EQUIDISTRIBUTION OF PRIMITIVE RATIONAL POINTS ON EXPANDING HOROSPHERES

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ABSTRACT. We confirm a conjecture of J. Marklof regarding the limiting distribution of certain sparse collections of points on expanding horospheres. These collections are obtained by intersecting the expanded horosphere with a certain manifold of complementary dimension and turns out to be of arithmetic nature. This result is then used along the lines suggested by J. Marklof to give an analogue of a result of W. Schmidt regarding the distribution of shapes of lattices orthogonal to integer vectors.

1. MAIN RESULTS

1.1. **The main theorem.** Given integers $m \geq n \geq 1$, let $d \stackrel{\text{def}}{=} n+m$ and consider the space $X_d \stackrel{\text{def}}{=} \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ on which $G \stackrel{\text{def}}{=} \mathrm{SL}_d(\mathbb{R})$ and its subgroups act. Unless otherwise stated, when we write an element $g \in G$ as a matrix $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, we shall mean that A, B, C, D represent matrices of dimensions $m \times m, m \times n, n \times m$, and $n \times n$ respectively. We refer to these as the *block components* of g. We shall denote the identity matrix in dimension k by I_k and often write just $I = I_k$ if the dimension is clear from the context. Similarly, 0 will denote the zero matrix in various dimensions.

Consider the following subgroups of G:

$$\begin{aligned}
U &\stackrel{\text{def}}{=} \left\{\left(\begin{smallmatrix} I & 0 \\ \mathbf{u} & I \end{smallmatrix}\right) : \mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{R})\right\}, \\
V &\stackrel{\text{def}}{=} \left\{\left(\begin{smallmatrix} I & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) : \mathbf{v} \in \mathrm{Mat}_{m\times n}(\mathbb{R})\right\}, \text{ and} \\
H &\stackrel{\text{def}}{=} \left\{\left(\begin{smallmatrix} h & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) : h \in \mathrm{SL}_m(\mathbb{R}), \mathbf{v} \in \mathrm{Mat}_{m\times n}(\mathbb{R})\right\}.
\end{aligned}$$

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Also define the diagonal matrices

$$a(y) \overset{\text{def}}{=} \begin{pmatrix} y^{-\frac{n}{m}} I_m & 0 \\ 0 & yI_n \end{pmatrix} \text{ for } y \in \mathbb{R}_{>0}.$$

We denote by $x_0 \in X_d$ the identity coset $\mathrm{SL}_d(\mathbb{Z})$ and set for $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$

$$x_{\mathbf{u}} \overset{\text{def}}{=} \left(\begin{smallmatrix} I & 0 \\ \mathbf{u} & I \end{smallmatrix}\right) x_0.$$

Definition 1.1. Given a subgroup $L < G$ and a point $x \in X_d$ we say that the orbit Lx is *periodic* if it supports an L-invariant probability measure. In this case we denote by $\mathbf{m}_{Lx}$ the unique L-invariant probability measure supported on Lx and refer to it as the *Haar* measure on the orbit.

The orbits Ux_0, Vx_0 and Hx_0 are periodic and their respective Haar measures will play a prominent role in our discussion. In our discussion the cases $m > n$ and $m = n$ will exhibit different phenomena and in order to be able to have unified statements we use the following notation throughout:

$$\boldsymbol{\theta} \overset{\text{def}}{=} \begin{cases} \mathbf{m}_{Vx_0} & \text{if } m = n \\ \mathbf{m}_{Hx_0} & \text{if } m > n. \end{cases}$$

Definition 1.2. We say that a matrix $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$ is *k-primitive*, where k is a positive integer, if the reduction modulo k of its columns span $(\mathbb{Z}/k\mathbb{Z})^n$.

Consider the finite[1] set

$$\widetilde{\mathcal{P}}_k \overset{\text{def}}{=} \{(x_{k^{-1}\mathbf{u}}, x_{k^{-1}\mathbf{u}}) : \mathbf{u} \text{ is } k\text{-primitive}\} \subset Ux_0 \times Ux_0, \tag{1.1}$$

and let $\widetilde{\boldsymbol{\mu}}_k$ denote the normalized counting measure on $\widetilde{\mathcal{P}}_k$; i.e.

$$\widetilde{\boldsymbol{\mu}}_k \overset{\text{def}}{=} \frac{1}{|\widetilde{\mathcal{P}}_k|} \sum_{(x,x)\in\widetilde{\mathcal{P}}_k} \delta_{(x,x)}. \tag{1.2}$$

Finally, let us denote $\widetilde{a}(k) = (e, a(k)) \in G \times G$. With these notations we can state our main result.

Theorem 1.3. *As $k \to \infty$, $\widetilde{a}(k)_* \widetilde{\boldsymbol{\mu}}_k \overset{\mathrm{w}^*}{\to} \mathbf{m}_{Ux_0} \times \boldsymbol{\theta}$.*

We remark that our main tool is [Sha98] which extends the fundamental work of Ratner [Ra91]. We also note that some of our arguments are similar to [MS95].

[1] Note that for two integer matrices, if $\mathbf{u}_1 = \mathbf{u}_2 \mod k$ then $x_{k^{-1}\mathbf{u}_1} = x_{k^{-1}\mathbf{u}_2}$.

1.2. **An application.** Theorem 1.3 is a generalization of a result by Marklof [Mar10, Theorem 6]. Marklof's result has various applications, most notably to the distribution of Frobenius numbers, circulant graphs and the shapes of co-dimension 1 primitive subgroups of $\mathbb{Z}^d$ (see [Mar10], [MS]). Naturally, in each of these discussions an application of Theorem 1.3 gives new results. We give one such application which is an analogue of a certain equidistribution result of Schmidt [Sch98]. We follow closely the viewpoint of [Mar10].

Assume that $n = 1, m \geq 2$ so that $d = m + 1 \geq 3$. The quotient $Z_m \overset{\text{def}}{=} \mathrm{SO}_m(\mathbb{R})\backslash X_m$ will be referred to as the *space of shapes of m-dimensional lattices*. We equip Z_m with the probability measure $\mathbf{m}_{Z_m}$ which is by definition the image of $\mathbf{m}_{X_m}$ under the natural projection.

For any integer vector $v \in \mathbb{Z}^d$ let us denote by $\Lambda_v \overset{\text{def}}{=} \mathbb{Z}^d \cap \{v\}^\perp$; that is Λ_v is the lattice of integer points in the m-dimensional orthocomplement of v in $\mathbb{R}^d$. We may choose a matrix $k_v \in \mathrm{SO}_d(\mathbb{R})$ so that $k_v\Lambda_v \subset \mathbb{R}^m \times \{0\} \subset \mathbb{R}^d$, and so that $k_v v$ lies on the positive half of the d-th coordinate axis. After normalizing the covolume of $k_v\Lambda_v$ to be 1, we obtain a point in X_m. As the choice of k_v is only well defined up to the action of $\mathrm{SO}_m(\mathbb{R})$ we obtain a well defined point in Z_m which we denote hereafter by $[\Lambda_v]$. There is a certain redundancy in considering Λ_v if $v \in \mathbb{Z}^d$ is non-primitive (that is if it is an integer multiple of another integer vector). We therefore denote by $\widehat{\mathbb{Z}}^d$ the subset of *primitive* integer vectors. Note that a vector $(u_1, \ldots, u_m, k) \in \mathbb{Z}^d$ is primitive if and only if the vector $(u_1, \ldots, u_m) \in \mathbb{Z}^m$ is k-primitive as in Definition 1.2. Finally, let $B_\infty \overset{\text{def}}{=} \left\{x \in \mathbb{R}^d : \|x\|_\infty \leq 1\right\}$ and ∂B_∞ denote its boundary. As an application of Theorem 1.3 we prove the following result in §8.

Theorem 1.4. *Let $\mathcal{F} \subset \partial B_\infty$ be a measurable set such that its boundary in ∂B_∞ has measure 0 with respect to the m-dimensional Lebesgue measure on ∂B_∞. For any positive integer k let*

$$\eta_k \overset{\text{def}}{=} \frac{1}{|\widehat{\mathbb{Z}}^d \cap k\mathcal{F}|} \sum_{v \in \widehat{\mathbb{Z}}^d \cap k\mathcal{F}} \delta_{[\Lambda_v]}.$$

Then, $\eta_k \overset{\mathrm{w}^*}{\to} \mathbf{m}_{Z_m}$ as $k \to \infty$.

Note that the choice of the ℓ_2-norm yields a much more elegant statement as we have the following relation

$$\left\{\Lambda_v : v \in \widehat{\mathbb{Z}}^d, \|v\|_2 = k\right\} = \left\{\begin{smallmatrix}\text{Primitive } m\text{-dimensional subgroups} \\ \text{of } \mathbb{Z}^d \text{ of covolume } = k\end{smallmatrix}\right\}$$

We suggest the following.[2]

Conjecture 1.5. Let $d \geq 3$ and let $B_2(r)$ denote the Euclidean ball of radius $r > 0$ in $\mathbb{R}^d$. Then if $r_n \to \infty$ is a sequence of radii such that $|\partial B_2(r_n) \cap \widehat{\mathbb{Z}}^d| \to \infty$, then the collections $\left\{[\Lambda_v] : v \in \partial B_2(r_n) \cap \widehat{\mathbb{Z}}^d\right\}$ equidistribute in Z_m.

2. Outline and initial steps

2.1. **A motivating low dimensional example.** We now prove Theorem 1.3 in the case $n = m = 1$ to which the techniques of the later parts of this paper do not apply. We learned the argument we give here from Jens Marklof and to the best of our understanding this result was his reason to anticipate the validity of Theorem 1.3.

We identify both of the two orbits $Ux_0 = \{\left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) x_0 : s \in \mathbb{R}\}$ and $Vx_0 = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) x_0 : s \in \mathbb{R}\}$ with $\mathbb{R}/\mathbb{Z}$ and use the parameters s, t to describe them respectively. We wish to analyze for which values of y

$$a(y)Ux_0 \cap Vx_0 \neq \varnothing \tag{2.1}$$

and in case this happens, we wish to understand the asymptotics of the joint distribution of the set

$$\left\{(s,t) \in (\mathbb{R}/\mathbb{Z})^2 : a(y) \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) x_0 = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) x_0\right\}. \tag{2.2}$$

Following the definitions one sees that (s,t) is in the set in (2.2) if and only if there exists $\left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ which solve the equation

$$\left(\begin{smallmatrix} y^{-1} & 0 \\ ys & y \end{smallmatrix}\right) \left(\begin{smallmatrix} k & \ell \\ m & n \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right). \tag{2.3}$$

Equation (2.3) has a solution if and only if the following three conditions on the variables y, s, t are satisfied (note that we may assume that $s, t \in (0, 1]$): (i) $y = k$ is a positive integer, (ii) $t = \frac{\ell}{k}$ for some $1 \leq \ell \leq k$ with $\gcd(\ell, k) = 1$, and (iii) $s = \frac{\ell^*}{k}$, where ℓ^* is the number $1 \leq \ell^* \leq k$ satisfying $\ell \cdot \ell^* = 1 \bmod k$.

To summarize, if (2.1) holds then $y = k$ for some positive integer k and the set in (2.2) equals

$$\left\{(\ell/k, \ell^*/k) \in (\mathbb{R}/\mathbb{Z})^2 : 1 \leq \ell \leq k, \gcd(\ell, k) = 1\right\}. \tag{2.4}$$

Taking into account both descriptions (2.2), (2.4) and our identification $(\mathbb{R}/\mathbb{Z})^2 \cong Ux_0 \times Vx_0$ we see that this set is exactly $\widetilde{a}(k)\widetilde{\mathcal{P}}_k$ which appears as the support of the measure in Theorem 1.3. If we denote the normalized counting measure on (2.4) by θ_k then the statement of Theorem 1.3 is interpreted as saying that θ_k equidistributes in $(\mathbb{R}/\mathbb{Z})^2$

[2]Significant progress towards the suggested conjecture was obtained recently in [AES], [AESb].

as $k \to \infty$. This equidistribution statement translates to estimating Kloosterman sums

$$K(a,b,k) = \sum_{\substack{1\le x\le k\\ \gcd(x,k)=1}} e^{2\pi i(ax+bx^*)} = \phi(k)\int_{(\mathbb{R}/\mathbb{Z})^2} e^{2\pi i(at+bs)} d\theta_k, \tag{2.5}$$

where $\phi(k)$ is the Euler function. The known estimates for Kloosterman sums (see for example [Iwa02, p. 48 eq. (2.25)]), imply that for any choice of $(a,b) \in \mathbb{Z}^2$ not both 0, $\phi(k)^{-1}K(a,b,k) \to 0$ as $k \to \infty$ which establishes the desired equidistribution of the θ_k's.

This establishes the case $m = n = 1$ in Theorem 1.3. The main objective of this paper is to establish the case $m \ge 2$ using techniques from homogeneous dynamics.

2.2. **A basic observation and the structure of the proof.** Let us define similarly to (1.1), (1.2),

$$\mathcal{P}_k \stackrel{\text{def}}{=} \{x_{k^{-1}\mathbf{u}} : \mathbf{u} \text{ is } k\text{-primitive}\} \subset Ux_0 \subset X_d, \tag{2.6}$$

$$\boldsymbol{\mu}_k \stackrel{\text{def}}{=} \frac{1}{|\mathcal{P}_k|}\sum_{x\in\mathcal{P}_k} \delta_x. \tag{2.7}$$

Using the fact that the $a(y)$-action is mixing on X_d one can show that that the pushed periodic orbit $a(y)Ux_0$ equidistributes in X_d when $y \to \infty$, as U is the expanding horospherical subgroup of $a(y)$ (see e.g. [Mar04]). On the other hand, the collection $\mathcal{P}_k \subset Ux_0$ is composed of rational points x for which the trajectories $\{a(y)x\}_{y>1}$ are divergent in X_d. It is therefore natural to investigate the tension between these two facts and to analyze the distribution of $a(y_k)\mathcal{P}_k$ in X_d for various choices of sequences $y_k \to \infty$. Setting $y_k = k$ as in Theorem 1.3 is natural because of the following lemma (which is in some sense the starting point of our discussion).[3]

Lemma 2.1 (Basic Lemma)**.** *For any positive integer k, $a(k)\mathcal{P}_k \subset Hx_0$.*

We prove Lemma 2.1 towards the end of this section. It shows that we cannot expect the sequence $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_k$ to equidistribute in $Ux_0 \times X_d$ as any limit point of this sequence is clearly a measure[4] supported in $Ux_0 \times Hx_0$.

The space $X_m = \mathrm{SL}_m(\mathbb{R})/\mathrm{SL}_m(\mathbb{Z})$ is naturally embedded in $Hx_0 \subset X_d$, simply by identifying $\mathrm{SL}_m(\mathbb{R})$ as a subgroup of H in the obvious way. Throughout we alternate between thinking of X_m as a subset of

[3]See Remark 2.5 for other natural choices of y_k's.

[4]Apriori the limit measures is not even known to be a probability measure.

X_d and as the space of m-dimensional unimodular lattices in $\mathbb{R}^m$. When thinking of X_m as the space of m-dimensional lattices, the identity coset x_0 corresponds to the lattice $\mathbb{Z}^m$. There is a natural projection $\pi_3 : Hx_0 \to X_m$ defined as follows:

$$\text{For } x = \left(\begin{smallmatrix} h & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0 \in Hx_0,\ \pi_3(x) \overset{\text{def}}{=} \left(\begin{smallmatrix} h & 0 \\ 0 & I \end{smallmatrix}\right) \mathrm{SL}_d(\mathbb{Z})$$

which corresponds to the m-dimensional unimodular lattice $h\mathbb{Z}^m$. Let

$$\boldsymbol{\nu}_k \overset{\text{def}}{=} (\pi_3)_* a(k)_* \boldsymbol{\mu}_k, \tag{2.8}$$

and consider the diagram with natural projection maps

$$\begin{tikzcd}[column sep=small]
 & (Ux_0 \times Hx_0, \widetilde{a}(k)_* \widetilde{\boldsymbol{\mu}}_k) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
(Ux_0, \boldsymbol{\mu}_k) & & (Hx_0, a(k)_* \boldsymbol{\mu}_k) \arrow[d, "\pi_3"] \\
 & & (X_m, \boldsymbol{\nu}_k)
\end{tikzcd} \tag{2.9}$$

We describe the structure of the proof of Theorem 1.3 using the diagram (2.9):

Step 1. Establish the convergence $\boldsymbol{\mu}_k \overset{\text{w}^*}{\to} \mathbf{m}_{Ux_0}$, which takes place on the left side of diagram (2.9).

Step 2. Show the convergence $\boldsymbol{\nu}_k \overset{\text{w}^*}{\to} (\pi_3)_*\boldsymbol{\theta}$, which equals $\mathbf{m}_{X_m}$ in case $m > n$, and the dirac measure δ_{x_0} in case $m = n$. This convergence takes place on the right side at the bottom of diagram (2.9).

Step 3. Use the second step to establish the convergence $a(k)_*\boldsymbol{\mu}_k \overset{\text{w}^*}{\to} \boldsymbol{\theta}$ taking place on the right side of diagram (2.9). Many of the ideas appearing in the argument of Step 3 already appear in simplified versions in the proof of Step 1.

Step 4. Combine Step 1 and Step 3 and use a disjointness argument to prove the convergence $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_k \overset{\text{w}^*}{\to} \mathbf{m}_{Ux_0} \times \boldsymbol{\theta}$ taking place at the top of diagram (2.9).

2.3. **The method of proof.** From this point and on we will assume that $m \geq 2$. As described above in the course of the proof of Theorem 1.3 we will frequently need to establish a convergence $\eta_n \overset{\text{w}^*}{\to} \eta$ of probability measures. It will turn out that all the measures involved are Λ-invariant under natural actions of the group[5]

$$\Lambda \overset{\text{def}}{=} \left\{ \left(\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right) : \delta_1 \in \mathrm{SL}_m(\mathbb{Z}), \delta_2 \in \mathrm{SL}_n(\mathbb{Z}) \right\}.$$

The following is the strategy we will use for proving such a claim: One starts by *classifying* the Λ-ergodic measures and shows that there are only countably many such. Say $\{\sigma_i\}_{i=0}^{\infty}$, with $\eta = \sigma_0$. Then, given an accumulation point η' of η_n, the ergodic decomposition of η' is given by $\eta' = \sum_{i=0}^{\infty} c_i \sigma_i$ with $c_i \geq 0$ and $\sum c_i = 1$. One then appeals to a *non-accumulation* result to show that the only possibility of c_i being positive is that $\mathrm{supp}(\eta_n) \subset \mathrm{supp}(\sigma_i)$ for infinitely many n's. One then verifies that the only i for which such an inclusion is possible is $i = 0$.

The above strategy is built out of (i) a non-accumulation result and (ii) a measure classification result. In §3 we prove the non-accumulation result. In §4 we prove various measure classification results. In §6 we use the above strategy to prove Theorem 1.3 through Steps 1– 4 which are described after (2.9).

2.4. **Elementary divisors.** In this subsection we collect some further preliminaries that will be needed throughout the paper. We shall use the following theorem, the proof of which can be found in [SD01, Appendix Lemma A2, Theorem A1].

Theorem 2.2 (Elementary divisors)**.** *Let $\Sigma \neq \{0\}$ be a subgroup of $\mathbb{Z}^m$. Then, there is an integer $1 \leq r \leq m$ and positive integers $\ell_1 \dots \ell_r$ such that $\ell_i | \ell_{i+1}$ and such that one can find a basis $v_1 \dots v_m$ of $\mathbb{Z}^m$ for which $\ell_1 v_1, \dots, \ell_r v_r$ form a basis for Σ. Furthermore, the numbers r, ℓ_i with this property are unique and are called the rank and the elementary divisors of Σ with respect to $\mathbb{Z}^m$ respectively.*

The following restatement of Theorem 2.2 will be more convenient to us.

Lemma 2.3. *Consider the action of $\left(\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right) \in \Lambda$ on $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$ given by $\left(\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{smallmatrix}\right) \mathbf{u} = \delta_2 \mathbf{u} \delta_1^{-1}$. Each Λ-orbit in $\mathrm{Mat}_{n\times m}(\mathbb{Z})$ contains an element of the form*

$$\mathbf{u} = \begin{pmatrix} \ell_1 & & 0 & \dots & 0 \\ 0 & \ddots & 0 & \dots & 0 \\ 0 & & \ell_n & \dots & 0 \end{pmatrix} \tag{2.10}$$

such that the ℓ_i's are integers satisfying $\ell_1 | \ell_2 | \dots | \ell_n$. Moreover, the integers ℓ_i are unique up to sign.

[5] Note that the assumption that $m \geq 2$ is equivalent to Λ being non-trivial.

Proof. The lemma follows from an application of Theorem 2.2 to the group Σ generated by the rows of $\mathbf{u}$ in $\mathbb{Z}^m$. Note that if $\mathrm{rank}(\Sigma) = r < n$ then the tuple $(\ell_1 \dots \ell_n)$ ends with $n-r$ zeros. Note also that the only case where we cannot require that $\ell_i \geq 0$ is when $n = m$ and $\mathbf{u}$ is an invertible matrix with negative determinant. □

Let us refer to the tuple $(\ell_1, \dots, \ell_n)$ attached to the Λ-orbit of $\mathbf{u}$ as its *elementary divisors tuple*.

As Definition 1.2 of k-primitivity is invariant under the Λ-action, it is clear that $\mathbf{u}$ is k-primitive if and only if ℓ_n is coprime to k. For the proof of Lemma 2.1 we will use the following characterization of k-primitivity.

Lemma 2.4. *The matrix $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$ is k-primitive if and only if there exists a matrix $\gamma \in \mathrm{SL}_d(\mathbb{Z})$ whose bottom n rows coincide with the $n \times d$ matrix $\begin{pmatrix}\mathbf{u} & kI_n\end{pmatrix}$.*

Proof. It is straightforward to show that the property described in the statement of the lemma is also invariant under the action of Λ. It follows that it is enough to verify the validity of the statement for matrices $\mathbf{u}$ in the form (2.10). Clearly if the elementary divisor ℓ_n is not coprime to k, i.e. if $\mathbf{u}$ is not k-primitive, then the bottom row of $\begin{pmatrix}\mathbf{u} & kI_n\end{pmatrix}$ is not a primitive vector in $\mathbb{Z}^d$ and so the existence of γ as in the statement is ruled out. If on the other hand $\mathbf{u}$ is k-primitive then the elementary divisors ℓ_i are all coprime to k. It follows that there are integers e_i, f_i so that $\det\begin{pmatrix} e_i & f_i \\ \ell_i & k\end{pmatrix} = 1$. The determinant of the $d \times d$ matrix

$$\gamma \stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathrm{diag}\,(e_1 \dots e_n) & 0 & \mathrm{diag}\,(f_1 \dots f_n) \\ 0 & I_{n-m} & 0 \\ \mathrm{diag}\,(\ell_1 \dots \ell_n) & 0 & \mathrm{diag}\,(k \dots k) \end{pmatrix} \tag{2.11}$$

equals $\prod_1^n \det\begin{pmatrix} e_i & f_i \\ \ell_i & k\end{pmatrix} = 1$. □

2.5. Proof of the Basic Lemma.

Proof of Lemma 2.1. Note that in order to show that a point $x = gx_0 \in X_d$ lies in Hx_0, one needs to show that there exists $\gamma \in \Gamma$ such that $g\gamma \in H$. Let $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{Z})$ be k-primitive and $a(k)x_{k^{-1}\mathbf{u}} \in a(k)\mathcal{P}_k$ be the corresponding point. By Lemma 2.4 $\mathbf{u}$ is k-primitive if and only if there exists a matrix $\gamma^{-1} \in \Gamma$ whose bottom n rows are given by the rows of the $n \times d$ matrix $(\mathbf{u}\; kI_n)$. It follows that if we denote by

$A_\gamma, B_\gamma, C_\gamma, D_\gamma$ the block components of γ, then

$$a(k)\begin{pmatrix} I & 0 \\ k^{-1}\mathbf{u} & I \end{pmatrix}\gamma = \begin{pmatrix} k^{-\frac{n}{m}}I & 0 \\ \mathbf{u} & kI \end{pmatrix}\begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} = \begin{pmatrix} k^{-\frac{n}{m}}A_\gamma & k^{-\frac{n}{m}}B_\gamma \\ 0 & I \end{pmatrix}. \quad (2.12)$$

The above equation shows that the point $a(k)x_{k^{-1}\mathbf{u}}$ belongs to Hx_0 as desired. $\square$

Equation (2.12) (which is an analogouse to (2.3)), is fundamental for our discussion and deserves some attention. We note two things:

(1) Let $\mathbf{u}$ be k-primitive and suppose γ solves (2.12). By considering determinants we see that $A_\gamma \in \mathrm{Mat}_m(\mathbb{Z})$ must have determinant k^n. This means that when considered as a lattice in $\mathbb{R}^m$, $\pi_3(a(k)x_{k^{-1}\mathbf{u}})$ equals $k^{-\frac{n}{m}}A_\gamma\mathbb{Z}^m$, and up to homothety equals $A_\gamma\mathbb{Z}^m$ which is a subgroup of index k^n of $\mathbb{Z}^m$. In Lemma 6.1 we show that the collection $\{\pi_3(a(k)x_{k^{-1}\mathbf{u}}) : \mathbf{u} \text{ is } k\text{-primitive}\}$ consists of all such lattices of a given *Hecke-type* (see Definition 5.1 for terminology).

(2) Assume again that $\mathbf{u}$ is k-primitive and that γ solves (2.12). The first m columns of γ form a basis for the discrete group of rank m which we denote by $\Lambda_{(\mathbf{u},k)}$, consisting of integer vectors in the orthocomplement of the linear space spanned by the rows of the $n \times d$ matrix $\begin{pmatrix}\mathbf{u} & kI\end{pmatrix}$. It follows that $\pi_3(a(k)x_{k^{-1}\mathbf{u}})$, as a lattice in $\mathbb{R}^m$, is (up to homothety) the projection of $\Lambda_{(\mathbf{u},k)}$ onto the copy of $\mathbb{R}^m$ given by the first m-coordinates. This is what furnishes the link with Schmidt's theorem and its strengthening given in Theorem 1.4.

Remark 2.5. Let $y_k \to \infty$ be given. By Lemma 2.1 $a(y_k)\mathcal{P}_k = a(y_k/k)a(k)\mathcal{P}_k \subset a(y_k/k)Hx_0$. Thus, if we set $y_k = k^\alpha$ for some positive α, then we obtain that $a(k^\alpha)\mathcal{P}_k \subset a(k^{\alpha-1})Hx_0$. In the case $\alpha > 1$, the collection $a(k^\alpha)\mathcal{P}_k$ is therefore contained in the uniformly divergent periodic orbit $a(k^{\alpha-1})Hx_0$, and in particular, the sequence of measures $a(k^\alpha)_*\boldsymbol{\mu}_k$ converges to the zero measure on X_d. In case $\alpha < 1$, the collection $a(k^\alpha)\mathcal{P}_k$ is therefore contained in the equidistributing periodic orbit $a(k^{\alpha-1})Hx_0$. It turns out that one can prove an analogue of Theorem 1.3 in this context and show that for $0 < \alpha < 1$, $\widetilde{a}(k^\alpha)_*\widetilde{\boldsymbol{\mu}}_k \overset{\mathrm{w}^*}{\to} \mathbf{m}_{Ux_0} \times \mathbf{m}_{X_d}$ (for any $m \geq n \geq 1$ including the case $n = m = 1$). This analysis is non-trivial and we plan on elaborating on this in a future manuscript.

3. Non-accumulation

Our goal in this section is to prove a certain non-accumulation result – Theorem 3.1 – that will be used in various steps in the proof of

Theorem 1.3. Although we do not aim at greatest generality, we still choose to state and prove the results in a somewhat abstract setting (if only to isolate the necessary features that are needed for the result to hold). To this end, in this section (and in it only) we abandon the notation presented so far and assume the following: Let G be a real Lie group, let $\Gamma < G$ be a lattice, and let $\Lambda, L < G$ be closed subgroups with Λ being discrete and generated by finitely many Ad-unipotent elements. Assume furthermore that there is a decomposition

$$\mathrm{Lie}(G) = \mathrm{Lie}(L) \oplus W \tag{3.1}$$

such that W is invariant under the action of Λ via the adjoint representation. Let $X = G/\Gamma$ and $z \in X$ be a point such that the orbit Lz is periodic and Λ-invariant.

Theorem 3.1 (Non-accumulation)**.** *Let $G, \Gamma, \Lambda, L, z, W$ and X be as above and assume that the Λ-representation on W does not contain any fixed vectors. Let $P_k \subset X$ be a sequence of finite Λ-invariant sets and μ_k the normalized counting measure on P_k. If $P_k \cap Lz = \varnothing$ for all k then any weak* accumulation point σ of $\{\mu_k\}_{k=1}^{\infty}$ satisfies $\sigma(Lz) = 0$.*

Below we use the absolute value symbol $|\cdot|$ to denote the usual absolute value of a real number as well as the Lebesgue measure of a set and the cardinality of a finite set. This should not cause any confusion. In the proof of Theorem 3.1 we will need to use some elementary properties of polynomials which we now recall. The following lemma may be found at [KSS02, Proposition 3.2.2].

Lemma 3.2. *For any degree d there exist a constant $c_d > 0$ such that for any polynomial $p : \mathbb{R} \to \mathbb{R}$ of degree bounded by d, for any interval $I \subset \mathbb{R}$ we have that if $\rho = \max\{|p(x)| : x \in I\}$ then for any $0 < \epsilon \le \rho$*

$$\frac{|\{x \in I : |p(x)| \le \epsilon\}|}{|I|} \le c_d \left(\frac{\epsilon}{\rho}\right)^{\frac{1}{d}}. \tag{3.2}$$

We deduce the following integer value version of this lemma.

Lemma 3.3. *For any degree d there exist a constant $c_d > 0$ such that for any polynomial $p : \mathbb{N} \to \mathbb{R}$ of degree bounded by d, for any interval $J \subset \mathbb{N}$ we have that if $\rho = \max\{|p(n)| : n \in J\}$ then for any $0 < \epsilon \le \rho$*

$$\frac{|\{n \in J : |p(n)| \le \epsilon\}|}{|J|} \le c_d \left(\frac{\epsilon}{\rho}\right)^{\frac{1}{d}} + \frac{d}{|J|}. \tag{3.3}$$

Proof. Let $I \subset \mathbb{R}$ denote the real interval defined as the convex hull of J. For $\epsilon > 0$ let $I_\epsilon \overset{\mathrm{def}}{=} \{x \in I : |p(x)| \le \epsilon\}$ and $J_\epsilon \overset{\mathrm{def}}{=} \{n \in J : |p(n)| \le \epsilon\}$.

The set I_ϵ is a disjoint union of finitely many closed intervals $I_{\epsilon,\ell}$ for $\ell = 1...k$. We have that

$$\begin{aligned}|J_\epsilon| =&|I_\epsilon \cap \mathbb{Z}| = \sum_\ell |I_{\epsilon,\ell} \cap \mathbb{Z}| \\ =& \sum_{\ell:|I_{\epsilon,\ell}\cap\mathbb{Z}|=1} 1 + \sum_{\ell:|I_{\epsilon,\ell}\cap\mathbb{Z}|>1} |I_{\epsilon,\ell} \cap \mathbb{Z}| \\ \leq& d + \sum_{\ell:|I_{\epsilon,\ell}\cap\mathbb{Z}|>1} 2|I_{\epsilon,\ell}| \leq d + 2I_\epsilon,\end{aligned}$$

where we used that if $I_{\epsilon,\ell}$ contains more than one integer then

$$|I_{\epsilon,\ell} \cap \mathbb{Z}| \leq 2|I_{\epsilon,\ell}|,$$

and also, the number of ℓ's for which $I_{\epsilon,\ell}$ contains a single integer is bounded by the degree d of the polynomial because between each such two intervals there must be a zero of the derivative.

The inequality (3.3) now follows from (3.2) (with a slightly bigger constant c_d). □

Proof of Theorem 3.1. Let σ be a weak* accumulation point as in the statement. Let $\Omega_1 \subset X$ be a compact set. We will show that for $K_1 \overset{\text{def}}{=} \Omega_1 \cap Lz$ one has $\sigma(K_1) = 0$. This is enough as Lz is a countable union of such sets. Choose some norm on $\mathrm{Lie}(G)$ and denote by B^W_ϵ the ball of radius ϵ around 0 in W. Choose an open set $\widetilde{\Omega}_1$ and a compact set Ω_2 such that $\Omega_1 \subset \widetilde{\Omega}_1 \subset \Omega_2$ so that $\widetilde{\Omega}_1 \cap \big(\exp B^W_1 \cdot (X \smallsetminus \Omega_2)\big) = \varnothing$; in other words, Ω_2 is big enough so that one cannot reach $\widetilde{\Omega}_1$ by acting on points outside of Ω_2 by elements of the form $\exp w$, where $w \in W$ is of norm ≤ 1. Let $\widetilde{K}_1 \overset{\text{def}}{=} \widetilde{\Omega}_1 \cap Lz$ and $K_2 \overset{\text{def}}{=} \Omega_2 \cap Lz$ and for any $\epsilon > 0$ any subset $F \subset Lz$ denote $\mathcal{T}(\epsilon, F) \overset{\text{def}}{=} \{\exp(w)x : x \in F, w \in W, \text{ and } \|w\| \leq \epsilon\}$ the *ϵ-tube around F*.

There exist $0 < \epsilon_0 < 1/2$ small enough so that the map $(w, x) \mapsto \exp(w)x$ from $B^W_{\epsilon_0} \times K_2 \to \mathcal{T}(\epsilon_0, K_2)$ is a homeomorphism onto its image. This gives a natural coordinate system on the ϵ_0-tube around K_2; we denote for $y \in \mathcal{T}(\epsilon_0, K_2)$ by $w_y \in B^W_{\epsilon_0}$ the *W-coordinate* and by $x_y \in Lz$ the *orbit-coortinate* so that the identity $y = \exp(w_y)x_y$ holds for $y \in \mathcal{T}(\epsilon_0, K_2)$.

Let $u_1, \dots, u_r \in \Lambda$ be Ad-unipotent elements that generate Λ. Let $0 \leq \epsilon < \epsilon_0$ and set

$$\mathcal{S}_j(\epsilon, \widetilde{K}_1) \overset{\text{def}}{=} \left\{ y \in \mathcal{T}(\epsilon, \widetilde{K}_1) : \mathrm{Ad}_{u_j}(w_y) \neq w_y \right\},$$

$j = 1 \dots r$, so that $\mathcal{T}(\epsilon, \widetilde{K}_1) \smallsetminus \widetilde{K}_1 = \cup_1^r \mathcal{S}_j(\epsilon, \widetilde{K}_1)$. The inclusion $\supset$ is clear. The other inclusion holds because Λ is generated by the u_j's and W contains no Λ-fixed non-zero vectors. We will find a function $\psi(\epsilon) \to_{\epsilon \to 0} 0$ such that for any j

$$\frac{|P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)|}{|P_k|} \leq \psi(\epsilon) \quad \text{for any } k \ . \tag{3.4}$$

Since $P_k \cap Lz = \varnothing$ this implies that

$$\frac{|P_k \cap \mathcal{T}(\epsilon, \widetilde{K}_1)|}{|P_k|} \leq \sum_1^r \frac{|P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)|}{|P_k|} \leq r\psi(\epsilon) \to_{\epsilon \to 0} 0.$$

In turn, this implies that $\sigma(K_1) = 0$ as desired because $\mathcal{T}(\epsilon, \widetilde{K}_1)$ is an open set containing the compact set K_1.

To this end, fix $1 \leq j \leq r$ and denote for each $y \in \mathcal{S}_j(\epsilon, \widetilde{K}_1)$,

$$p_y(n) \stackrel{\text{def}}{=} \left\| \mathrm{Ad}_{u_j}^n(w_y) \right\|^2 .$$

Note that as u_j is Ad-unipotent it follows that (for an appropriate choice of a norm $\|\cdot\|$), $p_y(n)$ is a non-constant polynomial in n of degree $\leq d$ for some integer d depending on $\dim G$ only. Let us use the following notation

- $n_y \stackrel{\text{def}}{=} \max\{n \geq 0 : p_y(k) \leq \epsilon_0^2 \text{ for } 0 \leq k \leq n\}$.
- $J_y \stackrel{\text{def}}{=} [0, n_y] \cap \mathbb{Z}$.
- $V_y \stackrel{\text{def}}{=} \left\{u_j^n y : n \in J_y\right\}$.
- $M \stackrel{\text{def}}{=} \max_{j=1\dots r} \left\| \mathrm{Ad}_{u_j}^{\pm 1} \right\|$.

Observe that n_y is finite as $p_y(n)$ is non-constant. We will shortly show that the following properties hold for $y, y_1, y_2 \in \mathcal{S}_j(\epsilon, \widetilde{K}_1)$.

(1) $V_{y_1} \cap V_{y_2} \neq \varnothing \Longrightarrow V_{y_1} \subset V_{y_2}$ or $V_{y_2} \subset V_{y_1}$.
(2) $V_y \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1) \subset \left\{u_j^n y : p_y(n) \leq \epsilon^2\right\}$.
(3) $p_y(n_y) \geq (\frac{\epsilon_0}{M})^2$.
(4) $|J_y| \geq \log(\frac{\epsilon_0}{\epsilon}) / \log M$.

We now conclude the proof using properties (1)-(4). Given any k, we choose a finite collection $\{y_i\} \subset P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)$ so that the V_{y_i}'s are maximal with respect to inclusion among $\left\{V_y : y \in P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)\right\}$, and such that $P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1) \subset \bigcup_i V_{y_i}$. By property (1) we deduce that the V_{y_i}'s are disjoint. Since $V_{y_i} \subset P_k$, we deduce that $\sum_i \frac{|J_{y_i}|}{|P_k|} \leq 1$. It follows that

$$\frac{|P_k \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)|}{|P_k|} = \frac{|\cup_i V_{y_i} \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)|}{|P_k|} \leq \sum_i \frac{|J_{y_i}|}{|P_k|} \cdot \frac{|V_{y_i} \cap \mathcal{S}_j(\epsilon, \widetilde{K}_1)|}{|J_{y_i}|}$$

$$\stackrel{\text{by (2)}}{\leq} \sum_i \frac{|J_{y_i}|}{|P_k|} \cdot \frac{|\{n \in J_{y_i} : p_{y_i}(n) \leq \epsilon^2\}|}{|J_{y_i}|}$$

$$\stackrel{\text{by (3),(4) and (3.3)}}{\leq} c_d \left(\frac{\epsilon}{\epsilon_0/M}\right)^{2/d} + \frac{d}{\log(\frac{\epsilon_0}{\epsilon})/\log M}.$$

As this last expression goes to 0 as $\epsilon \to 0$ we conclude that (3.4) holds with this last expression as $\psi(\epsilon)$. It remains to verify the validity of (1)-(4).

(1). Assume $y_1, y_2 \in \mathcal{S}_j(\epsilon, K_1)$ are such that $V_{y_1} \cap V_{y_2} \neq \varnothing$; that is, there exists $n_i \in J_{y_i}$ such that $u_j^{n_1} y_1 = u_j^{n_2} y_2$. Assume without loss of generality that $n_2 \leq n_1$ and so $u_j^{n_1 - n_2} y_1 = y_2$ and $n_1 - n_2 \in J_{y_1}$. Following the definitions we see that $V_{y_2} \subset V_{y_1}$ as desired.

(2). Let $y \in \mathcal{S}_j(\epsilon, \widetilde{K}_1)$ and assume $n \in J_y$ is such that $p_y(n) > \epsilon^2$ so that we know that $\epsilon < \left\| \mathrm{Ad}^n_{u_j}(w_y) \right\| \leq \epsilon_0$. We need to show that $u_j^n y \notin \mathcal{S}_j(\epsilon, \widetilde{K}_1)$. We have that

$$u_j^n y = u_j^n \exp(w_y) u_j^{-n} u_j^n x_y = \exp\left(\mathrm{Ad}^n_{u_j}(w_y)\right) u_j^n x_y. \tag{3.5}$$

If $u_j^n x_y \notin \Omega_2$ then (3.5) implies that $u_j^n y \in \exp B^W_{\epsilon_0}(X \smallsetminus \Omega_2)$ which is disjoint from $\mathcal{T}(\epsilon_0, \widetilde{K}_1)$ by choice of Ω_2. So in particular, $u_j^n y \notin \mathcal{S}_j(\epsilon, \widetilde{K}_1)$. If on the other hand $u_j^n x_y \in \Omega_2$, then as $u_j^n x_y \in Lz$ (because Lz is Λ-invariant), we deduce that $u_j^n x_y \in K_2$ which in turn implies by (3.5) that $u_j^n y \in \mathcal{T}(\epsilon_0, K_2)$ and the orbit and W coordinates of $u_j^n y$ are given by $\mathrm{Ad}^n_{u_j}(w_y)$ and $u_j^n x_y$ respectively. By the lower bound on the W-coordinate we deduce that $u_j^n y \notin \mathcal{T}(\epsilon, K_2)$ and in particular, $u_j^n y \notin \mathcal{S}_j(\epsilon, \widetilde{K}_1)$.

(3). We have that $\epsilon_0 \leq \left\| \mathrm{Ad}^{n_y+1}_{u_j}(w_y) \right\| \leq M\sqrt{p_y(n_y)}$.

(4). Similarly $\epsilon_0 \leq \left\| \mathrm{Ad}^{n_y+1}_{u_j}(w_y) \right\| \leq M^{n_y+1}\epsilon$.

□

Corollary 3.4. *The conclusion of Theorem 3.1 remains valid if the assumption $P_k \cap Lz = \varnothing$ is relaxed to $\frac{|P_k \cap Lz|}{|P_k|} \to 0$ as $k \to \infty$.*

Proof. We split $\mu_k = (1-\alpha_k)\mu_k^1 + \alpha_k \mu_k^2$ with μ_k^i, $i = 1, 2$ being the normalized counting measures on $P_k \smallsetminus Lz$ and $P_k \cap Lz$ respectively. In this case $\alpha_k = \frac{|P_k \cap Lz|}{|P_k|} \to 0$ as $k \to \infty$ by assumption and so the accumulation points of μ_k are the same as those of μ_k^1 for which Theorem 3.1 applies. □

4. Λ-invariance and measure classifications

In this section we will show that all the measures appearing in our discussions are invariant under a certain group Λ. We will then classify all the Λ-invariant and ergodic probability measures in certain situations. This will serve us in the proof of Theorem 1.3 along the lines described in §2.3.

4.1. **Invariance.** We return to use the notation introduced in §1, §2. In particular, recall that the subgroup $\Lambda \simeq \mathrm{SL}_m(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_d(\mathbb{Z})$ is defined by

$$\Lambda \overset{\text{def}}{=} \left\{ \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} : \delta_1 \in \mathrm{SL}_m(\mathbb{Z}), \delta_2 \in \mathrm{SL}_n(\mathbb{Z}) \right\}. \tag{4.1}$$

Further, let $\Lambda_\Delta < G \times G$ denote the diagonal embedding of Λ in $G \times G$.

Lemma 4.1. *The following periodic orbits in either X_d or $X_d \times X_d$ and probability measures supported on them are Λ-invariant or Λ_Δ-invariant respectively.*

(1) *The periodic orbit Ux_0 and the measures $\mathbf{m}_{Ux_0}$, $\{\boldsymbol{\mu}_k : k \in \mathbb{Z}_{>0}\}$.*
(2) *The periodic orbit Hx_0 and the measures $\mathbf{m}_{Hx_0}$, $\{a(k)_*\boldsymbol{\mu}_k : k \in \mathbb{Z}_{>0}\}$.*
(3) *The periodic orbit Vx_0 and the measure $\mathbf{m}_{Vx_0}$.*
(4) *The periodic orbit X_m and the measures $\mathbf{m}_{X_m}$,$\{\boldsymbol{\nu}_k : k \in \mathbb{Z}_{>0}\}$.*
(5) *The periodic orbit $Ux_0 \times Hx_0$ and the measures $\mathbf{m}_{Ux_0} \times \mathbf{m}_{Hx_0}$, $\{\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_k : k \in \mathbb{Z}_{>0}\}$.*
(6) *The periodic orbit $Ux_0 \times Vx_0$ and the measure $\mathbf{m}_{Ux_0} \times \mathbf{m}_{Vx_0}$.*

Proof. The Λ-action on Ux_0, Hx_0 and X_m are given by

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathbf{u} & I \end{pmatrix} x_0 = \begin{pmatrix} I & 0 \\ \delta_2 \mathbf{u} \delta_1^{-1} & I \end{pmatrix} x_0. \tag{4.2}$$

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} h & \mathbf{v} \\ 0 & I \end{pmatrix} x_0 = \begin{pmatrix} \delta_1 h & \delta_1 \mathbf{v} \delta_2^{-1} \\ 0 & I \end{pmatrix} x_0. \tag{4.3}$$

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & I \end{pmatrix} x_0 = \begin{pmatrix} \delta_1 h & 0 \\ 0 & I \end{pmatrix} x_0.$$

The Λ-action on Vx_0 is given by a similar formula. This shows that the corresponding periodic orbits are indeed Λ-invariant. Also, as conjugation by Λ fixes the volume form on the groups giving rise to these periodic orbits, the Haar measures on these periodic orbits are preserved.

The measures $\widetilde{\boldsymbol{\mu}}_k$ are Λ_Δ-invariant because it follows from (4.2) and Definition 1.2 that $\widetilde{\mathcal{P}}_k$ is Λ_Δ-invariant. In turn, because the Λ_Δ-action on $X_d \times X_d$ commutes with that of $\widetilde{a}(k)$, we conclude that $\widetilde{a}(k)_* \widetilde{\boldsymbol{\mu}}_k$ is Λ_Δ-invariant. Similarly $a(k)_* \boldsymbol{\mu}_k$ is Λ-invariant. The invariance of the measure $\boldsymbol{\nu}_k$ now follows from that of $\boldsymbol{\mu}_k$ as the projection π_3 in (2.9) intertwines the Λ-actions on Hx_0 and X_m. □

4.2. **Rationality issues.** We will need the following lemmas in order to establish various rationality statements when classifying measures. These rationality statements are important to us because they imply the countability of the measures we classify in each discussion. This countability is used later along the lines described in §2.3.

Lemma 4.2. *Let N be an integer and let $\lambda \in \mathrm{SL}_N(\mathbb{Z})$ be matrix acting naturally on the torus $\mathbb{T}^N = \mathbb{R}^N/\mathbb{Z}^N$. Assume that all the eigenvalues of λ are not roots of unity. Then, if $\mathbf{w} \in \mathbb{T}^N$ has a finite λ-orbit then $\mathbf{w}$ is a rational point (that is, any vector representing it is rational).*

Proof. Assume that $\lambda^j \mathbf{w} = \mathbf{w}$. If $w \in \mathbb{R}^N$ projects to $\mathbf{w}$ then this means that there is an integer vector e such that $\lambda^j w = w + e$ or said otherwise $(\lambda^j - I)w = e$. By assumption $\lambda^j - I$ is invertible and its inverse is a rational matrix so $w = (\lambda^j - I)e$ is rational as well. □

We recall the definition of the commensurator group: Let G' be a topological group and $\Gamma' < G'$ a closed subgroup. Let $\mathrm{comm}_{G'}(\Gamma') \stackrel{\mathrm{def}}{=} \{g \in G' : g\Gamma' g^{-1}, \Gamma' \text{ are commensurable}\}$.

Lemma 4.3. *Let G' be a topological group and Γ', Λ' closed subgroups.*

1. *Let $g \in G'$ and $q \in \mathrm{comm}_{G'}(\Gamma')$. Then, if the orbit $\Lambda g\Gamma' \subset G'/\Gamma'$ is finite, then so is $\Lambda gq\Gamma'$.*
2. *If $\Gamma' < G'$ is a lattice then $q \in \mathrm{comm}_{G'}(\Gamma')$ if and only the orbit of $\Gamma' g\Gamma' \subset G'/\Gamma'$ is finite.*

Proof. (1). Note that $\Lambda' g\Gamma' \subset G'/\Gamma'$ is finite if and only if $\mathrm{Stab}_{\Lambda'} g\Gamma' = \Lambda' \cap g\Gamma' g^{-1} < \Lambda'$ is of finite index. As $q \in \mathrm{comm}_{G'}(\Gamma')$ we deduce that the intersection $gq\Gamma' q^{-1} g^{-1} \cap g\Gamma' g^{-1}$ is of finite index in both groups. In particular, $\Lambda' \cap gq\Gamma' q^{-1} g^{-1} < \Lambda'$ is of finite index. Arguing in reverse this implies now that $\Lambda' gq\Gamma' \subset G'/\Gamma'$ is finite.

(2). One direction of implication follows by applying part (1) to $\Lambda' = \Gamma'$. In the other direction, if the orbit $\Gamma' q\Gamma'$ is finite then as before, this implies that the group $\mathrm{Stab}_{\Gamma'} q\Gamma' = \Gamma' \cap q\Gamma' q^{-1}$ is of finite index in Γ'. In particular, it is a lattice in G', which forces its index in $q\Gamma' q^{-1}$ to be finite as well; i.e. $q \in \mathrm{comm}_{G'}(\Gamma')$. □

Lemma 4.4. *If $\left(\begin{smallmatrix} g & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0$ has a finite Λ-orbit then*

$$g \in \mathrm{comm}_{\mathrm{SL}_m(\mathbb{R})}(\mathrm{SL}_m(\mathbb{Z})) \text{ and } \mathbf{v} \text{ is rational.}$$

A similar statement holds for $\left(\begin{smallmatrix} g & 0 \\ \mathbf{u} & I \end{smallmatrix}\right) x_0$.

Proof. We prove the first statement. The second statement follows by applying the involution on X_d induced by the transpose inverse operation. Assume that $\Lambda\left(\begin{smallmatrix} g & \mathbf{v} \\ 0 & I \end{smallmatrix}\right)x_0$ is finite. Projecting to $X_m \cong \mathrm{SL}_m(\mathbb{R})/\mathrm{SL}_m(\mathbb{Z})$ we deduce by Lemma 4.3(2) that

$$g \in \mathrm{Comm}_{\mathrm{SL}_m(\mathbb{R})}(\mathrm{SL}_m(\mathbb{Z})).$$

In turn, this implies that $q = \left(\begin{smallmatrix} g & 0 \\ 0 & I \end{smallmatrix}\right) \in \mathrm{comm}_G(\mathrm{SL}_d(\mathbb{Z}))$. It now follows from Lemma 4.3(1) that $\Lambda\left(\begin{smallmatrix} g & \mathbf{v} \\ 0 & I \end{smallmatrix}\right)q^{-1}x_0 = \Lambda\left(\begin{smallmatrix} I & \mathbf{v} \\ 0 & I \end{smallmatrix}\right)x_0$ is finite as well. Thus we have reduced the discussion to the situation where $g = I$.

We now identify Vx_0 with the torus $\mathbb{R}^N/\mathbb{Z}^N$ with $N = m{\cdot}n$ and apply Lemma 4.2 to conclude that $\mathbf{v}$ is rational. Indeed it is straightforward to verify the existence of $\lambda \in \Lambda$ that acts on $\mathbb{R}^N$ without roots of unity as eigenvalues. $\square$

4.3. Measure classifications in general.

Theorem 4.5. *Let μ be a Λ-invariant and ergodic probability measure on X_d. Then there exists an intermediate subgroup $\Lambda < L < G$ and a periodic L-orbit $Lx \subset X_d$, such that μ is the L-invariant probability measure $\mathbf{m}_{Lx}$.*

This theorem is a particular case of a more general measure classification by Shah [Sha98] which uses and generalizes Ratner's measure classification theorem [Ra91], [Ra91b]. It is applicable since $\Lambda = \mathrm{SL}_m(\mathbb{Z})\times\mathrm{SL}_n(\mathbb{Z})$ is generated by unipotent elements. In fact, because Λ is a lattice in a semisimple Lie subgroup of G with no compact factors, using the suspension technique [Wit94, Corollary 5.8] it is straightforward to deduce Theorem 4.5 directly from Ratner's measure classification theorem for the actions of semisimple groups without compact factors (for a simplified proof for this case see [Ein06]).

For a closed subgroup $L < G$ we denote by L° the connected component of the identity of L. We have the following corollary which will be more convenient for us.

Corollary 4.6. *Let μ be a Λ-invariant and ergodic probability measure on X_d and let L, x be the group and the point that arise by applying Theorem 4.5 so that $\mu = \mathbf{m}_{Lx}$. Then there exist $x_1 \dots x_N \in Lx$ such that $Lx = \bigsqcup_{i=1}^N L^\circ x_i$, each orbit $L^\circ x_i$ is periodic, Λ acts transitively by permuting the collection of orbits $\{L^\circ x_i\}$, and $\mu = \frac{1}{N}\sum_1^N \mathbf{m}_{L^\circ x_i}$.*

Proof. As $L^\circ \trianglelefteq L$ is open and closed, the orbit Lx decomposes into a union of (relatively) open and closed L°-orbits. As μ is finite, this decomposition is finite $Lx = \bigsqcup_{i=1}^N L^\circ x_i$. As L° is a normal subgroup of

L, Λ acts on the L°-orbits by permuting them. Moreover, the ergodicity assumption implies that this action is transitive. It follows that $\mu(L^\circ x_i) = N^{-1}$ and the formula $\mu = \frac{1}{N}\sum_1^N \mathbf{m}_{L^\circ x_i}$ follows. □

For convenience of reference we also state the following elementary lemma whose proof we omit.

Lemma 4.7. *Let F_1, F_2 be closed subgroups of G and $x \in X_d$. If $F_1 x \subset F_2 x$ then $F_1^\circ \subset F_2$.*

The rest of this section is devoted to classifying Λ-invariant and ergodic measures in various situations that will be encountered in the course of the proof of Theorem 1.3.

4.4. Measures supported in Ux_0.

Let us define

$$\mathrm{Tor}_k(Ux_0) \overset{\mathrm{def}}{=} \left\{x_{k^{-1}\mathbf{u}} \in Ux_0 : \mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{Z})\right\}. \tag{4.4}$$

Lemma 4.8. *The ergodic Λ-invariant probability measures on Ux_0 are exactly the normalized counting measures on finite Λ-orbits and $\mathbf{m}_{Ux_0}$. Moreover, any finite Λ-orbit is contained in $\mathrm{Tor}_k(Ux_0)$ for some positive integer k.*

Proof. Let[6] μ be an ergodic Λ-invariant measure supported in the orbit Ux_0. Applying Theorem 4.5 we conclude the existence of a closed subgroup $\Lambda < L < G$ such that μ is the L-invariant probability measure supported on a periodic L-orbit. Applying Lemma 4.7 we conclude that $L^\circ < U$. Viewing L° as a subspace of $\mathrm{Mat}_{n\times m}(\mathbb{R}) \cong U$, the fact that Λ normalizes L° translates into this subspace being invariant under the linear representation of Λ on $\mathrm{Mat}_{n\times m}(\mathbb{R})$ (here $\left(\begin{smallmatrix} A & 0 \\ 0 & D\end{smallmatrix}\right) \in \mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ acts on $\mathbf{u} \in \mathrm{Mat}_{n\times m}(\mathbb{R})$ by $D\mathbf{u}A^{-1}$). As this representation of $\mathrm{SL}_m(\mathbb{R}) \times \mathrm{SL}_n(\mathbb{R})$ is irreducible and Λ is Zariski dense in the former group, we deduce that L° is either the trivial group or U.

If $L^\circ = U$ then clearly $\mu = \mathbf{m}_{Ux_0}$. If L° is trivial, an application of Corollary 4.6 gives that μ is the normalized counting measure on a finite Λ-orbit $\Lambda x_{\mathbf{u}} \subset Ux_0$. Lemma 4.4 implies now that $\mathbf{u}$ is in fact a rational matrix. If k is a common denominator for its entries then $x_{\mathbf{u}} \in \mathrm{Tor}_k(Ux_0)$ as desired. □

[6]Lemma 4.8 could be proved using Fourier arguments but we choose to appeal to Theorem 4.5 as this is more compatible with the later arguments. Furthermore, this is by no means a new result and the proof is included for expository reasons. For a similar argument see for example [Wit94, Example 5.9].

4.5. **Measures supported in Hx_0.** Our next objective is to classify the Λ-invariant ergodic probability measures supported in Hx_0. By Theorem 4.5 and Corollary 4.6 such a measure is always of the form $\frac{1}{N}\sum \mathbf{m}_{Lx_i}$, where $L < G$ is a closed connected subgroup normalized by Λ. As $Lx_i \subset Hx_0$ and L is connected we conclude from Lemma 4.7 that $L < H$. We have the following classification of groups.

Lemma 4.9. *Let $L < H$ be a closed connected subgroup normalized by Λ. Then there are four possibilities*

(1) $L = \{e\}$,
(2) $L = V = \left\{\left(\begin{smallmatrix} I & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) : \mathbf{v} \in \mathrm{Mat}_{m\times n}(\mathbb{R})\right\}$,
(3) $L = \mathrm{SL}_m(\mathbb{R}) = \left\{\left(\begin{smallmatrix} h & 0 \\ 0 & I \end{smallmatrix}\right) : h \in \mathrm{SL}_m(\mathbb{R})\right\}$,
(4) $L = H$.

Proof. Consider the projection of L in the simple group $\mathrm{SL}_m(\mathbb{R}) = H/V$. As L is normalized by Λ, this projection is a connected subgroup normalized by $\mathrm{SL}_m(\mathbb{Z})$ and by Zariski density we deduce that it is a connected normal subgroup. It therefore follows that this projection is either trivial or $\mathrm{SL}_m(\mathbb{R})$. In the first case, $L < V$. As the adjoint action of Λ on the Lie algebra of V (which is isomorphic to V itself) is irreducible, it follows that either $L = V$ or L is the trivial group.

Assume then that the projection of L is onto and consider the subgroup $V' = L \cap V$ which is normalized by Λ. The same irreducibility argument as before implies that either $V' = V$ or V' is trivial. In the first case $L = H$. In the second case the projection of L onto H/V must also be injective so there exists a map $\psi : \mathrm{SL}_m(\mathbb{R}) \to \mathrm{Mat}_{m\times n}(\mathbb{R})$ such that $L = \left\{\left(\begin{smallmatrix} g & \psi(g) \\ 0 & I \end{smallmatrix}\right) : g \in \mathrm{SL}_m(\mathbb{R})\right\}$. If $\lambda = \left(\begin{smallmatrix} \delta & 0 \\ 0 & I \end{smallmatrix}\right) \in \Lambda$ then $\lambda \left(\begin{smallmatrix} \delta & \psi(\delta) \\ 0 & I \end{smallmatrix}\right) \lambda^{-1} = \left(\begin{smallmatrix} \delta & \delta\psi(\delta) \\ 0 & I \end{smallmatrix}\right) \in L$ so that $\delta\psi(\delta) = \psi(\delta)$. By Zariski density we deduce that this formula holds for any $\delta \in \mathrm{SL}_m(\mathbb{R})$. In turn, if 1 is not an eigenvalue of δ, this implies that the columns of $\psi(\delta)$ must be zero. Continuity now gives that ψ vanishes and so L is the standard copy of $\mathrm{SL}_m(\mathbb{R})$ in H. □

In light of Lemma 4.9 it makes sense to make (for the sake of the current discussion), the following

Definition 4.10. We say that a Λ-invariant and ergodic measure on Hx_0 is of *type* (1)–(4) in accordance to which one of the four groups that appear in Lemma 4.9 is attached to it.

Corollary 4.11. *The following is a classification of the Λ-invariant and ergodic probability measures on Hx_0.*

(1) Measures of type (1) *are simply normalized counting measures on a (necessarily finite) Λ-orbit of a point of the form $\left(\begin{smallmatrix} h & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0$,*

where $h \in \mathrm{comm}_{\mathrm{SL}_m(\mathbb{R})}(\mathrm{SL}_m(\mathbb{Z}))$ and $\mathbf{v}$ is rational. If such a measure projects to δ_{x_0} under $(\pi_3)_*$ then we can choose $h = I$.

(2) *Measures of type (2) are of the form $\frac{1}{N}\sum_{i=1}^{N} \mathbf{m}_{Vx_i}$, where $\{x_i\}$ is the Λ-orbit of $x_1 = \left(\begin{smallmatrix} h & 0 \\ 0 & I \end{smallmatrix}\right) x_0$, where $h \in \mathrm{comm}_{\mathrm{SL}_m(\mathbb{R})}(\mathrm{SL}_m(\mathbb{Z}))$. There is only one such measure that projects to δ_{x_0} and that is $\mathbf{m}_{Vx_0}$.*

(3) *Measures of type (3) are of the form $\frac{1}{N}\sum_{1}^{N} \mathbf{m}_{\mathrm{SL}_m(\mathbb{R})x_i}$, where $x_i = \left(\begin{smallmatrix} I & \mathbf{v}_i \\ 0 & I \end{smallmatrix}\right) x_0$ and the $\mathbf{v}_i$'s are rational. All such measures project under $(\pi_3)_*$ to $\mathbf{m}_{X_m}$.*

(4) *There is only one measure of type (4), namely $\mathbf{m}_{Hx_0}$ and it projects under $(\pi_3)_*$ to $\mathbf{m}_{X_m}$.*

Proof. Let μ denote a Λ-invariant and ergodic probability measure on Hx_0 and present it as $\mu = \frac{1}{N}\sum_{1}^{N} \mathbf{m}_{Lx_i}$ with L connected and determining the type as explained above. We prove statements (1)–(3) as (4) is clear.

(1). If μ is of type (1) we deduce from the ergodicity that $\{x_i\}_1^N$ forms a finite Λ-orbit and Lemma 4.4 gives the commensurability and rationality statements. The statement regarding the projection is clear.

(2). Suppose μ is of type (2) and write $x_i = \left(\begin{smallmatrix} h_i & \mathbf{v}_i \\ 0 & I \end{smallmatrix}\right) x_0$. Using the V-invariance we may assume without loss of generality that $\mathbf{v}_i = 0$ and so $\pi_3(x_i) = x_i$. Then $(\pi_3)_*\mu = \frac{1}{N}\sum_{1}^{N} \delta_{x_i}$ is a finitely supported Λ-invariant and ergodic measure and so the x_i's form an orbit. Lemma 4.4 gives now that $h_1 \in \mathrm{comm}_{\mathrm{SL}_m(\mathbb{R})}(\mathrm{SL}_m(\mathbb{Z}))$ as required.

(3). Assume μ is of type (3). As the periodic orbits $\mathrm{SL}_m(\mathbb{R})x_i$ are closed and the $\mathrm{SL}_m(\mathbb{R})$ orbits are transverse to the V-orbits (all of which are periodic in Hx_0), we deduce that the intersection $\mathrm{supp}(\mu) \cap Vx_0$, which is Λ-invariant and closed, is finite. Lemma 4.4 now implies that the points x_i which without loss of generality might be assumed to belong to $\mathrm{supp}(\mu) \cap Vx_0$, have rational representatives. □

4.6. Measures supported in $Ux_0 \times Hx_0$.

Theorem 4.12. *The only Λ_Δ-invariant probability measure on $Ux_0 \times Hx_0$ which projects under π_1 and π_2 to $\mathbf{m}_{Ux_0}$, $\mathbf{m}_{Hx_0}$ respectively, is the product measure $\mathbf{m}_{Ux_0} \times \mathbf{m}_{Hx_0}$.*

Proof. Let $u \in \Lambda$ be a unipotent element that belongs to the subgroup $\mathrm{SL}_m(\mathbb{Z}) < \Lambda$ (such an element always exists as $m \geq 2$). Note the following two facts

(1) The action of u on $(Hx_0, \mathbf{m}_{Hx_0})$ is mixing by the Howe-Moore theorem applied to the action of $\mathrm{SL}_m(\mathbb{R})$.

(2) The action of u on the torus $\mathbb{T}^{n\times m} \cong Ux_0$ is by a unipotent automorphism and so the ergodic components of $\mathbf{m}_{Ux_0}$ with respect to the action of u are minimal rotation on compact abelian groups.

The theorem now follows from the following two lemmas. □

Lemma 4.13 (cf. [Gla03, Theorem 6.27])**.** *Let $(X,\mu,u),(Y,\nu,u)$ be two dynamical systems. If (X,μ,u) is a minimal rotation on a compact abelian group and (Y,ν,u) is mixing, then the two systems are disjoint.*

Lemma 4.14. *Let $(X,\mu,u),(Y,\nu,u)$ be two dynamical systems such that ν is ergodic and let $\mu = \int \mu_x d\mu(x)$, be the ergodic decomposition of μ . Then, if for μ-a.e x the systems (X,μ_x,u), (Y,ν,u) are disjoint, then (X,μ,u), (Y,ν,u) are disjoint as well.*

We give both proofs for the sake of completeness.

Proof of Lemma 4.13. Let η be a joining of μ,ν. As η is a joining, the projections $\mathrm{p}_X : (X\times Y,\eta) \to (X,\mu)$, $\mathrm{p}_Y : (X\times Y,\eta)\to(Y,\nu)$ induce injections of the L^2-spaces; that is, the Hilbert spaces $\mathcal{H}_X \stackrel{\mathrm{def}}{=} L^2(X,\mu), \mathcal{H}_Y \stackrel{\mathrm{def}}{=} L^2(Y,\nu)$ are isometrically embedded in $\mathcal{H} \stackrel{\mathrm{def}}{=} L^2(X\times Y,\eta)$. We denote by $\mathcal{H}^0,\mathcal{H}^0_X,\mathcal{H}^0_Y$ the subspaces orthogonal to the constant functions in each of the spaces. Showing that η is the product measure is the same as showing that the two subspaces $\mathcal{H}^0_X,\mathcal{H}^0_Y$ of $\mathcal{H}^0$ are orthogonal. For this, note that the mixing assumption for the u-action on (Y,ν) is equivalent to saying that for each $v\in\mathcal{H}^0_Y$ the sequence $T_u^n v$ converges weakly to 0, where we write T_u for the unitary operator on $\mathcal{H}$ induced by the u-action. If the two subspaces are not orthogonal, then there exists a vector $0\neq v\in\mathcal{H}^0_Y$ that has a non-trivial projection to $\mathcal{H}^0_X$. Let us denote this projection of v by $w\in\mathcal{H}^0_X$. We claim that $T_u^n w$ must converge weakly to 0 as well. This simply follows from the fact that the projection commutes with T_u as the subspace we project on is T_u-invariant and T_u is unitary. However, as (X,μ,u) is isomorphic to a minimal rotation on a compact group there exists an integer sequence $n_k\to\infty$ such that $T_u^{n_k}w\to w$ as $k\to\infty$. This contradiction implies the lemma. □

Proof of Lemma 4.14. Let η be a joining of μ,ν and let

$$\eta = \int \eta_{(x,y)} d\eta(x,y)$$

be the ergodic decomposition of η. Let $\mathrm{p}_X,\mathrm{p}_Y$ denote the projections from $X\times Y$ to X,Y respectively. For η-a.e (x,y) we have that $(\mathrm{p}_Y)_*\eta_{(x,y)}$ is an ergodic u-invariant probability measure on Y and as

η projects to ν we deduce that $\nu = \int (\mathrm{p}_Y)_* \eta_{(x,y)} d\eta(x,y)$. Since ν is ergodic we conclude that $\nu = (\mathrm{p}_Y)_* \eta_{(x,y)}$ for η-almost any (x,y). By a similar reasoning one can argue that for η-a.e (x,y) $(\mathrm{p}_X)_* \eta_{(x,y)} = \mu_x$. It follows that for η-almost any (x,y) the ergodic component $\eta_{(x,y)}$ is a joining of μ_x and ν and therefore by our disjointness assumption we have $\eta_{(x,y)} = \mu_x \times \nu$ for η-almost any pair (x,y). This implies the lemma.

□

4.7. **Measures supported in $Ux_0 \times Vx_0$.** We will use the following notation in the special case where $m = n = 2$. Let us denote by $\mathbf{u} \mapsto \mathbf{u}^*$ the linear isomorphism of $\mathrm{Mat}_2(\mathbb{R})$ given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^* = \left(\begin{smallmatrix} d & -c \\ -b & a \end{smallmatrix}\right)$. For co-prime integers p, q we denote

$$L_{p,q} \stackrel{\mathrm{def}}{=} \left\{ \left(\left(\begin{smallmatrix} I & 0 \\ p\mathbf{u} & I \end{smallmatrix}\right), \left(\begin{smallmatrix} I & q\mathbf{u}^* \\ 0 & I \end{smallmatrix}\right) \right) : \mathbf{u} \in \mathrm{Mat}_2(\mathbb{R}) \right\} < U \times V. \tag{4.5}$$

Theorem 4.15. *Assume $n = m$. Let η be a Λ_Δ-invariant and ergodic probability measure supported on $Ux_0 \times Vx_0$ and suppose that $(\pi_1)_*\eta = \mathbf{m}_{Ux_0}$, $(\pi_2)_*\eta = \mathbf{m}_{Vx_0}$. Then $\eta = \mathbf{m}_{Ux_0} \times \mathbf{m}_{Vx_0}$, or $n = 2$ and $\eta = \frac{1}{N}\sum_1^N \mathbf{m}_{L_{p,q}(x_i,y_i)}$, where p, q are co-prime integers and the points x_i, y_i may be chosen to have rational matrix representatives.*

Proof. We abuse notation and write Λ for Λ_Δ. We identify $Ux_0 \times Vx_0$ with the product torus $\mathbb{T}^{n^2} \times \mathbb{T}^{n^2}$ and recall that the $\Lambda = \mathrm{SL}_n(\mathbb{Z}) \times \mathrm{SL}_n(\mathbb{Z})$-action on it is induced by the Λ-representation on $\mathrm{Mat}_n(\mathbb{R}) \times \mathrm{Mat}_n(\mathbb{R})$ given by (see (4.2), (4.3)),

$$(\delta_1, \delta_2)(\mathbf{u}, \mathbf{v}) = (\delta_2 \mathbf{u} \delta_1^{-1}, \delta_1 \mathbf{v} \delta_2^{-1}). \tag{4.6}$$

As Λ is generated by unipotents, a suitable application of Theorem 4.5 implies that there exists a Λ-invariant subspace $W \subset \mathrm{Mat}_n(\mathbb{R}) \times \mathrm{Mat}_n(\mathbb{R})$ and finitely many periodic orbits $\{W(\mathbf{u}_i, \mathbf{v}_i)\}_1^N \subset \mathbb{T}^{n^2} \times \mathbb{T}^{n^2}$ such that $\eta = \frac{1}{N}\sum_1^N \mathbf{m}_{W(\mathbf{u}_i,\mathbf{v}_i)}$. From our assumption that η is a joining of $\mathbf{m}_{Ux_0}$ and $\mathbf{m}_{Vx_0}$ we conclude that W projects onto $\mathrm{Mat}_n(\mathbb{R})$ under both left and right projections.

We will need the following general representation theoretic lemma. Its proof is straightforward and left to the reader.

Lemma 4.16. *Let ρ_i for $i = 1, 2$ be two irreducible representations of a group Λ on the vector spaces V_i and let $\rho = \rho_1 \oplus \rho_2$. If $\{0\} \subsetneq W \subsetneq V_1 \oplus V_2$ is a Λ-invariant subspace, then either $W = V_1 \times \{0\}$, or $W = \{0\} \times V_2$, or there is an isomorphism $\varphi : (V_1, \rho_1) \to (V_2, \rho_2)$ of Λ-representations such that $W = \{(v, \varphi(v)) : v \in V_1\}$.*

We apply this Lemma with $V_1 = V_2 = \mathrm{Mat}_n(\mathbb{R})$, with the representations ρ_i of Λ which are given by restricting to left and right coordinates of the formula (4.6), and with W being the subspace identified above which is attached to η. If W is the whole space then $\eta = \mathbf{m}_{Ux_0} \times \mathbf{m}_{Vx_0}$. Otherwise, we deduce from the lemma (and the fact that W projects onto each factor), the existence of the isomorphism φ. Then, applying (4.6) to the diagonal copy of $\mathrm{SL}_n(\mathbb{Z})$ in Λ, we deduce that the image of the line $\{sI : s \in \mathbb{R}\} \subset V_1$ must be stable under conjugation by every element of $\mathrm{SL}_n(\mathbb{Z})$, which implies that this line must be mapped to itself. Therefore, there exists a scalar ρ_φ such that $\varphi(I) = \rho_\varphi I$. It then follows from (4.6) that the restriction of φ to the set $\{s\delta : s \in \mathbb{R}, \delta \in \mathrm{SL}_n(\mathbb{Z})\}$ is given by the formula

$$\varphi(s\delta) = \rho_\varphi s \delta^{-1}. \tag{4.7}$$

For $n > 2$ this is not a linear map and so the existence of φ is ruled out and so indeed $\eta = \mathbf{m}_{Ux_0} \times \mathbf{m}_{Vx_0}$ as claimed.

In the case $n = 2$ on the other hand, formula (4.7) is given by

$$\mathbf{u} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \xrightarrow{\varphi} \rho_\varphi \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \rho_\varphi \mathbf{u}^*, \tag{4.8}$$

which is linear, and by Zariski density, this must be the formula for φ on $V_1 = \mathrm{Mat}_2(\mathbb{R})$. By the above lemma we have that

$$W = \{(\mathbf{u}, \rho_\varphi \mathbf{u}^*) : \mathbf{u} \in \mathrm{Mat}_n(\mathbb{R})\}$$

and since it has periodic orbits in $\mathbb{T}^{n^2} \times \mathbb{T}^{n^2}$, ρ_φ must be rational, say $\rho_\varphi = p/q$ for some co-prime integers p, q. Under our identifications W corresponds to to the subgroup $L_{p,q}$ from (4.5).

Finally, we need to justify the rationality of the points $(\mathbf{u}_i, \mathbf{v}_i)$. Consider the action of Λ on the quotient torus $\mathbb{T}^{n^2} \times \mathbb{T}^{n^2}/W$ which we identify with the standard torus $\mathbb{T}^N$ for a suitable N. The measure η projects there to a finitely supported Λ-invariant measure. It is straightforward to show the existence of $\lambda \in \Lambda$ which acts on $\mathbb{R}^N \cong \mathrm{Mat}_n \times \mathrm{Mat}_n / W$ without roots of unity as eigenvalues and therefore by Lemma 4.2 we conclude the rationality of the images of $(\mathbf{u}_i, \mathbf{v}_i)$ in $\mathbb{T}^N$ and in turn the desired rationality before projecting (because W is a rational space). ☐

5. Hecke friends

5.1. **Hecke friends.** For any positive integer ℓ and an m-dimensional lattice $x \in X_m$ we define the set of *ℓ-Hecke friends* of x to be the collection of lattices

$$\left\{\ell^{-\frac{1}{m}}\Delta : \ \Delta \text{ is a subgroup of index } \ell \text{ of } x\right\} \subset X_m.$$

Clearly, it is enough to understand the collection of ℓ-Hecke friends of $\mathbb{Z}^m$, for if $x = h\mathbb{Z}^m$ for some $h \in \mathrm{SL}_m(\mathbb{R})$, then the collection of ℓ-Hecke friends of x is simply the image under h of the corresponding collection for $\mathbb{Z}^m$.

Given a subgroup $\Delta < \mathbb{Z}^m$ of index ℓ, we choose $\mathbf{u} \in \mathrm{Mat}_m(\mathbb{Z})$ whose columns form a basis for Δ and with $\det \mathbf{u} > 0$. By Lemma 2.3 (and its proof) there is a unique positive elementary divisors tuple $(\ell_1 \dots \ell_m)$ attached to it. Consequently, $\ell = \prod_1^m \ell_i$. It is straightforward to show that the tuple does not depend on the choice of $\mathbf{u}$ and so we make the following.

Definition 5.1. Given an ℓ-Hecke friend of $\mathbb{Z}^m$, we define its *Hecke-type* to be the corresponding m-tuple of elementary divisors.

The above discussion could be restated as follows.

Lemma 5.2. *The collection of ℓ-Hecke friends of $\mathbb{Z}^m$ is partitioned into $\mathrm{SL}_m(\mathbb{Z})$-orbits in the following way*

$$\{\ell\text{-}\textit{Hecke friends of } \mathbb{Z}^m\} = \bigsqcup_{(\ell_1 \dots \ell_m)} \mathrm{SL}_m(\mathbb{Z}) \,\mathrm{diag}\,(\ell_1, \dots, \ell_m)\, \mathbb{Z}^m,$$

where the union is taken over the ℓ-Hecke-types; that is, all positive tuples $(\ell_1 \dots \ell_m)$ such that $\ell_i | \ell_{i+1}$ and $\prod_1^m \ell_i = \ell$.

The following equidistribution result will be needed in the proof of Theorem 1.3.

Theorem 5.3. *Let $\{(\ell_{i1} \dots \ell_{im})\}_{i=1}^\infty$ be a sequence of types of $\boldsymbol{\ell}_i$-Hecke friends of $\mathbb{Z}^m$, where $\boldsymbol{\ell}_i \stackrel{\text{def}}{=} \prod_{j=1}^m \ell_{ij}$, and let ν_i denote the normalized counting measure supported on the collection of $\boldsymbol{\ell}_i$-Hecke friends of $\mathbb{Z}^m$ of type $(\ell_{i1} \dots \ell_{im})$. Then $\nu_i \xrightarrow{\mathrm{w}^*} \mathbf{m}_{X_m}$ if and only if $\frac{\ell_{im}}{\ell_{i1}} \longrightarrow \infty$.*

Proof. This theorem follows from [COU01]. More conveniently, it is a special case of [EO06, Theorem 1.2]. This result states that for $q_i \in \mathrm{GL}_m(\mathbb{Q})$, the normalized counting measure on the (finite) orbit $\frac{1}{(\det q_i)^{1/m}} \mathrm{SL}_m(\mathbb{Z}) q \mathbb{Z}^m$ equidistribute to $\mathbf{m}_{X_m}$ once $\deg q_i \to \infty$ where $\deg q_i$ is the size of the corresponding orbit. We apply this for $q_i = \mathrm{diag}\,(\ell_{i1}, \dots, \ell_{im})$. We need to explain why the condition $\deg(\mathrm{diag}\,(\ell_{i1} \dots \ell_{im})) \to \infty$ is equivalent to the requirement $\frac{\ell_{im}}{\ell_{i1}} \longrightarrow \infty$.

Without loss of generality we may assume on scaling by $1/\ell_{i1}$ that $\ell_{i1} = 1$. If ℓ_{im} is bounded along some subsequence of i's then clearly the degrees do not diverge to ∞. On the other hand, if $(1, \dots, \ell_m)$ is a Hecke type and $\ell_m > M$, then there exists some $1 \le r < m$ such that $\frac{\ell_{r+1}}{\ell_r} > M^{1/m}$. Then the orbit $\mathrm{SL}_m(\mathbb{Z}) \,\mathrm{diag}\,(1, \dots, \ell_m)\, \mathbb{Z}^m$ contains an embedded copy of the orbit $\mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \ell_{r+1}/\ell_r \end{smallmatrix}\right) \mathbb{Z}^2$ which contains at least $p^{e-1}(p+1)$ points where p^e is the largest prime power dividing ℓ_{r+1}/ℓ_r. As this prime power goes to ∞ with M, we are done. $\square$

6. Proof of Theorem 1.3

We follow the scheme presented after (2.9), divide the proof into Steps 1-4.

6.1. **Step 1.** In this step we establish the convergence $\boldsymbol{\mu}_k \overset{\mathrm{w}^*}{\to} \mathbf{m}_{Ux_0}$. Let σ be a weak* limit of the sequence $\{\boldsymbol{\mu}_k\}$. By Lemma 4.1(1) σ is a Λ-invariant probability measure. By Lemma 4.8 there are only countably many Λ-invariant ergodic probability measures on Ux_0. We let $\sigma_0 \overset{\mathrm{def}}{=} \mathbf{m}_{Ux_0}$ and let $\{\sigma_i\}_{i=1}^{\infty}$ be any enumeration of the measures supported on finite Λ-orbits. By the ergodic decomposition we may write $\sigma = \sum_{i=0}^{\infty} c_i\sigma_i$ where $c_i \geq 0$ and $\sum_i c_i = 1$. The proof of Step 1 will be concluded once we show that $c_0 = 1$. By Lemma 4.8 each σ_i, $i > 0$ is supported in the torsion points $\mathrm{Tor}_{k_i}(Ux_0)$ for some integer k_i. It is straightforward to show that for $k > k_i$ we have that $\mathcal{P}_k \cap \mathrm{Tor}_{k_i}(Ux_0) = \varnothing$, and so for all large enough k the support of $\boldsymbol{\mu}_k$ is disjoint from the orbit on which σ_i is supported. We now apply the Non-accumulation Theorem 3.1 and deduce that $c_i = 0$ for $i > 0$ as desired. The application of Theorem 3.1 is done with the following choices: $G = \Lambda \ltimes U$, $\Gamma = \Lambda \ltimes \mathrm{Mat}_{n\times m}(\mathbb{Z})$, $L = \Lambda$, z being one of the points in the support of σ_i. The space W, being a linear complement of $\mathrm{Lie}(L) = \{0\}$, equals $\mathrm{Lie}(U) = \mathrm{Mat}_{n\times m}(\mathbb{R})$ and indeed there are no Λ-fixed vectors in W other than zero.

6.2. **Step 2.** In this step we establish the convergence $\boldsymbol{\nu}_k \overset{\mathrm{w}^*}{\to} (\pi_3)_*\boldsymbol{\theta}$. This follows from the following lemma.

Lemma 6.1. *The measure $\boldsymbol{\nu}_k \overset{\mathrm{def}}{=} (\pi_3)_* a(k)_*\boldsymbol{\mu}_k$ is the normalized counting measure on the collection of k^n-Hecke friends of $\mathbb{Z}^m$ of type $(\underbrace{1\dots 1}_{m-n}\underbrace{k\dots k}_{n})$. In particular, if $m > n$, $\boldsymbol{\nu}_k$ equidistribute to $\mathbf{m}_{X_m}$ and if $m = n$ then $\boldsymbol{\nu}_k$ is the Dirac measure supported on x_0.*

Proof. The last sentence in the statement simply follows from Theorem 5.3 (and the observation that $\mathbb{Z}^m$ is the only k^n-Hecke friend of type $(k\dots k)$ of itself). We are thus left to verify the first sentence of the statement.

The set $\mathcal{P}_k$ decomposes into Λ-orbits and each such orbit is mapped by $\pi_3 \circ a_k$ to a single Λ-orbit in X_m. Let $\Omega \subset \mathcal{P}_k$ be such an orbit. By Lemma 2.3 there exists $x_{k^{-1}\mathbf{u}} \in \Omega$ with $\mathbf{u}$ of the form (2.10). By Lemma 5.2, if we will show that $\pi_3(a_k x_{k^{-1}\mathbf{u}})$ is a Hecke friend of $\mathbb{Z}^m$ of the type prescribed in the statement, it would follow that the normalized counting measure on Ω is pushed by $(\pi_3)_*(a_k)_*$ to the normalized counting measure on the set of Hecke friends on $\mathbb{Z}^m$ of this type[7]. This implies the same statement regarding $\boldsymbol{\nu}_k$.

To this end, let $x_{k^{-1}\mathbf{u}} \in \mathcal{P}_k$ with $\mathbf{u}$ k-primitive of the form (2.10). We now calculate $\pi_3(a(k)x_{k^{-1}\mathbf{u}})$. Let $\gamma = \left(\begin{smallmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{smallmatrix}\right)$ be a matrix solving (2.12) so that

$$\pi_3(a(k)x_{k^{-1}\mathbf{u}}) = k^{-\frac{n}{m}} A_\gamma \mathbb{Z}^m. \tag{6.1}$$

It follows from the proofs of Lemmas 2.1, 2.4 that a possible choice for γ is the inverse of the matrix δ appearing in (2.11). A short computation shows that (using the formula for the inverse of a 2×2 matrix) this inverse is

$$\delta^{-1} = \begin{pmatrix} \operatorname{diag}(k \dots k) & 0 & -\operatorname{diag}(f_1 \dots f_n) \\ 0 & I_{n-m} & 0 \\ -\operatorname{diag}(\ell_1 \dots \ell_n) & 0 & \operatorname{diag}(e_1 \dots e_n) \end{pmatrix}. \tag{6.2}$$

In other words $\pi_3(a(k)x_{k^{-1}\mathbf{u}})$ is the m-dimensional unimodular lattice

$$k^{-\frac{n}{m}} \operatorname{diag}\left(\underbrace{k,\dots,k}_{n}, \underbrace{1,\dots,1}_{m-n}\right)\mathbb{Z}^m$$

which is a Hecke-friend of $\mathbb{Z}^m$ of the desired type. $\square$

6.3. **Step 3.** In this step we establish the convergence $a(k)_*\boldsymbol{\mu}_k \overset{\mathrm{w}^*}{\to} \boldsymbol{\theta}$. Before turning to the proof we need notation and a lemma. The orbit Hx_0 breaks into periodic V-orbits which are the π_3-fibers. In order to state the next lemma we need to define the notions of k-torsion and k-primitive points in such a fiber. For $x \in Hx_0$, we write $x = \left(\begin{smallmatrix} g & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0$ and set

$$\operatorname{Tor}_k(Vx) \overset{\text{def}}{=} \left\{ \left(\begin{smallmatrix} g & 0 \\ 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} I & k^{-1}\mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0 \in Vx : \mathbf{v} \in \operatorname{Mat}_{m\times n}(\mathbb{Z}) \right\}. \tag{6.3}$$

$$\operatorname{Tor}_k^{\text{prim}}(Vx) \overset{\text{def}}{=} \left\{ \left(\begin{smallmatrix} g & 0 \\ 0 & I \end{smallmatrix}\right) \left(\begin{smallmatrix} I & k^{-1}\mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0 \in Vx : \mathbf{v}^t \text{ is } k\text{-primitive} \right\}. \tag{6.4}$$

Note that although g is not well defined, the coset $g\,\mathrm{SL}_m(\mathbb{Z})$ is well defined and therefore $\operatorname{Tor}_k(Vx), \operatorname{Tor}_k^{\text{prim}}(Vx)$ are well defined as well.

Lemma 6.2. *For any positive integer k,*

$$a_k \mathcal{P}_k \subset \bigcup_{x \in X_m} \operatorname{Tor}_k^{\text{prim}}(\pi_3^{-1}(x)). \tag{6.5}$$

[7]Note that from here follows a stronger statement than the one sought in Step 2 which is the reason for the strengthening of Theorem 1.3 discussed in §7.

Proof. We first observe that the sets on both sides of (6.5) are Λ-invariant. Regarding $a_k\mathcal{P}_k$ this follows from Lemma 4.1(2). Regarding the set on the right, given $\lambda = \left(\begin{smallmatrix}\delta_1 & 0\\ 0 & \delta_2\end{smallmatrix}\right) \in \Lambda$ and a point $\left(\begin{smallmatrix}g & 0\\ 0 & I\end{smallmatrix}\right)\left(\begin{smallmatrix}I & k^{-1}\mathbf{v}\\ 0 & I\end{smallmatrix}\right)x_0 \in \mathrm{Tor}_k^{\mathrm{prim}}(\pi_3^{-1}(x))$, where $x = \left(\begin{smallmatrix}g & 0\\ 0 & I\end{smallmatrix}\right)x_0$ and $\mathbf{v}^t$ k-primitive, then

$$\lambda\left(\begin{smallmatrix}g & 0\\ 0 & I\end{smallmatrix}\right)\left(\begin{smallmatrix}I & k^{-1}\mathbf{v}\\ 0 & I\end{smallmatrix}\right)x_0 = \begin{pmatrix}\delta_1 g\delta_1^{-1} & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & k^{-1}\delta_1\mathbf{v}\delta_2^{-1}\\ 0 & I\end{pmatrix}x_0,$$

which belongs to $\mathrm{Tor}_k^{\mathrm{prim}}(\pi_3^{-1}(\lambda x))$ because $(\delta_1\mathbf{v}\delta_2^{-1})^t$ is k-primitive.

It now follows from Lemma 2.3 that it is enough to show that $a_k x_{k^{-1}\mathbf{u}}$ is in the right hand side of (6.5) for $\mathbf{u}$ as in (2.10). For such a choice of $\mathbf{u}$ we have seen in the course of the proof of Lemma 6.1 that the matrix γ solving equation (2.12) may be chosen to be as in (6.2). Thus (2.12) now becomes

$$a_k x_{k^{-1}\mathbf{u}} = \begin{pmatrix}k^{-m/n}A_\gamma & k^{-m/n}B_\gamma\\ 0 & I\end{pmatrix}x_0 = \begin{pmatrix}k^{-m/n}A_\gamma & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}I & A_\gamma^{-1}B_\gamma\\ 0 & I\end{pmatrix}x_0, \quad (6.6)$$

and indeed the concrete form of γ (6.2) gives by a straightforward calculation that

$$kA_\gamma^{-1}B_\gamma = \begin{pmatrix}-\mathrm{diag}(f_1\ldots f_n)\\ 0_{m-n\times m-n}\end{pmatrix}, \quad (6.7)$$

so indeed $(kA_\gamma^{-1}B_\gamma)^t$ is k-primitive. $\square$

In particular, we record the following corollary for future reference.

Corollary 6.3. *For any $k > k_0$ and any $x \in X_m$,* $\mathrm{supp}\,(a(k)_*\boldsymbol{\mu}_k) \cap \mathrm{Tor}_{k_0}(\pi_3^{-1}(x)) = \varnothing$.

We now turn to the proof of Step 3. Let σ be a weak* accumulation point of $\{a(k)_*\boldsymbol{\mu}_k\}_{k=1}^{\infty}$. By Lemmas 2.1, 4.1, σ is a Λ-invariant measure supported in Hx_0. By Step 2 we conclude that σ is a probability measure that projects under π_3 to either $\mathbf{m}_{X_m}$ or δ_{x_0} according to whether $m > n$, $m = n$ respectively. By Corollary 4.11, there are only countably many ergodic Λ-invariant probability measures supported in Hx_0 and so let $\{\sigma_i\}_{i=0}^{\infty}$ be an enumeration of them. By the ergodic decomposition we may represent $\sigma = \sum_{i=0}^{\infty} c_i\sigma_i$, where $c_i \geq 0$ and $\sum_{i=0}^{\infty} c_i = 1$.

<u>Assume $m = n$</u>. Let us denote $\sigma_0 = \mathbf{m}_{Vx_0}$. With this notation we will conclude this case once we show that $c_i = 0$ for $i > 0$. Fixing $i > 0$ and using the terminology of Definition 4.10, σ_i is of type (1)–(4). According to the type we will show that $c_i = 0$.

It is impossible that σ_i is of type (2) because since by Step 2 $(\pi_3)_*\sigma = \delta_{x_0}$, it would follow from Corollary 4.11(2) that $\sigma_i = \mathbf{m}_{Vx_0} = \sigma_0$. Similarly, Corollary 4.11 implies that if σ_i is of type (3) or (4) then $c_i = 0$. Finally, if i is such that σ_i is of type (1) then by Corollary 6.3 we have that the finite Λ-orbit on which σ_i is supported on is disjoint, for large values of k, from the support of $a(k)_*\boldsymbol{\mu}_k$. By the non-accumulation Theorem 3.1 we deduce that $c_i = 0$. Note that, similarly to the discussion at the end of §6.1, the application of Theorem 3.1 was done with the following choices: $G = \Lambda \ltimes V, \Gamma = \Lambda \ltimes \mathrm{Mat}_{m\times n}(\mathbb{Z})$, $L = \Lambda$, z being a point in the support of σ_i and $W = \mathrm{Lie}(V) = \mathrm{Mat}_{m\times n}(\mathbb{R})$ which indeed has no nonzero Λ-fixed vectors. This concludes the proof of Step 3 for the case $n = m$.

Assume $m > n$. Let us denote $\sigma_0 = \mathbf{m}_{Hx_0}$ – the only measure of type (4). With this notation we will conclude this case once we show that $c_i = 0$ for all $i > 0$. We do this by considering the possible types. By Step 2 $(\pi_3)_*\sigma = \mathbf{m}_{X_m}$ so we conclude from Corollary 4.11 that for any i such that σ_i is of type (1) or (2) we have that $c_i = 0$. Let $i > 0$ be such that σ_i is of type (3). Note that by Corollary 4.11, the intersection of the support of a measure of type (3) with any π_3-fiber $\pi_3^{-1}(x)$ is contained in $\mathrm{Tor}_{k_0}(\pi_3^{-1}(x))$ for a fixed k_0 that does not depend on the choice of $x \in X_m$. By Corollary 6.3 we conclude that for $k > k_0$ the support of $a(k)_*\boldsymbol{\mu}_k$ is disjoint from the periodic $\Lambda \cdot \mathrm{SL}_m(\mathbb{R})$-orbit on which σ_i is supported. It now follows from the Non-accumulation Theorem 3.1 that $c_i = 0$. The application of Theorem 3.1 is done with the following choices: $G = \Lambda \cdot H$, $\Gamma = \Lambda \ltimes \mathrm{Mat}_{m\times n}(\mathbb{Z})$, $L = \Lambda \cdot \mathrm{SL}_m(\mathbb{R})$, z any point in the support of σ_i, and $W = \mathrm{Lie}(V) = \mathrm{Mat}_{m\times n}(\mathbb{R})$. This concludes the proof of Step 3.

6.4. **Step 4.** In this step we conclude the proof of Theorem 1.3; that is, we establish the convergence $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_k \xrightarrow{\mathrm{w}^*} \mathbf{m}_{Ux_0} \times \boldsymbol{\theta}$. At some point close to the end of the proof we will need the following lemma[8].

Lemma 6.4. *In the case*[9] $m = n$,

$$\widetilde{a}_k\widetilde{\mathcal{P}}_k = \left\{\left(\left(\begin{smallmatrix} I & 0 \\ k^{-1}\mathbf{u} & I \end{smallmatrix}\right) x_0, \left(\begin{smallmatrix} I & k^{-1}\mathbf{u}^* \\ 0 & I \end{smallmatrix}\right) x_0\right) : \mathbf{u} \textit{ is } k\textit{-primitive}\right\},$$

where $\mathbf{u}^* \in \mathrm{Mat}_n(\mathbb{Z})$ *is an inverse of* $\mathbf{u}$ *modulo* k.

Proof. The proof is a further inspection of the arguments in Lemmas 6.1, 6.2 so we keep it terse. We need to show that $a_k x_{k^{-1}\mathbf{u}} = \left(\begin{smallmatrix} I & k^{-1}\mathbf{u}^* \\ 0 & I \end{smallmatrix}\right) x_0$ and by acting with Λ we see that it is enough to show this for $\mathbf{u}$ as in (2.10). For such $\mathbf{u}$ this follows from (6.6), (6.7), taking into account the concrete form of γ given by (6.2). $\square$

We now turn to the proof of Step 4. Let $\widetilde{\sigma}$ be a weak* accumulation point of the sequence $\{\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_k\}$. By Lemma 4.1(5), $\widetilde{\sigma}$ is Λ_Δ-invariant. By Step 3 we have that $(\pi_2)_*\widetilde{\sigma} = \boldsymbol{\theta}$ (this follows since π_2 is proper).

[8]This lemma is needed only in the case $m = n = 2$.

[9]Compare this with (2.4).

In particular, $\widetilde{\sigma}$ is a probability measure. Although the projection π_1 in diagram (2.9) is not proper, it is straightforward to argue that Step 1 implies now that $(\pi_1)_*\widetilde{\sigma} = \mathbf{m}_{Ux_0}$. That is, the dynamical system $(Ux_0 \times Hx_0, \widetilde{\sigma}, \Lambda)$ is a joining of $(Ux_0, \mathbf{m}_{Ux_0}, \Lambda)$, $(Hx_0, \boldsymbol{\theta}, \Lambda)$. By Theorems 4.12, 4.15, we deduce that if $(m,n) \neq (2,2)$ then $\widetilde{\sigma} = \mathbf{m}_{Ux_0} \times \boldsymbol{\theta}$ and the proof is concluded. We are thus left to deal with the case $n = m = 2$.

6.4.1. *The case $n = m = 2$.* As the projections $\mathbf{m}_{Ux_0}, \mathbf{m}_{Vx_0}$ are Λ-ergodic we conclude that (almost) any ergodic component of $\widetilde{\sigma}$ is a joining of $\mathbf{m}_{Ux_0}, \mathbf{m}_{Vx_0}$ as well. By Theorem 4.15 there are only countably many such ergodic components and we will be done once we prove the following lemma in which we use the notation of (4.5).

Lemma 6.5. *Let p, q be co-prime integers and let $(x, y) \in Ux_0 \times Vx_0$. Then, $\widetilde{\sigma}(L_{p,q}(x, y)) = 0$.*

Proof. By Corollary 3.4 we deduce that it is enough to prove that

$$\frac{|\widetilde{a}_k\widetilde{\mathcal{P}}_k \cap L_{p,q}(x,y)|}{|\widetilde{\mathcal{P}}_k|} \longrightarrow 0 \text{ as } k \to \infty. \tag{6.8}$$

Here the application of Corollary 3.4 is done with the following choices: $G = \Lambda_\Delta \ltimes (U \times V)$, $\Gamma = \Lambda_\Delta \ltimes (U(\mathbb{Z}) \times V(\mathbb{Z}))$. $L = L_{p,q}$ and $z = (x, y)$. The existence of the Λ_Δ-invariant linear complement of $\mathrm{Lie}(L_{p,q})$ follows from the semi-simplicity of the group $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and the fact that $\mathrm{Lie}(L_{p,q})$ is an invariant subspace of $\mathrm{Mat}_2(\mathbb{R}) \times \mathrm{Mat}_2(\mathbb{R})$ under the representation (4.6).

Let us denote for $\mathbf{v} \in \mathrm{Mat}_2(\mathbb{R})$, $y_{\mathbf{v}} \stackrel{\text{def}}{=} \left(\begin{smallmatrix} I & \mathbf{v} \\ 0 & I \end{smallmatrix}\right) x_0 \in Vx_0$, and for $\mathbf{u} \in \mathrm{GL}_2(\mathbb{Z}/k\mathbb{Z})$ we write (using the notation of Lemma (6.4)),

$$\Phi(\mathbf{u}) \stackrel{\text{def}}{=} (x_{k^{-1}\mathbf{u}}, y_{k^{-1}\mathbf{u}^*}). \tag{6.9}$$

Lemma 6.4 gives that Φ is a bijection between $\mathrm{GL}_2(\mathbb{Z}/k\mathbb{Z})$ and

$$\widetilde{a}_k\widetilde{\mathcal{P}}_k = \left\{\Phi(\mathbf{u}) : \mathbf{u} \in \mathrm{GL}_2(\mathbb{Z}/k\mathbb{Z})\right\}. \tag{6.10}$$

The left and right actions of $\mathrm{GL}_2(\mathbb{Z}/k\mathbb{Z})$ on itself induce (via Φ) actions of it on $\widetilde{\mathcal{P}}_k$. Fixing $\mathbf{u} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}/k\mathbb{Z})$ we will establish (6.8) by analyzing the orbit $\Phi\left(\{g_\ell \mathbf{u} : \ell \in (\mathbb{Z}/k\mathbb{Z})^\times\}\right)$, where $g_\ell \stackrel{\text{def}}{=} \left(\begin{smallmatrix} \ell & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which is of size $\phi(k)$ (here ϕ is the Euler function), and proving that

$$\frac{|\left\{\ell : \Phi(g_\ell\mathbf{u}) \text{ is in the same } L_{p,q}\text{-orbit of } \Phi(\mathbf{u})\right\}|}{\phi(k)} \stackrel{k\to\infty}{\longrightarrow} 0. \tag{6.11}$$

Following the definition we see that $\Phi(\mathbf{u}), \Phi(g_\ell\mathbf{u})$ are in the same $L_{p,q}$-orbit if and only if the difference $(g_\ell\mathbf{u}, \mathbf{u}^*g_\ell^*) - (\mathbf{u}, \mathbf{u}^*)$ (thought of as an element of $\mathrm{Mat}_2(\mathbb{R})\times\mathrm{Mat}_2(\mathbb{R})$) is in $\mathrm{Lie}(L_{p,q})+\mathrm{Mat}_2(k\mathbb{Z})\times\mathrm{Mat}_2(k\mathbb{Z})$. In other words, if we let $j=\det\mathbf{u}$ and recall that $\mathbf{u}^*=j^*\left(\begin{smallmatrix} d & -b\\ -c & a\end{smallmatrix}\right)$, $g_\ell^*=\ell^*\left(\begin{smallmatrix}1&0\\0&\ell\end{smallmatrix}\right)$, and the definition of $L_{p,q}$, we see that $\Phi(\mathbf{u}),\Phi(g_\ell\mathbf{u})$ are in the same $L_{p,q}$-orbit if and only if

$$p\begin{pmatrix} 0 & (1-\ell)b\\ 0 & (\ell-1)a\end{pmatrix}=q\begin{pmatrix}(\ell^*-1)j^*d & 0\\ (1-\ell^*)j^*c & 0\end{pmatrix}\mod k\mathbb{Z}. \tag{6.12}$$

We claim that (6.12) implies $\ell=1\mod k\mathbb{Z}$. For that purpose we split $k=k_1k_2$ with k_1 coprime to p and k_2 coprime to q. Using the second column in (6.12) we get $(\ell-1)b\in k_1\mathbb{Z}$ and $(\ell-1)a\in k_1\mathbb{Z}$. However, since a,b form the first row of an invertible matrix modulo k we see that the greatest common divisor of a,b,k_1 is one and we conclude $\ell=1\mod k_1\mathbb{Z}$. Using the first column of (6.12) and invertibility of j^* modulo k_2 in the same way we get $\ell^*=1\mod k_2\mathbb{Z}$. Together this shows $\ell=1\mod k\mathbb{Z}$ and so (6.11) (since $\phi(k)\to\infty$). This concludes the proof the lemma and by that the proof of Theorem 1.3. □

7. A strengthening of Theorem 1.3

Before we turn to the closing section of this paper we wish to comment that the proof presented above of Theorem 1.3 actually establishes more than stated. So far we looked at (various images of) the counting measure on $\widetilde{\mathcal{P}}_k$ and used its Λ_Δ-invariance but $\widetilde{\mathcal{P}}_k$ splits into Λ_Δ-orbits and we could restrict our attention to single orbits. Let

$$I_k\overset{\mathrm{def}}{=}\left\{\vec{\ell}=(\ell_1,\dots,\ell_n)\in\mathbb{Z}^n:\ell_i|\ell_{i+1},\gcd(\ell_i,k)=1\right\}, \tag{7.1}$$

and denote by $[I_k]$ the image of I_k in $(\mathbb{Z}/k\mathbb{Z})^n$. We refer to the elements of $[I_k]$ as *k-primitive elementary divisors tuples* and sometimes abuse notation and do not distinguish between elements in $[I_k]$ and their integer representatives. It follows from Lemma 2.3 that a Λ_Δ-orbit in $\widetilde{\mathcal{P}}_k$ is a set of the form

$$\mathcal{P}_{k,\vec{\ell}}\overset{\mathrm{def}}{=}\{(x_{k^{-1}\mathbf{u}},x_{k^{-1}\mathbf{u}}):\text{ the elementary divisors tuple of }\mathbf{u}\text{ equals }\vec{\ell}\bmod k\}.$$

Let $\widetilde{\boldsymbol{\mu}}_{k,\vec{\ell}}$ be the normalized counting measure on $\widetilde{\mathcal{P}}_{k,\vec{\ell}}$ so that $\widetilde{\boldsymbol{\mu}}_k$ is an average of the $\widetilde{\boldsymbol{\mu}}_{k,\vec{\ell}}$'s. An examination of the proof presented above for Theorem 1.3 establishes the following:

Theorem 7.1. *If*[10] $(m,n)\neq(2,2)$. *Then, as* $k\to\infty$ *and* $\vec{\ell}\in I_k$ *is arbitrary* $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,\vec{\ell}}\overset{\mathrm{w}^*}{\to}\mathbf{m}_{Ux_0}\times\boldsymbol{\theta}$.

[10] Recall that we exclude the case $(m,n)=(1,1)$ from our discussion.

The reason for excluding the case $(m,n)=(2,2)$ is that the only place in the proof of Theorem 1.3 where we used something other than the Λ-invariance, was in the proof of the case $(m,n)=(2,2)$ (towards the end of step 4 where GL_2 played a role). In fact, if $n=m=2$ and we choose for any k the k-primitive elementary divisors tuple to be $\vec{\ell}=(1,1)$ then the set $\operatorname{supp}(\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,\vec{\ell}})=\widetilde{a}(k)\widetilde{\mathcal{P}}_{k,\vec{\ell}}$, which is the Λ_Δ-orbit of $\Phi\left(\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right)$ (Φ as in (6.9)), is contained in the image in the fixed orbit $L_{1,1}(x_0,x_0)\subset Ux_0\times Vx_0$ (notation being as in (4.5)). In particular, $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,\vec{\ell}}$ cannot be expected to converge to $\mathbf{m}_{Ux_0}\times\mathbf{m}_{Vx_0}$. This shows that the statement of Theorem 7.1 is simply false in this case.

8. Proof of a strengthening of Theorem 1.4

Below we state and prove a generalization of Theorem 1.4. Recall the notation of §1.2 and in particular, that $n=1$ and so $d=m+1$. Theorem 7.1 obtains a rather concrete meaning: The set $[I_k]$ from (7.1) is simply $(\mathbb{Z}/k\mathbb{Z})^\times$ and the elementary divisor tuple of a vector $\mathbf{u}$ is simply $\gcd(\mathbf{u})$ so that for $c_k\in(\mathbb{Z}/k\mathbb{Z})^\times$, the measure $\widetilde{\boldsymbol{\mu}}_{k,c_k}$ is simply the normalized counting measure on the diagonal embedding of the collection $\{x_{k^{-1}\mathbf{u}}:\gcd(\mathbf{u})=c_k\}$ in $Ux_0\times Ux_0$. Theorem 7.1 states that $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,c_k}\overset{\mathrm{w}^*}{\to}\boldsymbol{m}_{Ux_0}\times\mathbf{m}_{Hx_0}$ for any choice of $c_k\in(\mathbb{Z}/k\mathbb{Z})^\times$. In particular, it follows that if $\boldsymbol{\mu}_{k,c_k}\overset{\mathrm{def}}{=}(\pi_2)_*\widetilde{\boldsymbol{\mu}}_{k,c_k}$, then $a(k)_*\boldsymbol{\mu}_{k,c_k}\to\mathbf{m}_{Hx_0}$. We shall need the following corollary of Theorem 7.1 which gives a significant strengthening of this implication.

Corollary 8.1. *Let $F\subset Ux_0$ be a subset satisfying* (i) $\mathbf{m}_{Ux_0}(F)>0$, (ii) $\mathbf{m}_{Ux_0}(\partial F)=0$ *(where ∂F denote the boundary of F in Ux_0), and let $f\overset{\mathrm{def}}{=}\frac{1}{\mathbf{m}_{Ux_0}(F)}\chi_F$. For any sequence $c_k\in(\mathbb{Z}/k\mathbb{Z})^\times$, $a(k)_*f\boldsymbol{\mu}_{k,c_k}\overset{\mathrm{w}^*}{\to}\boldsymbol{m}_{Hx_0}$.*

Proof. Given a test function $\varphi\in C_c(X_d)$ we consider the function $(f\times\varphi)$ on $Ux_0\times X_d$ given by $(f\times\varphi)(x,y)=f(x)\varphi(y)$. Although $(f\times\varphi)$ is not continuous, assumption (ii) implies that the points of discontinuity are of $\mathbf{m}_{Ux_0}\times\mathbf{m}_{Hx_0}$-measure zero. This in turn gives that the weak* convergence $\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,c_k}\to\mathbf{m}_{Ux_0}\times\mathbf{m}_{Hx_0}$ from Theorem 7.1 implies the convergence

$$\begin{aligned}\int_{X_d\times X_d}(f\times\varphi)d\widetilde{a}(k)_*\widetilde{\boldsymbol{\mu}}_{k,c_k}&\to\int_{X_d\times X_d}(f\times\varphi)d\left(\mathbf{m}_{Ux_0}\times\mathbf{m}_{Hx_0}\right)\\&=\int_{X_d}\varphi d\mathbf{m}_{Hx_0}.\end{aligned}$$

We conclude that indeed $a(k)_*fd\boldsymbol{\mu}_{k,c_k}\overset{\mathrm{w}^*}{\to}\mathbf{m}_{Hx_0}$ as desired. $\square$

In the course of proving Theorem 1.4 it is clearly harmless to assume that the set $\mathcal{F}$ is contained in the face $\left\{(u_1,\dots,u_m,1)\in\mathbb{R}^d : |u_i|<1\right\}\subset\partial B_\infty$. Recall that η_k is the normalized counting measure on the set $\left\{[\Lambda_v] : v\in k\mathcal{F}, v=(\mathbf{u},k), \gcd(\mathbf{u},k)=1\right\}$. We split this set (and η_k) according to the Λ-orbit of the various $\mathbf{u}$'s: We denote for $1\le c\le k$ with $\gcd(c,k)=1$ by $\eta_{k,c}$ the normalized counting measure on $\left\{[\Lambda_v] : v\in k\mathcal{F}, v=(\mathbf{u},k), \gcd(\mathbf{u})=c\right\}$, then η_k is an average of the $\eta_{k,c}$'s. We conclude that Theorem 1.4 is implied by the following.

Theorem 8.2. *For any positive integer k choose $1\le c_k\le k$ with $\gcd(k,c_k)=1$. Let $\mathcal{F}\subset\left\{(u_1,\dots,u_m,1)\in\mathbb{R}^d : |u_i|<1\right\}$ be a measurable set such that its boundary in ∂B_∞ has measure 0 with respect to the m-dimensional Lebesgue measure on ∂B_∞. Then, $\eta_{k,c_k}\overset{\mathrm{w}^*}{\to}\mathbf{m}_{Z_m}$ as $k\to\infty$.*

Proof. Let us use the following ad-hoc notation: Given co-prime integers c,k we denote $\widehat{\mathbb{Z}}^d_{k,c}\overset{\mathrm{def}}{=}\left\{(\mathbf{u},k)\in\mathbb{Z}^d : \gcd(\mathbf{u})=c\right\}$. We need to show that for any $f\in C_c(Z_m)$, we have that

$$\frac{1}{|\widehat{\mathbb{Z}}^d_{k,c_k}\cap k\mathcal{F}|}\sum_{v\in\widehat{\mathbb{Z}}^d_{k,c_k}\cap k\mathcal{F}} f([\Lambda_v])\to\int_{Z_m} f d\mathbf{m}_{Z_m} \text{ as } k\to\infty. \tag{8.1}$$

Fix such a function f and $\epsilon>0$. Choose a finite partition $\mathcal{F}=\sqcup_{i=1}^{\ell}F_i$, where the sets F_i are measurable, have boundary of Lebesgue measure zero (relative to ∂B_∞), and have diameter $\le\delta$, where δ is chosen small enough in a manner soon to be explained. In order to conclude the proof we will show that once δ is chosen small enough, for any large enough k and for any $1\le i\le\ell$, we have that

$$\left|\int_{Z_m} f d\mathbf{m}_{Z_m}-\frac{1}{|\widehat{\mathbb{Z}}^d_{k,c_k}\cap kF_i|}\sum_{v\in\widehat{\mathbb{Z}}^d_{k,c_k}\cap kF_i} f([\Lambda_v])\right|\le\epsilon. \tag{8.2}$$

To this end, fix $1\le i\le\ell$ and denote $F=F_i$.

We identify $\left\{(u_1,\dots u_m,1)\in\mathbb{R}^d : 0\le u_i\le 1\right\}$ with $Ux_0\in X_d$ in the obvious manner; that is $(\mathbf{u},1)$ corresponds to $x_{\mathbf{u}}$. This way we may think of F as a subset of X_d or $\mathbb{R}^d$ at our convenience. It follows from the discussion in (2) after (2.12) that if we denote by $p:\mathbb{R}^d\to\mathbb{R}^m$ the projection onto the first m coordinates, then the measure $(\pi_3)_*a(k)_*\left(\chi_F d\boldsymbol{\mu}_{k,c_k}\right)$ on X_m (when normalized to be a probability measure), is the normalized counting measure supported on the collection

$$\left\{\frac{1}{\operatorname{covol}(p(\Lambda_v))}p(\Lambda_v)\in X_m : v\in\widehat{\mathbb{Z}}^d_{k,c_k}\cap kF\right\}$$

(where here we think of F as a subset of ∂B_∞ and of kF as a subset of $\mathbb{R}^d$). Corollary 8.1 tells us that these measures equidistribute to $\mathbf{m}_{X_m}$ as $k \to \infty$. In particular, after projecting to Z_m we see that

$$\frac{1}{|\widehat{\mathbb{Z}}^d_{k,c_k} \cap kF|} \sum_{v \in \widehat{\mathbb{Z}}^d_{k,c_k} \cap kF} \delta_{[p(\Lambda_v)]} \xrightarrow{\mathrm{w}^*} \mathbf{m}_{Z_m} \text{ as } k \to \infty. \tag{8.3}$$

This is not too far from implying (8.2); we only need to deal with the distortion that the orthogonal projection p brings into the picture. We will see that this distortion is controllable once δ (and hence the diameter of F) is small enough.

Choose a reference point $v_0 \in F$ and let S be the inverse of the restriction of the projection p to the m-dimensional linear space $W_0 \overset{\text{def}}{=} \{v_0\}^\perp$. Consider the function $\varphi \in C_c(X_m)$ defined by $\varphi(\Lambda) \overset{\text{def}}{=} f([S(\Lambda)])$. Here we think of a point $\Lambda \in X_m$ as a lattice in the copy of $\mathbb{R}^m$ given by $p(\mathbb{R}^d)$ and so $S(\Lambda)$ is an m-dimensional discrete subgroup of W_0 and so $[S(\Lambda)] \in Z_m$ is well defined. Note that $\int_{X_m} \varphi d\mathbf{m}_{X_m} = \int_{Z_m} f d\mathbf{m}_{Z_m}$. Applying (8.3) to the function φ we conclude that

$$\lim_k \left| \frac{1}{|\widehat{\mathbb{Z}}^d_{k,c_k} \cap kF|} \sum_{v \in \widehat{\mathbb{Z}}^d_{k,c_k} \cap kF} f([S(p(\Lambda_v))] - \int_{X_m} f d\mathbf{m}_{X_m} \right| = 0. \tag{8.4}$$

Let $\mathrm{d}_{Z_m}(\cdot,\cdot)$ denote the distance function on Z_m. Let $\mathrm{Cone}(F) \overset{\text{def}}{=} \{tF : t > 0\}$. It is straightforward to show

$$\sup_{v \in \mathbb{Z}^d \cap \mathrm{Cone}(F)} \{\mathrm{d}_{Z_m}([\Lambda_v], [S(p(\Lambda_v))])\} \to 0 \text{ as } \operatorname{diam}(F) \to 0. \tag{8.5}$$

Using (8.5) and the uniform continuity of f we see that once $\delta > 0$ is chosen small enough, $|f([S(p(\Lambda_v))]) - f([\Lambda_v])| \le \epsilon$. Plugging this into (8.4) gives (8.2) as desired. □

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References

[AESa] M. Aka, M. Einsiedler, and U. Shapira, *Integer points on spheres and their orthogonal grids*. Preprint.

[AESb] M. Aka, M. Einsiedler, and U. Shapira, *Integer points on spheres and their orthogonal lattices (with an appendix by Ruixiang Zhang)*. Preprint.

[COU01] L. Clozel, H. Oh, and E. Ullmo, *Hecke operators and equidistribution of Hecke points*, Invent. Math. **144** (2001), no. 2, 327–351. MR1827734 (2002m:11044)

[Ein06] M. Einsiedler, *Ratner's theorem on* $\mathrm{SL}(2,\mathbb{R})$*-invariant measures*, Jahresber. Deutsch. Math.-Verein. **108** (2006), no. 3, 143–164. MR2265534 (2008b:37048)

[EO06] A. Eskin and H. Oh, *Ergodic theoretic proof of equidistribution of Hecke points*, Ergodic Theory Dynam. Systems **26** (2006), no. 1, 163–167. MR2201942 (2006j:11068)

[Gla03] E. Glasner, *Ergodic theory via joinings*, Mathematical Surveys and Monographs, vol. 101, American Mathematical Society, Providence, RI, 2003. MR1958753 (2004c:37011)

[Iwa02] H. Iwaniec, *Spectral methods of automorphic forms*, Second, Graduate Studies in Mathematics, vol. 53, American Mathematical Society, Providence, RI, 2002. MR1942691 (2003k:11085)

[KSS02] D. Kleinbock, N. Shah, and A. Starkov, *Dynamics of subgroup actions on homogeneous spaces of Lie groups and applications to number theory*, Handbook of dynamical systems, Vol. 1A, 2002, pp. 813–930. MR1928528 (2004b:22021)

[Mar04] G. A. Margulis, *On some aspects of the theory of Anosov systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2004. With a survey by Richard Sharp: Periodic orbits of hyperbolic flows, Translated from the Russian by Valentina Vladimirovna Szulikowska. MR2035655 (2004m:37049)

[Mar10] J. Marklof, *The asymptotic distribution of Frobenius numbers*, Invent. Math. **181** (2010), no. 1, 179–207. MR2651383 (2011e:11133)

[MS95] S. Mozes and N. Shah, *On the space of ergodic invariant measures of unipotent flows*, Ergodic Theory and Dynamical Systems **15** (1995), no. 1, 149–159.

[MS] J. Marklof and A. Strömbersson, *Diameters of random circulant graphs.*

[Ra91a] M. Ratner, *On Raghunathan's measure conjecture*, Ann. of Math. (2) **134** (1991), no. 3, 545–607. MR1135878 (93a:22009)

[Ra91b] M. Ratner, *Raghunathan's topological conjecture and distributions of unipotent flows*, Duke Math. J. **63** (1991), no. 1, 235–280. MR1106945 (93f:22012)

[Sch98] W. M. Schmidt, *The distribution of sublattices of* $\mathbf{Z}^m$, Monatsh. Math. **125** (1998), no. 1, 37–81. MR1485976 (99c:11083)

[SD01] H. P. F. Swinnerton-Dyer, *A brief guide to algebraic number theory*, London Mathematical Society Student Texts, vol. 50, Cambridge University Press, Cambridge, 2001. MR1826558 (2002a:11117)

[Sha98] N. A. Shah, *Invariant measures and orbit closures on homogeneous spaces for actions of subgroups generated by unipotent elements*, Lie groups and ergodic theory (Mumbai, 1996), 1998, pp. 229–271. MR1699367 (2001a:22012)

[Wit94] D. Witte, *Measurable quotients of unipotent translations on homogeneous spaces*, Trans. Amer. Math. Soc. **345** (1994), no. 2, 577–594. MR1181187 (95a:22005)