R}\}$ is a (nontrivial) unipotent subgroup of G. Since $\exp tw_i \to \exp tw$ for all t and $\mu_i \to \mu$, it follows that μ is invariant under the action of U on X. $\square$

The next result says that every measure in $\mathcal{Q}(X)$ is ergodic with respect to a unipotent one-parameter subgroup of G.

Lemma 2.3 *Let W be a closed subgroup of G generated by unipotent one-parameter subgroups of G contained in W. Suppose that W acts ergodically with respect to a measure $\mu \in \mathcal{P}(X)$. Then there exists a unipotent one-parameter subgroup of G contained in W which acts ergodically with respect to μ.*

Proof. Let N be a maximal connected unipotent subgroup of G contained in W. Then no proper normal subgroup of W contains N (see [Sh91, Lemma 2.9]). Therefore by Mautner phenomenon, N acts ergodically with respect to μ (see [Mo80, Theorem 1.1] and [Ma91]). Since N is a nilpotent group, there exists a one-parameter subgroup of N which acts ergodically with respect to μ (see [Da89, Proposition 2.2] or [Ra91a, Proposition 5.1]). $\square$

The following result is useful in applying the theorem of Ratner to describe all finite invariant (possibly non-ergodic) measures for unipotent actions.

Proposition 2.1 *Let F be a connected Lie group, Δ be a lattice in F, and W be a subgroup which is generated by unipotent one-parameter subgroups of F contained in W. Let L be the smallest closed subgroup of F containing W such that the orbit $L\Delta/\Delta$ is closed in F/Δ. Then the following holds:*

1. *$L \cap \Delta$ is a lattice in L.*
2. *W acts ergodically on $L\Delta/\Delta \cong L/L\cap\Delta$ with respect to the L-invariant probability measure.*
3. *Let $\rho : F \to \mathrm{GL}(V)$ be a finite dimensional representation such that for every unipotent one-parameter subgroup U of F contained in W, $\rho(U)$ consists of unipotent transformations on V. Then $\rho(L \cap \Delta)$ is Zariski dense in $\rho(L)$.*

Proof. Statements 1) and 2) follow from [Sh91, Theorem 2.3]. Statement 3) follows from [Sh91, Proposition 2.11 and Corollary 2.13]. $\square$

Let $\mathcal{H}$ be the collection of all closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H and the subgroup generated by all the unipotent one-parameter subgroups of G contained in H acts ergodically on $H\Gamma/\Gamma$ with respect to the H-invariant probability measure.

In particular, by Proposition 2.1, $\mathrm{Ad}(H \cap \Gamma)$ is Zariski dense in $\mathrm{Ad}(H)$ for every $H \in \mathcal{H}$, where Ad denotes the Adjoint representation of G.

Theorem 2.1 ([Ra91a, Theorem 1.1]) *$\mathcal{H}$ is a countable collection.* $\square$

See [DM93, Proposition 2.1] for an alternative proof of this fact. Also compare [Sh91, Lemma 5.2].

Let W be a subgroup of G which is generated by unipotent one-parameter subgroups of G contained in W. For $H \in \mathcal{H}$, define

$$N(H, W) = \{g \in G : W \subset gHg^{-1}\},$$

$$S(H, W) = \bigcup_{H' \in \mathcal{H},\, H' \subset H,\, H' \neq H} N(H', W),$$

and

$$T_H(W) = \pi(N(H, W) \setminus S(H, W)).$$

Lemma 2.4 *For any $g \in N(H, W) \setminus S(H, W)$, the group gHg^{-1} is the smallest closed subgroup of G which contains W and whose orbit through $\pi(g)$ is closed in X.*

In particular, $T_H(W) = \pi(N(H, W)) \setminus \pi(S(H, W))$.

Proof. Let L be the smallest closed subgroup of G such that $W \subset L$ and $L\pi(g)$ is closed. Since $W \subset gHg^{-1}$ and the orbit $gHg^{-1} \cdot \pi(g) = g(H\Gamma/\Gamma)$ is closed, we have $L \subset gHg^{-1}$. Put $H' = g^{-1}Lg \subset H$. Then due to Lemma 2.1 and Proposition 2.1, $H' \in \mathcal{H}$. Since $g \in N(H', W)$ and $g \notin S(H, W)$, we have $H' = H$. Hence $gHg^{-1} = L$. $\square$

Corollary 2.1 *For any $H_1\, H_2 \in \mathcal{H}$, we have that*

$$\begin{aligned} T_{H_1}(W) \cap T_{H_2}(W) \neq \emptyset \quad &\Leftrightarrow \quad H_2 = \gamma H_1 \gamma^{-1} \text{ for some } \gamma \in \Gamma \\ &\Leftrightarrow \quad T_{H_1}(W) = T_{H_2}(W). \end{aligned}$$

Proof. Use Lemma 2.4. □

In the next result we give a description of a finite invariant measure of a unipotent flow, using the Ratner's classification of finite ergodic invariant measures of this flow.

Theorem 2.2 *Let $\mu \in \mathcal{P}(X)$ be a W-invariant measure. For every $H \in \mathcal{H}$, let μ_H denote the restriction of μ on $T_H(W)$. Then the following holds.*

1. *For all borel measurable subsets $A \subset X$,*

$$\mu(A) = \sum_{H \in \mathcal{H}^*} \mu_H(A),$$

where $\mathcal{H}^* \subset \mathcal{H}$ is a countable set consisting of one representative from each Γ-conjugacy class of elements in $\mathcal{H}$.

2. *Each μ_H is W-invariant. For any W-ergodic component $\nu \in \mathcal{P}(X)$ of μ_H, there exists a $g \in N(H, W)$ such that ν is the (unique) gHg^{-1}-invariant probability measure on the closed orbit $gH\Gamma/\Gamma$.*

Proof. Disintegrate μ into W-ergodic components. Due to Ratner's theorem [Ra91a], each one of them is an algebraic measure. Using Proposition 2.1, we can conclude that each one of them is of the form $g\nu_H$ for some $H \in \mathcal{H}$ and $g \in N(H, W)$. Now 1) and 2) can be obtained by using Theorem 2.1, Lemma 2.4, and Corollary 2.1. □

3 Dynamics of unipotent trajectories in thin neighbourhoods of $T_H(W)$

Let W be as in Section 2, and $H \in \mathcal{H}$. Let $\underline{\mathbf{g}}$ denote the Lie algebra of G and let $\underline{\mathbf{h}}$ denote its Lie subalgebra associated to H. For $d = \dim \underline{\mathbf{h}}$, put $V_H = \wedge^d \underline{\mathbf{g}}$, the d-th exterior power, and consider the linear G-action on V_H via the representation $\wedge^d \operatorname{Ad}$, the d-th exterior of the Adjoint representation of G on $\underline{\mathbf{g}}$. Fix $p_H \in \wedge^d \underline{\mathbf{h}} \setminus \{0\}$, and let $\eta_H : G \to V_H$ be the map defined by $\eta_H(g) = g \cdot p_H = (\wedge^d \operatorname{Ad} g) p_H$ for all $g \in G$. Note that

$$\eta_H^{-1}(p_H) = \{g \in \mathrm{N}(H) : \det(\operatorname{Ad} g|_{\underline{\mathbf{h}}}) = 1\},$$

where $N(H)$ denotes the normalizer of H in G.

Put $\Gamma_H = \mathrm{N}(H) \cap \Gamma$. Then for any $\gamma \in \Gamma_H$, we have $\gamma(H\Gamma/\Gamma) = H\Gamma/\Gamma$, and hence γ preserves the volume of $H\Gamma/\Gamma$. Therefore $|\det(\operatorname{Ad}\gamma|_{\underline{\mathbf{h}}})| = 1$, and hence $\gamma \cdot p_H = \pm p_H$.

In view of this we define $\bar{V}_H = V_H/\{1,-1\}$ if $\Gamma_H \cdot p_H = \{p_H, -p_H\}$, and define $\bar{V}_H = V_H$ if $\Gamma_H \cdot p_H = p_H$. The action of G factors through the quotient map of V_H onto $\bar{V}_H$. Let $\bar{p}_H$ denote that image of p_H in $\bar{V}_H$, and define $\bar{\eta}_H : G \to \bar{V}_H$ as $\bar{\eta}_H(g) = g \cdot \bar{p}_H$ for all $g \in G$. Then $\Gamma_H = \bar{\eta}_H^{-1}(\bar{p}_H) \cap \Gamma$.

Theorem 3.1 ([DM93, Theorem 3.4]) *The orbit $\Gamma \cdot \bar{p}_H$ is discrete in $\bar{V}_H$.*

In particular, the map $G/\Gamma_H \to G/\Gamma \times \bar{V}_H$, given by $g\Gamma_H \mapsto (\pi(g), \bar{\eta}_H(g))$ for all $g \in G$, is proper. □

Proposition 3.1 ([DM93, Proposition 3.2]) *Let A_H denote the Zariski closure of $\bar{\eta}_H(N(H,W))$ in $\bar{V}_H$. Then*

$$\bar{\eta}_H^{-1}(A_H) = N(H,W).$$

□

Proposition 3.2 *Let D be a compact subset of A_H. Define*

$$S(D) = \{g \in \bar{\eta}_H^{-1}(D) : g\gamma \in \bar{\eta}_H^{-1}(D) \textit{ for some } \gamma \in \Gamma \setminus \Gamma_H\}.$$

Then the following holds:

1. *$S(D) \subset S(H,W)$.*
2. *$\pi(S(D))$ is closed in X.*
3. *For any compact set $K \subset X \setminus \pi(S(D))$, there exists a neighbourhood Φ of D in $\bar{V}_H$ such that every $y \in \pi(\bar{\eta}_H^{-1}(\Phi)) \cap K$ has a unique representative in Φ; that is, the set $\bar{\eta}_H(\pi^{-1}(y)) \cap \Phi$ consists of a single element.*

Proof. The proof is essentially contained in [DM93, Sect. 3]. (Also compare [Sh91, Sec. 6]).

Let $g, g\gamma \in \bar{\eta}_H^{-1}(D)$ be such that $\gamma \in \Gamma \setminus \Gamma_H$. Then by Proposition 3.1, $g, g\gamma \in N(H,W)$. Hence by Lemma 2.4, either $g \in S(H,W)$ or $gHg^{-1} \subset (g\gamma)H(g\gamma)^{-1}$; but the latter is not possible since $\gamma \notin \Gamma_H$. This proves 1).

For proving 2) and 3), first let K be any compact subset of X. Suppose that there exist sequences $\{g_i\} \subset \pi^{-1}(K)$ and $\{\gamma_i\} \subset \Gamma \setminus \Gamma_H$ such that $\bar{\eta}_H(g_i) \to q$ and $\bar{\eta}_H(g_i\gamma_i) \to q'$ in $\bar{V}_H$ for some $q, q' \in D$. Due to Theorem 3.1, by passing to subsequences we may assume that, the sequences $\{g_i\Gamma_H\}$ and $\{g_i\gamma_i\Gamma_H\}$ are convergent in G/Γ_H. Therefore there exist $g \in G$ and $\gamma \in \Gamma$ such that $g_i\Gamma_H \to g\Gamma_H$ and $g_i\gamma_i\Gamma_H \to g\gamma\Gamma_H$ as $i \to \infty$. Since $\{\gamma_i\} \subset \Gamma \setminus \Gamma_H$, we have $\gamma \in \Gamma \setminus \Gamma_H$. Clearly $\pi(g) \in K$, $\bar{\eta}_H(g) = q$, and $\bar{\eta}_H(g\gamma) = q'$. Therefore $g \in S(D) \cap \pi^{-1}(K)$.

From the above discussion, we can conclude that the set $\pi(S(D)) \cap K$ is compact. This implies 2).

The same discussion also shows that if $K \cap \pi(S(D)) = \emptyset$, then there exists a neighbourhood Φ of D in $\bar{V}_H$ such that for any $g \in \pi^{-1}(K)$, if $\bar{\eta}_H(g) \in \Phi$ then $\bar{\eta}_H(g\gamma) \notin \Phi$ for any $\gamma \in \Gamma \setminus \Gamma_H$. This proves 3). $\square$

The above result allows us to relate a trajectory of a unipotent one-parameter subgroup of G in the set $K \cap \pi(\bar{\eta}_H^{-1}(\Phi))$ with a trajectory of it in Φ. Since the orbits of unipotent one-parameter subgroups on a finite dimensional vector space are polynomial curves of bounded degree, we can use properties of polynomial functions to obtain information about dynamical behaviour of unipotent trajectories in neighbourhoods of compact subsets of $T_H(W)$, which are of the form $K \cap \pi(\bar{\eta}_H^{-1}(\Phi))$. In view of Theorem 2.2, this will help us in describing the measure μ_H, and hence the measure μ.

The next result is very useful in understanding the behaviour of unipotent trajectories near the algebraic variety A_H.

Proposition 3.3 ([DM93, Proposition 4.2]) *Let a compact set $C \subset A_H$ and an $\epsilon > 0$ be given. Then there exists a compact set $D \subset A_H$ with $C \subset D$ such that for any neighbourhood Φ of D in $\bar{V}_H$, there exists a neighbourhood Ψ of C in $\bar{V}_H$ with $\Psi \subset \Phi$ such that the following holds: For a unipotent one-parameter subgroup $\{u(t)\}$ of G, an element $w \in \bar{V}_H$, and a bounded interval $I \subset \mathbf{R}$, if $u(t_0)w \notin \Phi$ for some $t_0 \in I$, then*

$$\ell\left(\{t \in I : u(t)w \in \Psi\}\right) \leq \epsilon \cdot \ell\left(\{t \in I : u(t)w \in \Phi\}\right). \tag{1}$$

$\square$

The next result is the main technical tool needed for our proof of Theorem 1.1.

Proposition 3.4 *Let a compact set $C \subset A_H$ and a $0 < \epsilon < 1$ be given. Then there exists a closed subset $\mathcal{S}$ of X contained in $\pi(S(H, W))$ such that the following holds: For a given compact set $K \subset X \setminus \mathcal{S}$, there exists a neighbourhood Ψ of C in $\bar{V}_H$ such that for any unipotent one-parameter subgroup $\{u(t)\}$ of G and any $x \in X$, at least one of the following conditions is satisfied:*

1. *There exists $w \in \bar{\eta}_H(\pi^{-1}(x)) \cap \overline{\Psi}$, such that*

$$\{u(t)\} \subset G_w := \{g \in G : gw = w\}.$$

2. *For all large $T > 0$,*

$$\ell\left(\{t \in [0, T] : u(t)x \in K \cap \pi(\bar{\eta}_H^{-1}(\Psi))\}\right) < \epsilon T.$$

Proof. It is possible to deduce this result from [DM93, Theorem 7.3]; the latter is stronger but its proof is technically more involved. Here we shall give a simpler proof along the line of proof of Theorem 1 in [DM93].

For the given C and ϵ, obtain a compact set $D \subset A_H$ as in Proposition 3.3. For this D, apply Proposition 3.2 to obtain a closed subset $\mathcal{S} = \pi(S(D))$ of X contained in $\pi(S(H, W))$. Now let K be any compact subset of $X \setminus \mathcal{S}$ and let Φ be an open neighbourhood of D in $\bar{V}_H$ as in 3) of Proposition 3.2. Finally let Ψ be a neighbourhood of C in $\bar{V}_H$ such that the Eq. 1 is satisfied.

Put $\Omega = \pi(\bar{\eta}_H^{-1}(\Psi)) \cap K$, and define

$$J = \{t \geq 0 : u(t)x \in \overline{\Omega}\}. \tag{2}$$

Then for every $t \in J$, there exists a unique $w \in \bar{\eta}_H(\pi^{-1}(x))$ such that $u(t)w \in \Phi$; in which case $u(t)w \in \overline{\Psi}$.

Since $s \mapsto u(s)w$ is a polynomial function, either it is constant or it is unbounded as $s \to \pm\infty$. In the first case the condition 1) is satisfied and we are through. Therefore now we can assume that for every $t \in J$, there exists the largest closed interval $I(t) =: [t^-, t^+]$ in $\mathbf{R}$ containing t such that the following three conditions are satisfied:

1. $u(s)w \in \overline{\Phi}$ for all $s \in I(t)$.

2. $u(t^-)w \in \overline{\Phi} \setminus \Phi$.

3. $t^+ \in J$.

Note that $I(s) = I(t)$ for every $s \in I(t) \cap J$, in particular for $s = t^+$. Due to this property, for any $t_1, t_2 \in J$,

$$\text{either } I(t_1) = I(t_2) \text{ or } I(t_1) \cap I(t_2) = \emptyset. \tag{3}$$

Since $u(t^-)w \notin \Phi$, due to Eq. 1, for any $T \in I(t)$,

$$\ell\left(\{s \in [t^-, T] : u(s)w \in \Psi\}\right) \leq \epsilon(T - t^-).$$

Therefore,

$$\ell\left(\{s \in [t^-, T] : u(s)x \in \Omega\}\right) \leq \epsilon(T - t^-). \tag{4}$$

Let $T_0 = \inf_{t \in J} t^+$. Then by Eqs. 2, 3, and 4, for any $T > T_0$,

$$\ell\left(\{s \in [T_0, T] : u(s)x \in \Omega\}\right) < \epsilon T.$$

Now for all $T > T_0/\epsilon$, the condition 2 is satisfied for 2ϵ in place of ϵ. $\square$

4 Proof of Theorem 1.1

Let W be the subgroup of G generated by all unipotent one-parameter subgroups of G contained in $\Lambda(\mu)$. If $\dim W = 0$, due to Lemma 2.2, $\{u_i(t)\} = \{e\}$ for all large $i \in \mathbf{N}$; in which case μ is also a point measure and the conclusion of the theorem is obvious. Therefore we may assume that $\dim W > 0$.

By Theorem 2.2 (1), there exists a $H \in \mathcal{H}$ such that $\mu(\pi(S(H, W))) = 0$ and $\mu(\pi(N(H, W))) > 0$. Hence there exists a compact set $C_1 \subset N(H, W) \setminus S(H, W)$ such that

$$\mu(\pi(C_1)) = \alpha > 0. \tag{5}$$

Let $g_i \to e$ be a sequence in G such that for every $i \in \mathbf{N}$, $g_i x \in \operatorname{supp}(\mu_i)$ and the trajectory $\{u_i(t)g_i x\}_{t>0}$ is uniformly distributed with respect to μ_i; note that, due to Birkhoff ergodic theorem, such a sequence always exists. Take any $y \in \operatorname{supp}(\mu) \cap \pi(C_1)$. Then for each $i \in \mathbf{N}$ there exists $y_i \in \{u_i(t)g_i x\}_{t \geq 0}$ such that as $i \to \infty$, $y_i \to y$. Let $h_i \to e$ be a sequence in G such that $h_i y_i = y$ for all $i \in \mathbf{N}$. Put $\mu_i' = h_i \mu_i$ and $u_i'(t) = h_i u_i(t) h_i^{-1}$ for all $t \in \mathbf{R}$ and all $i \in \mathbf{N}$. Then $\mu_i' \to \mu$ as $i \to \infty$. Also $y \in \operatorname{supp}(\mu_i')$ and the trajectory $\{u_i'(t)y : t > 0\}$ is uniformly distributed with respect to μ_i' for each $i \in \mathbf{N}$.

Apply Proposition 3.4 for $C = \bar{\eta}_H(C_1)$ and $\epsilon = \alpha/2$. Let the notation be as in the statement of the proposition. Since $\pi(C_1) \cap \pi(S(H, W)) = \emptyset$, we can choose a compact neighbourhood K of $\pi(C_1)$ in X such that $K \cap \mathcal{S} = \emptyset$. Put $\Omega = \pi(\bar{\eta}_H^{-1}(\Psi)) \cap K$. Since $\mu_i' \to \mu$, due to Eq. 5, there exists $k_0 \in \mathbf{N}$ such that $\mu_i'(\Omega) > \epsilon$ for all $i \geq k_0$. Therefore for any $i \geq k_0$ and all large $T > 0$,

$$\frac{1}{T}\ell\left(\{t \in [0, T] : u_i'(t)y \in \Omega\}\right) > \epsilon.$$

This shows that for each $i \geq k_0$, the condition 2) of Proposition 3.4 is violated for $\{u_i'(t)\}$ and y. Since $y \in \pi(C_1) \subset K$, there exists a unique representative w of y in Ψ. Therefore according to the condition 1) of Proposition 3.4, for each $i \geq k_0$,

$$\{u_i'(t)y\}_{t \in \mathbf{R}} \subset (G_w y)^0 = G_w^0 y,$$

where $(G_w y)^0$ denotes the connected component of $G_w y$ containing y and G_w^0 denotes the connected component of G_w containing the identity. Due to Theorem 3.1, the orbit $G_w y$ is closed in X.

We intend to prove the parts 1) and 2) of Theorem 1.1 by induction on $\dim G$.

First suppose that $\dim G_w^0 < \dim G$. Due to Lemma 2.1, we can treat $G_w^0 y$ as a homogeneous space of G_w^0. Also each $\{u_i'(t)\}$ is a unipotent subgroup of G_w^0 and each μ_i' is supported on $G_w^0 y$. Therefore by induction hypothesis applied to G_w^0, we obtain the following: $\mathrm{supp}(\mu) = (\Lambda(\mu) \cap G_w^0)y$ and there exists $j_0 \in \mathbf{N}$ such that for all $i \geq j_0$, $\mathrm{supp}(\mu_i') \subset \mathrm{supp}(\mu)$.

Next suppose that $\dim G_w^0 = \dim G$. In this case $G_w = G$, and hence H is a normal subgroup of G. Let $\bar{G} = G/H$ be the quotient group. Since $\dim H \geq \dim W > 0$, we have $\dim \bar{G} < \dim G$. We will project the measures on the homogeneous space $G/(H\Gamma)$ of $\bar{G}$ and apply induction.

We need some notation. Let $\rho : G \to \bar{G}$ be the quotient homomorphism. Since $H\Gamma$ is closed in G, the subgroup $\bar{\Gamma} = \rho(\Gamma)$ is closed (and hence discrete) in $\bar{G}$. Put $\bar{X} = \bar{G}/\bar{\Gamma}$, and let $\bar{\rho} : X \to \bar{X}$ be the natural quotient map. Define a map $\bar{\rho}_* : \mathcal{P}(X) \to \mathcal{P}(\bar{X})$ such that for any $\nu \in \mathcal{P}(X)$ and any borel measurable subset $A \subset \bar{X}$, $\bar{\rho}_*(\nu)(A) = \nu(\bar{\rho}^{-1}(A))$. Then $\bar{\rho}_*$ is continuous.

Put $\bar{y} = \bar{\rho}(y)$. Observe the following: for each $i \geq k_0$, 1) $\{\rho(u_i'(t))\}$ is a unipotent one-parameter subgroup of $\bar{G}$, 2) $\bar{\rho}_*(\mu_i)$ is ergodic $\{\rho(u_i'(t))\}$-invariant, 3) $\bar{y} \in \mathrm{supp}(\bar{\rho}_*(\mu_i'))$, and 4) the trajectory $\{\rho(u_i'(t))\bar{y}\}_{t>0}$ is uniformly distributed with respect to $\bar{\rho}_*(\mu_i')$. Also $\bar{\rho}_*(\mu_i) \to \bar{\rho}_*(\mu)$ as $i \to \infty$.

Therefore by induction hypothesis applied to $\bar{G}$, we obtain the following:

1. $\mathrm{supp}(\bar{\rho}_*(\mu)) = \Lambda(\bar{\rho}_*(\mu))\bar{y}$.
2. There exists $j_0 \geq k_0$ such that for all $i \geq j_0$,

$$\mathrm{supp}(\bar{\rho}_*(\mu_i')) \subset \mathrm{supp}(\bar{\rho}_*(\mu)).$$

We claim that

$$\rho^{-1}(\Lambda(\bar{\rho}_*(\mu))) = \Lambda(\mu).$$

Since H is normal in G, by Theorem 2.2 (2), each ergodic component of μ_H is H-invariant. Since $N(H,W) = G$ and $\mu(\pi(S(H,W))) = 0$, we have $\mu = \mu_H$. Therefore μ is H-invariant. Now the claim follows from Proposition 1.6 of [Da78] applied to the quotient space $G/H\Gamma$.

Hence for all $i \geq j_0$,

$$\mathrm{supp}(\mu_i') \subset \bar{\rho}^{-1}(\mathrm{supp}(\bar{\rho}_*(\mu))) = \rho^{-1}(\Lambda(\bar{\rho}_*(\mu)))y = \Lambda(\mu)y.$$

Thus if either $\dim G_w^0 < \dim G$ or $\dim G_w^0 = \dim G$, we have obtained the following conclusions: $\mathrm{supp}(\mu) = \Lambda(\mu)y$, and there exists $j_0 \in \mathbf{N}$ such that for all $i \geq j_0$, $\mathrm{supp}(\mu_i') \subset \mathrm{supp}(\mu)$. Thus $x \in \Lambda(\mu)y$, and hence

$$\mathrm{supp}(\mu) = \Lambda(\mu)x.$$

Since $h_i(g_i x) \in h_i \cdot \mathrm{supp}(\mu_i) = \mathrm{supp}(\mu_i')$ for all $i \in \mathbf{N}$, we have $(h_i g_i)x \in \Lambda(\mu)x$ for all $i \geq j_0$. Therefore since $h_i \to e$ and $g_i \to e$, there exists $i_0 \geq j_0$ such that for all $i \geq i_0$, $h_i g_i \in \Lambda(\mu)$. Hence for all $i \geq i_0$,

$$\mathrm{supp}(\mu_i) = h_i^{-1} \cdot \mathrm{supp}(\mu_i') \subset h_i^{-1} \cdot \Lambda(\mu)x = g_i \cdot \mathrm{supp}(\mu).$$

This proves parts 1) and 2) of Theorem 1.1 for G.

Now let L be defined as in part 3) of the theorem. Then $\Lambda(\mu)^0$ is the smallest closed subgroup of G which contains L and whose orbit through y is closed. Therefore by Proposition 2.1, L acts ergodically on $\Lambda(\mu)x$ with respect to μ. This completes the proof of the theorem. □

Proofs of Corollary 1.1 and Corollary 1.2. Use Lemma 2.3 and Theorem 1.1. □

Proofs of Corollary 1.3 and Corollary 1.4. First observe that due to Lemma 2.3, the Birkhoff ergodic theorem, and the theorem of Dani and Margulis mentioned in the introduction, for any $\nu \in \mathcal{Q}(X)$, if $\nu(C) > 0$ then $\nu(K) > 1 - \epsilon$ (notation as in the statement of the theorem of Dani and Margulis). Now use Corollary 1.1 and Corollary 1.2 to complete the proofs. $\square$

We need the following result for the proof of Corollary 1.5.

Theorem. (Ratner [Ra91b]) *Suppose that X admits a finite G-invariant measure. Let $U = \{u(t)\}$ be a unipotent one-parameter subgroup of G and $x \in X$. Then there exists a closed subgroup F of G containing U such that the orbit Fx is closed, it admits a finite F-invariant probability measure, say μ, and the trajectory $\{u(t)x : t > 0\}$ is uniformly distributed with respect to μ.* $\square$

(We remark that this theorem may now be deduced also using the theorem of Dani and Margulis stated in the introduction, Theorem 2.2, Proposition 3.4, as well as arguments similar to those in [Sh91, Corollary 7.1].)

Proof of Corollary 1.5. First suppose that x is regular for $\mathcal{W}$. In view of the above result, for each $i \in \mathbf{N}$, the trajectory $\{u_i(t)x : t > 0\}$ is uniformly distributed with respect to some $\mu_i \in \mathcal{Q}(x)$. By Corollary 1.4, there exists a sequence $i_k \to \infty$ such that $\mu_{i_k} \to \mu$ for some $\mu \in \mathcal{Q}(x)$. Then by Theorem 1.1, $U_{i_k} \subset \Lambda(\mu)$ for all large $k \in \mathbf{N}$. Since x is regular for $\mathcal{W}$, we have that $\Lambda(\mu) = G$. In particular, $\mu_i \to \mu = \mu_G$ as $i \to \infty$.

Let f be a given bounded continuous function on X. Then for each $i \in \mathbf{N}$, there exists $S_i > 0$ such that for every $T_i > S_i$,

$$\left| \int_X f \, d\mu_i - \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x) \, dt \right| < \epsilon/i.$$

Now since $\mu_i \to \mu_G$, we have

$$\lim_{i \to \infty} \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x) dt = \int f d\mu_G.$$

Thus x is generic for $\mathcal{W}$. The converse implication is obvious. $\square$

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