UNIPOTENT FLOWS ON HOMOGENEOUS SPACES

A THESIS

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BY

NIMISH A. SHAH

TATA INSTITUTE OF FUNDAMENTAL RESEARCH

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Dedicated to

Rao Saheb

who inspired me to become a mathematician.

Preface

This thesis is concerned with developing a technique for studying problems related to unipotent flows on homogeneous spaces of Lie groups. The methods are based on the work of S.G. Dani, G.A. Margulis and Marina Ratner on Raghunathan's conjecture.

My teachers and well-wishers have all contributed towards this thesis by the way of encouragement and guidance at different stages of my life. I thank them all.

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Contents

Introduction

Let G be a connected Lie group and Γ be a lattice in G; that is, Γ is a discrete subgroup of G such that the quotient space $X = G/\Gamma$, called a *homogeneous space* of G, admits a finite G-invariant measure. The actions of subgroups of G on X form a natural class of dynamical systems referred to as *flows on homogeneous spaces*.

For a closed subgroup F of G and a point $x \in X$, if the orbit $Y = Fx$ is closed and admits a finite F-invariant measure, say μ, then Y is called a *homogeneous subset* of X and μ is called a *homogeneous measure* on X.

A subgroup U of G is said to be *unipotent* if for every $u \in U$, the adjoint automorphism $\mathrm{Ad}u$ of the Lie algebra of G has all eigenvalues equal to 1. The flows defined by the actions of unipotent subgroups are called *unipotent flows*.

The dynamical behaviour of unipotent flows and its connection with problems in Diophantine approximation have generated a great deal of interest in the study of unipotent flows. In an attempt to deal with a long standing conjecture due to Oppenheim, on values of quadratic forms at integral points, through the study of subgroup actions on homogeneous spaces, Raghunathan conjectured that *the closures of orbits of unipotent flows are homogeneous sets.* For horospherical flows the latter conjecture was shown to be true by Dani [6]. Later Margulis [21] settled the Oppenheim's conjecture by verifying a special case of Raghunathan's conjecture. The reader is referred to the survey articles by Dani [4, 10], Margulis [20, 22], and Ratner [27, 32, 33] for the past and the recent developements in the area.

In a remarkable acheivement recently the Raghunathan's conjecture was fully proved by Ratner [31]. A major component of her proof [30] involves the following classification of invariant measures.

Theorem A (Description of invariant measures) *Any Borel probability measure invariant and ergodic under under a unipotent flow is a homogeneous measure.*

In fact, let L be a subgroup of G which is generated by the unipotent one-parameter subgroups of G contained in it. Then every L-invariant and L-ergodic Borel probability measure on X is a homogeneous measure.

Using this classification, Ratner [31] proved the following result, which in particular settles Raghunathan's conjecture.

A curve $c : [0, \infty) \to X$ is said to be *uniformly distributed* with respect to a probability measure μ on X, if for every bounded continuous function f on X,

$$\lim_{T\to\infty} \int_0^T f(c(t))\,dt = \int_X f\,d\mu.$$

Theorem B (Limit distributions of trajectories) *Let $u : \mathbf{R} \to G$ be a unipotent one-parameter subgroup. Then for any $x \in X$, the trajectory $\{u(t)x : t > 0\}$ is uniformly distributed with respect to a homogeneous measure μ on X such that $x \in \operatorname{supp}(\mu)$. In particular, $\overline{\{u(t)x : t > 0\}}$ is a homogeneous subset of X.*

One also deduces the following generalization of Raghunathan's conjecture from this result.

Theorem C (Closures of orbits) *Let L be a subgroup of G which is generated by the unipotent one-parameter subgroups of G contained in it. Then the closure of any orbit of L in X is a homogeneous set.*

For semisimple groups G of $\mathbf{R}$-rank 1, the present author [35] had independently derived theorem B from theorem A, using the approach of [15, 12].

In this thesis we take a closer look at the space of ergodic invariant measures in the context of theorem A, obtain new results in dynamics of polynomial trajectories on homogeneous spaces of linear Lie groups, and classify topological factors of G-actions on $Y \times G/P$, where G is a simple Lie group of $\mathbf{R}$-rank ≥ 2 with finite center, P is a parabolic subgroup of G, and Y is a homogeneous space of a Lie group containing G. The technique is motivated by the work of Dani and Margulis [14]. The details of the main results are described below, chapterwise.

Chapter 1 is devoted to certain growth properties of polynomials of several variables; we extend a theorem of Dani and Margulis [13, Theorem 1] proved for unipotent one-parameter subgroups.

In Chapter 2 theorem A is applied to show that the finite invariant measures of a unipotent flow other than the G-invariant ones are supported on the image in X of a countable union of certain algebraic subvarieties of G. We develop a 'linerization procedure' which allows us to study the behaviour of 'polynomial trajectories' near the images of these algebraic subvarieties in X. The results proved in the subsequent chapters are deduced from the technical results proved in this chapter.

In Chapter 3 we obtain the following result on the space of ergodic invariant measures of unipotent flows. Let $\mathcal{U}(X)$ denote the space of all Borel probability measures μ on X such that μ is invariant and ergodic with respect to the action of a unipotent one-parameter subgroup of G. By theorem A, $\mathcal{U}(X)$ consists of homogeneous measures.

Theorem 0.1 *The space $\mathcal{U}(X)$ is closed in the space of all Borel probability measures on X. More precisely, if $\{\mu_i\} \subset \mathcal{U}(X)$ is a sequence converging weakly to a Borel probability measure μ on X then $\mu \in \mathcal{U}(X)$ and there exists a sequence $g_i \to e$ such that $g_i \cdot \operatorname{supp}(\mu_i) \subset \operatorname{supp}(\mu)$ for all but finitely many i's.*

In view of a theorem of Dani and Margulis [14, Theorem 6.1] the above result implies the following.

Corollary 0.2 *Given a compact set $C \subset X$, the set $\{\mu \in \mathcal{U}(X) : \operatorname{supp}(\mu) \cap C \neq \emptyset\}$ is compact.*

In Chapter 4 we obtain the following results on limit distributions of polynomial trajectories on homogeneous spaces.

A map $\Theta : \mathbf{R}^k \to \mathrm{SL}_n(\mathbf{R})$ is called a *polynomial* map if every matrix coefficient of Θ is a polynomial on $\mathbf{R}^k$. A polynomial map $\Theta : \mathbf{R}^k \to \mathrm{SL}_n(\mathbf{R})$ is said to be of *split type* if $\Theta(t_1, \ldots, t_k) = \theta_k(t_k) \cdots \theta_1(t_1)$, $\forall (t_1, \ldots, t_k) \in \mathbf{R}^k$, where $\theta_i : \mathbf{R} \to \mathrm{SL}_n(\mathbf{R})$ are polynomial maps.

Theorem 0.3 *Suppose that G is a closed subgroup of $\mathrm{SL}_n(\mathbf{R})$ and let $\Theta : \mathbf{R}^k \to G$ be a polynomial map with $\Theta(0) = e$. Then for any $x \in X$, there exists a measure $\mu \in \mathcal{U}(X)$ such that $x \in \mathrm{supp}(\mu)$ and for any sequence $\{B_i\}$ of balls in $\mathbf{R}^k$ centered at 0 with $\mathrm{radius}(B_i) \to \infty$ and any $f \in \mathrm{C_b}(X)$,*

$$\lim_{i\to\infty} \frac{1}{\mathrm{vol}(B_i)} \int_{\mathbf{t}\in B_i} f(\Theta(\mathbf{t})\Gamma)\, d\mathbf{t} = \int f\, d\mu.$$

In particular, $\overline{\{\Theta(\mathbf{t})x : \mathbf{t} \in \mathbf{R}^k\}} = Fx$, where $F = \{g \in G : g \cdot \mu = \mu\}$.

Further, if we assume that Θ is of split type, the above result also holds for any sequence of boxes $B_i = [0, T_i^{(1)}] \times \cdots \times [0, T_i^{(k)}] \subset \mathbf{R}^k$ with each $T_i^{(l)} \to \infty$.

Note that a unipotent one-parameter subgroup of $\mathrm{SL}_n(\mathbf{R})$ is a polynomial map. Thus theorem 0.3 generalizes theorem B. We deduce the following result on limit distributions of orbits of higher dimensional unipotent flows, solving a problem proposed by Ratner in [31, p.236] and [32, Problem 2].

Corollary 0.4 *Let N be a simply connected unipotent subgroup of G. Let $\Theta : \mathbf{R}^k \to N$ be a map defined by $\Theta(t_1, \ldots, t_k) = (\exp t_k b_k) \cdots (\exp t_1 b_1)$, where $\{b_1, \ldots, b_k\}$ is a basis of the Lie algebra of N such that under Θ the Lebesgue measure on $\mathbf{R}^k$ projects to a Haar measure λ on N; (such bases always exist). Then for any $x \in X$, there exists a measure $\mu \in \mathcal{U}(X)$ such that $x \in \mathrm{supp}(\mu)$ and for any $f \in \mathrm{C_b}(X)$,*

$$\lim_{s_1,\ldots,s_k\to\infty} \frac{1}{\lambda(\Theta([0,s_1] \times \cdots \times [0,s_k]))} \int_{h\in\Theta([0,s_1]\times\cdots\times[0,s_k])} f(hx)\, d\lambda(h) = \int f\, d\mu.$$

The next result is a uniform version of theorem 0.3 and it generalizes [14, Theorem 3] proved for one-parameter unipotent subgroups.

Theorem 0.5 *Let $\Theta : \mathbf{R}^k \to G \subset \mathrm{SL}_n(\mathbf{R})$ be a polynomial map with $\Theta(0) = 0$. Let a compact set $K \subset X$, a function $f \in \mathrm{C_b}(X)$, and an $\epsilon > 0$ be given. Then there exist finitely many closed subgroups $H_1, \ldots, H_r$ of G, with each orbit $H_j\Gamma$ being homogeneous in X, and compact sets*

$$C_j \subset \{g \in G : \Theta(\mathbf{R}^k)g \subset gH_j\}, \quad j = 1, \ldots, r,$$

such that the following holds: For any compact set $K_1 \subset K \setminus \bigcup_{j=1}^r C_j\Gamma$ there exists $T_0 > 0$ such that for any $x \in K_1$ and any ball B in $\mathbf{R}^k$ centered at 0 with $\mathrm{Radius}(B) > T_0$,

$$\left| \frac{1}{\mathrm{vol}(B)} \int_{\mathbf{t}\in B} f(\Theta(\mathbf{t})x)\, d\mathbf{t} - \int f\, d\mu_G \right| < \epsilon,$$

where μ_G is the G-invariant probability measrue on X.

Further if we assume that Θ is of split type, then the above result holds for any box $B = [0, s^{(1)}] \times \cdots \times [0, s^{(k)}]$ with each $s^{(l)} > T_0$.

In Chapter 5, we prove certain results about actions of semisimple groups G on homogeneous spaces of larger Lie groups. These results were known earlier only for the actions of G on its own homogeneous spaces.

Theorem 0.6 *Let L be a Lie group, Λ a lattice in L, $\pi : L \to L/\Lambda$ the natural quotient map, and μ_L the L-invariant probability measure on L/Λ. Let $a \in G$ be a semisimple element and $U = \{u \in G : \lim_{n\to\infty} a^{-n}ua^n = e\}$ be the associated horospherical subgroup. Let Ω be a relatively compact neighbourhood of e in U such that π is injective on Ω. Let μ_Ω be the probability measure on $\pi(\Omega)$ which is the image of the restriction of a Haar measure on U to Ω. Assume that $\pi(G_1)$ is dense in L/Λ for any normal subgroup G_1 of G containing U. Then the sequence of measures $a^n \cdot \mu_\Omega$ converges weakly to μ_L. In particular, $\pi(\{a^n : n \in \mathbf{N}\} \cdot U)$ is dense in L/Λ.*

The fixed point set of an involution (an automorphism of order 2) of a semisimple group G is said to be *symmetric* subgroup of G. For example, $\mathrm{SO}(n)$ is the fixed point set of the involution of $\mathrm{SL}_n(\mathbf{R})$ given by $g \mapsto {}^{\mathrm{t}}g^{-1}$.

Corollary 0.7 *Let L, Λ, π, and μ_L be as in theorem 0.6. Let G be connected semisimple Lie subgroup of L with finite center. Let H be a symmetric subgroup of G such that the orbit $\pi(H)$ admits a (unique) H-invariant probability measure, say μ_H. Let $\{g_i\}$ be a sequence in G. Suppose that $\pi(G_1)$ is dense in L/Λ for any closed connected normal subgroup G_1 of G such that the image of $\{g_i\}$ in $G/(G_1H)$ has a convergent subsequence. Then the sequence of measures $g_i \cdot \mu_H$ converges weakly to μ_L.*

This result generalises a theorem of Duke, Rudnik and Sarnak [16] (cf. Eskin and McMullen [17]), where the case of $L = G$ is considered.

We also apply the results to study equivariant maps and address a question raised by Stuck and Zimmer [39, Problem C]. The following result is obtained in this respect.

Corollary 0.8 *Let L be a Lie group and Λ a lattice in L, G a connected semisimple Lie subgroup of L with finite center. Suppose that the action of G_1 on $X = L/\Lambda$ is minimal for any closed connected normal subgroup G_1 of G such that $\mathbf{R}\text{-rank}(G/G_1) \leq 1$. Let P be a parabolic subgroup of G, Y a Hausdorff space with a continuous G-action, and $X \times G/P \to Y \to X$ continuous surjective G-equivariat maps such that the composition is the projection on X. Then Y is G-equivariantly homeomorphic to $X \times G/P'$ for some parabolic subgroup P' of G containing P.*

This result extends a theorem of Dani [7], where the special case of $L = G$ is proved.

Chapter 1

Polynomials and returning to compact sets

1.1 Growth properties of polynomial functions

Certain growth properties of polynomials of bounded degrees observed by Margulis in [19] have played a key role in understanding the dynamics of individual orbits of unipotent flows. Here we generalize these properties for polynomials of several variables.

Notation 1.1 For $d \in \mathbf{N}$, let $\mathcal{P}_d$ denote the space of real polynomials of degree at most d.

Fix $d \in \mathbf{N}$. By Lagrange's interpolation formula, for any $t_0 < t_1 < \ldots < t_d$, any $f \in \mathcal{P}_d$, and $t \in \mathbf{R}$,

$$f(t) = \sum_{k=0}^{d} f(t_k) \prod_{i \neq k} \frac{t - t_i}{t_k - t_i}. \tag{1.1}$$

For any measurale set $E \subset \mathbf{R}^n$, the Lebesgue measure of E is denoted by $|E|$.

Lemma 1.2 *There exists a constant $M > 0$ such that for any nonempty bounded open interval $I \subset \mathbf{R}$ and any $f \in \mathcal{P}_d$,*

$$|I| \cdot \sup_{t \in I} \left| \frac{d}{dt} f(t) \right| \leq M \cdot \sup_{t \in I} |f(t)|.$$

Lemma 1.3 *There exists a constant $M \geq 1$ such that for any nonempty bounded open intervals $I \subset J$ in $\mathbf{R}$ and a function $f \in \mathcal{P}_d$,*

$$\sup_{t \in J} |f(t)| \leq M \cdot \frac{|J|^d}{|I|^d} \cdot \sup_{t \in I} |f(t)|.$$

Proofs of lemma 1.2 and lemma 1.3: By a linear change of variable, we may assume that $I = (0, 1)$. Put $t_i = i/d$ for $i = 0, \ldots, d$. Put $P_k(t) = \prod_{i \neq k} \frac{t - t_i}{t_k - t_i}$ for $k = 0, \ldots, d$.

By equation 1.1,

$$\frac{d}{dt}f(t) = \sum_{k=0}^{d} f(t_k)\frac{d}{dt}P_k(t).$$

Now

$$\sup\left\{\left|\frac{d}{dt}P_k(t)\right| : t \in (0,1),\ k = 0,\ldots,d\right\} \le d^{d+1}.$$

Therefore

$$\sup_{t\in(0,1)}\left|\frac{d}{dt}f(t)\right| \le (d+1)d^{d+1}\cdot \sup_{t\in(0,1)}|f(t)|.$$

This proves lemma 1.2.

Also $\sup\{|P_k(t)| : t \in J,\ k = 0,\ldots,n\} \le (d|J|)^d$. Therefore by equation 1.1, we have $\sup t \in J|f(t)| \le ((d+1)d^d)|J|^d \cdot \sup_{t\in(0,1)}|f(t)|$. This proves lemma 1.3. □

Lemma 1.4 *There exists a constant $M \ge 1$ such that for any $\epsilon > 0$, a bounded interval $J \subset \mathbf{R}$ and $f \in \mathcal{P}_d$, if we put*

$$E = \{s \in J : |f(s)| < (\epsilon^d/M)\cdot \sup_{t\in J}|f(t)|\},$$

then $|E| \le \epsilon \cdot |J|$.

Proof. The set E has at most $d_1 = [(d+2)/2]$ components. Let M_1 be the M as in lemma 1.3. Put $M = M_1 d_1^d$. For each component I of E, apply lemma 1.3. Then

$$\sup_{t\in J}|f(t)| \le M_1 \cdot \frac{|J|^d}{|I|^d}\cdot\left(\frac{\epsilon^d}{M}\cdot\sup_{t\in J}|f(t)|\right).$$

Therefore $|I| \le (\epsilon/d_1)|J|$, and hence $|E| \le \epsilon\cdot|J|$. □

Notation 1.5 Fix $d, m \in \mathbf{N}$. Let $\mathcal{P}_{d,m}$ denote the space of real polynomials of degree d in m variables.

Let S^m denote the unit ball in $\mathbf{R}^m$ around 0. For any open convex set $B \subset \mathbf{R}^m$, $\mathbf{t}_0 \in B$, and $\mathbf{v} \in S^m$, put

$$B_{\mathbf{v},\mathbf{t}_0} = \{t > 0 : t\mathbf{v} + \mathbf{t}_0 \in B\}.$$

Lemma 1.6 *There exists a constant $M \ge 1$ such that for any bounded open convex sets $D \subset B \subset \mathbf{R}^m$, $\mathbf{t}_0 \in D$, and $f \in \mathcal{P}_{d,m}$,*

$$\sup_{\mathbf{t}\in B}|f(\mathbf{t})| \le M\left(\sup_{\mathbf{v}\in S^m}|B_{\mathbf{v},\mathbf{t}_0}|/|D_{\mathbf{v},\mathbf{t}_0}|\right)^{md}\cdot\sup_{\mathbf{t}\in D}|f(\mathbf{t})|.$$

Proof. For any $\mathbf{v} \in S^m$, the function $t \mapsto f(t\mathbf{v} + t_0)$ is in $\mathcal{P}_{(md)}$. Therefore the result follows from lemma 1.3. □

Lemma 1.7 *Given $M > 1$ there exists $\lambda > 1$ such that for any $f \in \mathcal{P}_{d,m}$ and a bounded open convex set B, there exist an open convex set $D \subset B$ and $\mathbf{t}_0 \in D$ such that*

$$\sup_{\mathbf{v}\in S^m} |B_{\mathbf{v},\mathbf{t}_0}|/|D_{\mathbf{v},\mathbf{t}_0}| \leq \lambda \quad \textit{and} \quad \sup_{\mathbf{t}\in B} |f(t)| \leq M \cdot \inf_{\mathbf{t}\in D} |f(t)|. \tag{1.2}$$

Proof. Let $1 < M' < M$ and $\lambda = 2M_1/(1 - M'^{-1}) > 1$, where M_1 is a constant such that the contention of lemma 1.2 holds for all $f \in \mathcal{P}_{(dm)}$. Let $\mathbf{t}_0 \in B$ such that $|f(\mathbf{t}_0)| \geq (M'/M) \sup_{\mathbf{t}\in B} |f(\mathbf{t})|$. For every $\mathbf{v} \in S^m$, define $\phi_{\mathbf{v}}(t) = f(t\mathbf{v} + \mathbf{t}_0)$, $\forall t \in \mathbf{R}$. Then $\phi_{\mathbf{v}} \in \mathcal{P}_{(md)}$ and hence

$$\sup_{t\in B_{\mathbf{v},\mathbf{t}_0}} \left| \frac{d\phi_{\mathbf{v}}}{dt}(t) \right| \leq \frac{2M_1}{|B_{\mathbf{v},\mathbf{t}_0}|} |f(\mathbf{t}_0)|, \quad \forall \mathbf{v} \in S.$$

Then for any $0 \leq t \leq \lambda^{-1}|B_{\mathbf{v},\mathbf{t}_0}|$, we get

$$\begin{aligned} |\phi_{\mathbf{v}}(t)| &= \left| \phi_{\mathbf{v}}(0) + t \cdot \frac{d\phi_{\mathbf{v}}}{dt}(t_1) \right| \\ &\geq (1 - 2M_1\lambda^{-1})|f(\mathbf{t}_0)| \\ &= (1/M')|f(\mathbf{t}_0)|. \end{aligned}$$

Hence, if we put

$$D = \cup_{\mathbf{v}\in S^m} [0, \lambda^{-1} B_{\mathbf{v},\mathbf{t}_0})\mathbf{v} + \mathbf{t}_0,$$

then D is an open convex subset of B and equation 1.2 holds. □

1.2 Condition for returning to compact sets

In this section we extend an important result of Dani and Margulis [13] about large compact sets in finite volume homogeneous spaces, having relative measures close to 1 with respect to trajectories of unipotent flows.

First we describe some elementary results, which will also be used again at a latter stage. The results are therefore presented in a form more technical than is necessary for the immediate purpose.

Notation 1.8 For any set $E \subset \mathbf{R}$ and $m \in \mathbf{N}$, define $E^m := \{s^m : s \in E\}$.

Lemma 1.9 *Let E and F be Borel measurable subsets of a bounded interval $I \subset [0, \infty)$ and $m \in \mathbf{N}$ be such that for some $\epsilon_1, \epsilon_2 \in (0, 1/m)$,*

$$|E| \leq \epsilon_1 \cdot |I| \quad \textit{and} \quad |F| \geq (1 - \epsilon_2) \cdot |I|.$$

Then

$$|E^m| \leq (m\epsilon_1)(1 - m\epsilon_2)^{-1} \cdot |F^m|.$$

Proof. Suppose that $I = [a, b]$, where $0 \le a < b$. Then

$$\begin{aligned} |E^m| &= \int_{a^m}^{b^m} \chi_{E^m}(t)\, dt \\ &= m \cdot \int_a^b \chi_E(s) s^{m-1}\, ds \\ &\le m \cdot |E| b^{m-1} \\ &\le m \cdot |E|(b^m - a^m)/(b-a) \\ &\le (m\epsilon_1) \cdot |I^m|. \end{aligned} \tag{1.3}$$

Similarly, we have

$$|F^m| = |I^m| - |(I \setminus F)^m| \ge (1 - m\epsilon_2) \cdot |I^m|. \tag{1.4}$$

The lemma follows from eqs. 1.3 and 1.4. □

Lemma 1.10 *Let B be a bounded open convex subset of $\mathbf{R}^m$. Fix $\mathbf{t}_0 \in B$. Let E and F be measurable subsets of B. Suppose there are a subset $D \subset B$, containing E and F, and $\epsilon_1, \epsilon_2 \in (0, 1/m)$ such that, for every $\mathbf{x} \in S^m$, the set $D_{\mathbf{x},\mathbf{t}_0}$ is open in $[0, \infty)$, and for every connected component I of $D_{\mathbf{x},\mathbf{t}_0}$, we have*

$$|E_{\mathbf{x},\mathbf{t}_0} \cap I| \le \epsilon_1 \cdot |I| \quad \textit{and} \quad |F_{\mathbf{x},\mathbf{t}_0} \cap I| \ge (1 - \epsilon_2) \cdot |I|.$$

Then

$$|E| \le (m\epsilon_1)(1 - m\epsilon_2)^{-1} \cdot |F|.$$

Proof. Let σ denote the rotation invariant measure on S^m such that the volume of the unit ball in $\mathbf{R}^m$ is $\sigma(S^m)/m$. Using polar decomposition of B at the pole $\mathbf{t}_0$, we have

$$\begin{aligned} |E| &= \int_{\mathbf{x} \in S^m} d\sigma(\mathbf{x}) \cdot \int_0^{|B_{\mathbf{x},\mathbf{t}_0}|} \chi_{E_{\mathbf{x},\mathbf{t}_0}}(t) t^{m-1}\, dt \\ &= (1/m) \int_{\mathbf{x} \in S^m} d\sigma(\mathbf{x}) \cdot \int_0^{|B_{\mathbf{x},\mathbf{t}_0}|^m} \chi_{E_{\mathbf{x},\mathbf{t}_0}}(t^{1/m})\, dt \\ &= (1/m) \int_{\mathbf{x} \in S^m} |E^m_{\mathbf{x},\mathbf{t}_0}|\, d\sigma(\mathbf{x}) \\ &\le (m\epsilon_1)(1 - m\epsilon_2)^{-1} \cdot (1/m) \int_{\mathbf{x} \in S^m} |F^m_{\mathbf{x},\mathbf{t}_0}|\, d\sigma(\mathbf{x}) \\ &= (m\epsilon_1)(1 - m\epsilon_2)^{-1} \cdot |F|, \end{aligned}$$

where the inequality follows from lemma 1.9. This completes the proof. □

Notation 1.11 Let G be a Lie group and $\underline{\mathbf{g}}$ the Lie algebra associated to G. For $d, m \in \mathbf{N}$, let $\mathcal{P}_{d,m}(G)$ denote the set of continuous maps $\Theta : \mathbf{R}^m \to G$ such that for all $\mathbf{c}, \mathbf{a} \in \mathbf{R}^m$ and $X \in \underline{\mathbf{g}}$, the map

$$t \in \mathbf{R} \mapsto \mathrm{Ad} \circ \Theta(t\mathbf{c} + \mathbf{a})(X) \in \underline{\mathbf{g}}$$

is a polynomial of degree at most d in each co-ordinate of $\underline{\mathbf{g}}$ (with respect to any basis).

We shall write $\mathcal{P}_d(G)$ for the set $\mathcal{P}_{d,1}(G)$. Note that if $\theta \in \mathcal{P}_d(G)$ is a group homomorphism then θ is a Ad-unipotent one-parameter subgroup of G, and conversely any Ad-unipotent one-parameter subgroup θ belongs to $\mathcal{P}_d(G)$, where $d = \dim G - 1$.

Theorem 1.12 *Let G be a Lie group and Γ a lattice in G. Then given a compact set $C \subset G/\Gamma$, an $\epsilon > 0$, and a $d \in \mathbf{N}$, there exists a compact subset $K \subset G/\Gamma$ with the following property: For any $x \in G/\Gamma$, any $\Theta \in \mathcal{P}_{d,m}(G)$ and any bounded open convex set $B \subset \mathbf{R}^m$, one of the following conditions hold:*

1. $\frac{1}{|B|}|\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in K\}| \geq (1 - \epsilon)$.

2. $\Theta(B)x \cap C = \emptyset$.

Proof. In [14, Theorem 6.1], the result is stated for a one-parameter Ad-unipotent subgroup $u : \mathbf{R} \to G$, in the place of Θ as above. The proof uses only the property that $u \in \mathcal{P}_d(G)$ for $d = \dim G - 1$, rather than the condition that u is a one-parameter subgroup. Hence essentially the same proof applies for all $\theta \in \mathcal{P}_d(G)$.

Now choose a compact set K such that the conclusion of the theorem is valid for ϵ/m in place of ϵ and $\theta \in \mathcal{P}_d(G)$ in place of Θ.

Let B be as in the hypotheis. Suppose that condition (2) does not hold. Then there exists $\mathbf{t}_0 \in B$ such that $\Theta(\mathbf{t}_0)x \in C$. Define

$$E = \{\mathbf{t} \in B : \Theta(\mathbf{t})x \notin K\}.$$

Fix $\mathbf{x} \in S^m$. Define a map $\theta(t) = \Theta(t\mathbf{x} + \mathbf{t}_0)$ for all $t \in \mathbf{R}$. Then $\theta \in \mathcal{P}_d(G)$. Therefore

$$|E_{\mathbf{x},\mathbf{t}_0}| = |\{t \in B_{\mathbf{x},\mathbf{t}_0} : \theta(t)x \notin K\}| < (\epsilon/m) \cdot |B_{\mathbf{x},\mathbf{t}_0}|.$$

Therefore by lemma 1.10, for $F = D = B$, $\epsilon_1 = \epsilon/m$, and $\epsilon_2 = 0$, we get

$$|E| \leq \epsilon \cdot |B|.$$

This completes the proof. $\square$

The usefulness of the above result is enhanced by the following theorem which provides an algebraic condition as an alternative to the possibility $\Theta(B)x \cap C = \emptyset$.

Notation 1.13 Let G be a connected Lie group and $\underline{\mathbf{g}}$ denote the Lie algebra associated to G. Let $V = \oplus_{k=1}^{\dim \underline{\mathbf{g}}} \wedge^k \underline{\mathbf{g}}$, the direct sum of exterior powers of $\underline{\mathbf{g}}$, and consider the linear G-action on V via the representation $\oplus_{l=1}^{\dim \underline{\mathbf{g}}} \wedge^l \mathrm{Ad}$, the direct sum of exterior powers of the adjoint representation of G on $\underline{\mathbf{g}}$.

Fix any euclidean norm on $\underline{\mathbf{g}}$ and let $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_{\dim \underline{\mathbf{g}}}\}$ denote an orthonormal basis of $\underline{\mathbf{g}}$. There is a unique euclidean norm $\|\cdot\|$ on V such that the associated basis of V given by

$$\{\mathbf{e}_{l_1} \wedge \cdots \wedge \mathbf{e}_{l_r} : 1 \leq l_1 < \ldots < l_r \leq \dim \underline{\mathbf{g}},\ r = 1, \ldots, \dim \underline{\mathbf{g}}\}$$

is orthonormal. This norm is independent of the choice of $\mathcal{B}$.

To any Lie subgroup W of G and the associated Lie subalgebra $\underline{\mathbf{w}}$ of $\underline{\mathbf{g}}$ we associate a unit-norm vector $\mathbf{p}_W \in \wedge^{\dim \underline{\mathbf{w}}} \underline{\mathbf{w}} \in V$.

Theorem 1.14 *Let G be a connected Lie group, Γ a lattice in G, and $\pi : G \to G/\Gamma$ the quotient map. Let M be the smallest closed normal subgroup of G such that $\bar{G} = G/M$ is a semisimple group with trivial center and no compact factors. Let $q : G \to \bar{G}$ be the quotient homomorphism. Then there exist finitely many closed subgroups $W_1, \ldots, W_r$ of G such that each W_i is of the form $q^{-1}(U_i)$ with U_i for the unipotent radical of a maximal parabolic subgroup of $\bar{G}$, $\pi(W_i)$ is compact and the following holds: Given $d, m \in \mathbf{N}$ and reals $\alpha, \epsilon > 0$, there exists a compact set $C \subset G/\Gamma$ such that for any $x \in G/\Gamma$, $\Theta \in \mathcal{P}_{d,m}(G)$, and a bounded open convex set $B \subset \mathbf{R}^m$, one of the following conditions is satisfied:*

1. *$\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in C\}| \geq (1 - \epsilon)|B|$.*
2. *There exist $g \in \pi^{-1}(x)$ and $i \in \{1, \ldots, r\}$ such that*

$$\sup_{\mathbf{t} \in \mathbf{B}} \|\Theta(\mathbf{t})g \cdot \mathbf{p}_{W_i}\| < \alpha.$$

Proof. By Auslander's theorem [26, 8.24] and Borel's density theorem [26, 5.24], $\bar{\Gamma} = q(\Gamma)$ is a lattice in $\bar{G}$ and the fibres of the map $\bar{q} : G/\Gamma \to \bar{G}/\bar{\Gamma}$ are compact M-orbits. Therefore to prove this result, without loss of generality, we may assume that $\bar{G} = G$.

Then there are finitely many normal connected subgroups $G_1, \ldots, G_r$ of G such that $G = G_1 \times \cdots \times G_r$ and each $\Gamma_i = G_i \cap \Gamma$ is an irreducible lattice in G_i (see [26, 5.22]). Therefore without loss of generality we may replace Γ by its finite-index subgroup $\Gamma_1 \times \cdots \times \Gamma_r$. In order to prove the theorem for G, it is enough to prove it for each G_i separately. Thus without loss of generality we may assume that Γ is an irreducible lattice.

Then by the arithmeticity theorem of Margulis [23, 40], if $\mathbf{R}$-rank of G is at least 2 then Γ is an arithmetic lattice. That is, there exist a semisimple algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$ and a surjective homomorphism $\rho : \mathbf{G}(\mathbf{R})^0 \to G$ with compact kernel such that for $\Lambda = \mathbf{G}(\mathbf{Z}) \cap \mathbf{G}(\mathbf{R})^0$ the subgroup $\Gamma \cap \rho(\Lambda)$ is a subgroup of finite index in both Γ and $\rho(\Lambda)$. Again in this case without loss of generality we may replace G by $\mathbf{G}(\mathbf{R})^0$ and Γ by Λ.

We shall prove the result by considering the cases of (1) arithmatic lattices, and (2) G of $\mathbf{R}$-rank 1, separately.

1.2.1 Case of arithmetic lattices

Let $G = \mathbf{G}(\mathbf{R})^0$ for a semisimple algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$. Let $\Gamma = \mathbf{G}(\mathbf{Z}) \cap G$ and $\pi : G \to G/\Gamma$ be the natural quotient map. Let r be the $\mathbf{Q}$-rank of $\mathbf{G}$. We can assume that $r \geq 1$, since otherwise by Godement's compactness criterion (see [1, Theorem 8.4]), G/Γ is cocompact and the results of this section are trivial.

Let $\mathbf{P}$ denote a minimal $\mathbf{Q}$-parabolic subgroup of $\mathbf{G}$. Then by [1, Theorem 15.6], there exists a finite set $F \subset \mathbf{G}(\mathbf{Q})$ such that

$$\mathbf{G}(\mathbf{Q}) = \mathbf{P}(\mathbf{Q}) \cdot F \cdot \Gamma.$$

Let $\mathbf{S}$ be a maximal $\mathbf{Q}$-split torus of $\mathbf{G}$ contained in $\mathbf{P}$. The subgroup $\mathbf{P}$ determines an order on the set of $\mathbf{Q}$-roots of $\mathbf{S}$. Let $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ be the corresponding system of simple $\mathbf{Q}$-roots. Take $i \in \{1, \ldots, r\}$. Let $\mathbf{P}_i$ denote the standard maximal parabolic subgroup associated to the set of simple roots $\Delta \setminus \{\alpha_i\}$. Let $\mathbf{U}_i$ be the unipotent radical of $\mathbf{P}_i$ and put $U_i = \mathbf{U}_i(\mathbf{R})$. Then for any $g \in \mathbf{P}_i(\mathbf{R})$, we have

$$g \cdot \mathbf{p}_{U_i} = \det(\mathrm{Ad} g|_{\underline{\mathbf{u}}_i}) \cdot \mathbf{p}_{U_i}.$$

Define a function $d_i : G \to \mathbf{R}^*$ as

$$d_i(g) = \|g \cdot \mathbf{p}_{U_i}\|^2 \quad \forall g \in G.$$

Theorem 1.15 *Given $d, m \in \mathbf{N}$ and $\alpha > 0$ there exists a compact set $C \subset G/\Gamma$ such that for any bounded open convex set $B \subset \mathbf{R}^m$ and any $\Theta \in \mathcal{P}_{d,m}(G)$, one of the following conditions is satisfied:*

1. *There exists $i \in \{1, \ldots, r\}$ and $\lambda \in F\Gamma$ such that*

$$d_i(\Theta(\mathbf{t})\lambda^{-1}) = \|\phi(\mathbf{t})\lambda^{-1} \cdot \mathbf{p}_{U_i}\|^2 < \alpha, \quad \forall \mathbf{t} \in B.$$

2. *$\pi(\Theta(B)) \cap C \neq \emptyset$.*

To prove this result we need to set up some more notation and recall a result from [13].

For $I \subset \{1, \ldots, r\}$, put $J = \{1, \ldots, r\} \setminus I$. Define

$$\begin{aligned} \mathbf{P}_I &= \bigcap_{i \in I} \mathbf{P}_i \\ \mathbf{Q}_I &= \{g \in \mathbf{P}_I : d_i(g) = 1, \forall i \in I\} \end{aligned}$$

Note that $\mathbf{P} \cap \mathbf{Q}_I$ is a minimal $\mathbf{Q}$-parabolic subgroup of $\mathbf{Q}_I$. Therefore by [1, Theorem 15.6], there exists a finite set $F_I \subset \mathbf{Q}_I(\mathbf{Q})$ such that

$$\mathbf{Q}_I(\mathbf{Q}) = (\mathbf{P} \cap \mathbf{Q}_I)(\mathbf{Q}) F_I (\Gamma \cap \mathbf{Q}_I(\mathbf{R})). \tag{1.5}$$

We define

$$\Lambda(I) = (\Gamma \cap Q_I){F_I}^{-1},$$

where $Q_I = \mathbf{Q}_I(\mathbf{R})$. Note that $\mathbf{P}_\emptyset = \mathbf{Q}_\emptyset = \mathbf{G}$ and $\Lambda(\emptyset) = \Gamma F^{-1}$.

Lemma 1.16 *Let $j \in \{1, \ldots, r\}$, $I \subset \{1, \ldots, r\} \setminus \{j\}$, and $I' = I \cup \{j\}$. Then there exists a finite set $E \subset \mathbf{P}(\mathbf{Q})$ such that*

$$\Lambda(I)\Lambda(I') \subset \Lambda(I)E.$$

Proof. By definition

$$\Lambda(I)\Lambda(I') = (Q_I \cap \Gamma){F_I}^{-1} \cdot (Q_{I'} \cap \Gamma){F_{I'}}^{-1}.$$

There exists a finite set $L \subset \mathbf{Q}_I(\mathbf{Q})$ such that

$${F_I}^{-1}(Q_{I'} \cap \Gamma) \subset {F_I}^{-1}(Q_I \cap \Gamma) \subset (Q_I \cap \Gamma)L.$$

Now by eq. 1.5, there exists a finite set $E_1 \subset (\mathbf{P} \cap \mathbf{Q}_I)(\mathbf{Q})$ such that

$$(LF_{I'}{}^{-1})^{-1} \subset E_1 F_I(\Gamma \cap Q_I).$$

Now

$$\begin{aligned}\Lambda(I)\Lambda(I') &\subset (Q_I \cap \Gamma)(LF_{I'}{}^{-1}) \\ &\subset (Q_I \cap \Gamma)F_I{}^{-1}E_1{}^{-1} \\ &= \Lambda(I)E,\end{aligned}$$

where $E = E_1{}^{-1} \subset \mathbf{P}(\mathbf{Q})$. □

Notation 1.17 Let $\mathcal{I}$ be the collection of all permutations of elements of subsets of $\{1, \ldots, r\}$. Let $I = (i_1, \ldots, i_p) \in \mathcal{I}$. Then by lemma 1.16 there exists a finite set $L(I) \subset \mathbf{G}(\mathbf{Q})$ such that

$$\Lambda(\emptyset)\Lambda(\{i_1\}) \cdots \Lambda(\{i_1, \ldots, i_{p-1}\}) = \Gamma L(I).$$

We define $L(\emptyset) = \{e\}$.

For positive reals $0 < a < b$ and $\alpha > 0$, and any $\lambda \in \Gamma L(I)$, define

$$\begin{aligned}W_{\alpha,a,b}(I, \lambda) = \{g \in G \quad : \quad & a \leq d_i(g\lambda) \leq b, \quad \forall i \in I \text{ and} \\ & d_j(g\lambda\theta) > \alpha, \quad \forall j \in \{1, \ldots, r\} \setminus I, \forall \theta \in \Lambda(I)\}.\end{aligned}$$

Note that for any $\gamma \in \Gamma$,

$$W_{\alpha,a,b}(I, \gamma\lambda) = W_{\alpha,a,b}(I, \lambda)\gamma^{-1}.$$

Define the following subsets of G/Γ:

$$W_{\alpha,a,b}(I) = \bigcup_{\lambda \in L(I)} \overline{\pi(W(I,\lambda))} = \bigcup_{\lambda \in \Gamma L(I)} \overline{\pi(W(I,\lambda))}.$$

Proposition 1.18 (Dani and Margulis [13, Proposition 1.8]) *The set $W_{\alpha,a,b}(I)$ is compact.*

Notation 1.19 Fix $d, m \in \mathbf{N}$. The map $V \ni \mathbf{v} \mapsto \|\mathbf{v}\|^2$ is a polynomial function on V. Therefore there exists $d' \in \mathbf{N}$ such that for any $i \in \{1, \ldots, r\}$, $\phi \in \mathcal{P}_{d,m}(G)$ and $g \in G$, the map $f : \mathbf{R}^m \to \mathbf{R}$ defined as $f(\mathbf{t}) = d_i(\phi(\mathbf{t})g)$, $\forall \mathbf{t} \in \mathbf{R}^m$, is in $\mathcal{P}_{d',m}(G)$.

Let B be an open convex subset of $\mathbf{R}^m$ and $\Theta \in \mathcal{P}_{d,m}(G)$. When condition (1) in theorem 1.15 fails to hold, using the following proposition we shall find constants $0 < a < b$ and $\alpha > 0$ (independent of Θ and B), $I \in \mathcal{I}$, and $\mathbf{t} \in B$ such that $\pi(\Theta(\mathbf{t})) \in W_{\alpha,a,b}(I)$.

Proposition 1.20 (cf.[13]) *Let $\alpha > 0$ and $D \subset \mathbf{R}^m$ be a bounded open convex subset. Suppose a family $\mathcal{F} \subset \mathcal{P}_{d',m}(G)$ satisfies the following conditions:*

1. *For any* $\mathbf{t} \in D$ *and any* $\beta > 0$,

$$\#\{f \in \mathcal{F} : |f(\mathbf{t})| < \beta\} < \infty.$$

2. *For every* $f \in \mathcal{F}$,

$$\sup_{\mathbf{t} \in D} |f(\mathbf{t})| > \alpha.$$

Then one of the following conditions is satisfied:

(a) $|f(\mathbf{t}_0)| > \alpha$ *for all* $f \in \mathcal{F}$ *and* $\mathbf{t}_0 \in D$.

(b) *There exist an open convex subset* $D_1 \subset D$ *and* $f_0 \in \mathcal{F}$ *such that the following holds:*

   (i) $f_0(D_1) \subset (\alpha/2, \alpha)$.

   (ii) *For all* $f \in \mathcal{F}$,

$$\sup_{\mathbf{t} \in D_1} |f(\mathbf{t})| > \alpha/M,$$

   where $M \geq 1$ *is a constant depending only on* d' *and* m.

Proof. If (a) does not hold then by condition (1), there exists $\mathbf{t}_0 \in D$ and a finite set $\mathcal{F}_1 \subset \mathcal{F}$ such that $|f(\mathbf{t}_0)| \geq \alpha$ for all $f \in \mathcal{F} \setminus \mathcal{F}_1$. There exists $s \in (0,1)$ such that if we put $E = \{(1-s)\mathbf{t}_0 + s\mathbf{v} : \mathbf{v} \in D\}$ then E is an open convex subset of D containing $\mathbf{t}_0$ and the following holds:

(1) $\sup_{\mathbf{t} \in E} |f(\mathbf{t})| \geq \alpha, \quad \forall f \in \mathcal{F}_1$, and

(2) there exists $f_0 \in \mathcal{F}_1$ such that $\sup_{\mathbf{t} \in E} |f_0(\mathbf{t})| = \alpha$.

By lemma 1.7, there exists a constant $\lambda > 1$ (depending only on d' and m) and an open convex subset D_1 of E containing $\mathbf{t}_0$ such that

$$\frac{|E_{\mathbf{v},\mathbf{t}_0}|}{|(D_1)_{\mathbf{v},\mathbf{t}_0}|} \leq \lambda, \quad \forall \mathbf{v} \in S^m$$

and

$$\inf_{\mathbf{t} \in D_1} |f_0(\mathbf{t})| \geq \alpha/2.$$

Now by lemma 1.6, there exists a constant $M \geq 1$ (depending only on λ, d' and m) such that for any $f \in \mathcal{F}$ we have

$$\sup_{\mathbf{t} \in D_1} |f(\mathbf{t})| \geq \alpha/M.$$

This completes the proof. $\square$

Proof of theorem 1.15. Let $\alpha > 0$ be given. Let $B \subset \mathbf{R}^m$ be a bounded open convex set and let $\Theta \in \mathcal{P}_{d,m}(G)$. Suppose that the condition (1) of the theorem fails to hold.

By a stepwise construction we shall obtain $I \in \mathcal{I}$, $\lambda \in \Gamma L_I$, and constants $0 < a_I < b_I$ and $\alpha_I > 0$ depending only on I and α such that

$$\pi(\Theta(B)) \cap \pi(W_{\alpha_I,a_I,b_I}(I,\lambda)) \neq \emptyset.$$

In view of proposition 1.18 this will imply that the second condition of the theorem holds.

First note the following procedure: Suppose $I \in \mathcal{I}$, $\lambda \in \Gamma L_I$, an open convex set $D \subset \mathbf{R}^m$, and constants $0 < a_I < b_I$ are such that

$$\text{(A)} \qquad d_i(\Theta(B)\lambda) \subset (a_I, b_I), \quad \forall i \in I.$$

Let $\mathcal{F}(I,\lambda)$ denote the family of all functions $f : \mathbf{R}^m \to \mathbf{R}_{>0}$ of the form $f(\mathbf{t}) = d_j(\Theta(\mathbf{t})\lambda\theta)$ for all $\mathbf{t} \in \mathbf{R}^m$, where $\theta \in \Lambda(I)$ and $j \in J = \{1,\ldots,r\} \setminus I$. Suppose further that for some $\alpha_I > 0$, we have

$$\text{(B)} \qquad \sup_{\mathbf{t}\in D} |f(\mathbf{t})| > \alpha_I, \quad \forall f \in \mathcal{F}(I,\lambda).$$

Observe that condition (1) of proposition 1.20 is satisfied for the family $\mathcal{F}(I,\lambda)$, because the set $\Gamma \cdot L_I\Lambda(I) \cdot \mathbf{p}_j$ is discrete in V for every $j \in J$. The condition (2) of proposition 1.20 follows from the condition (B) as above. Therefore due to the proposition, one of the following holds:

(a) There exists $\mathbf{t}_0 \in D$ such that $d_j(\Theta(\mathbf{t}_0)\lambda\theta) \geq \alpha_I$ for all $\theta \in \Lambda(I)$ and all $j \in J$.

   In this case by condition (A) we have $\Theta(\mathbf{t}_0)\Gamma \in W_{\alpha_I,a_I,b_I}(I)$. We fix this $I \in \mathcal{I}$, $\mathbf{t}_0 \in B$, and constants $0 < a_I < b_I$ and $\alpha_I > 0$ and stop the procedure.

(b) There exist $j_0 \in J$, $\theta_0 \in \Lambda(I)$, and an open convex subset $D_1 \subset D$ such that the following holds:

   (i) $d_{j_0}(\Theta(D_1)\lambda\theta_0) \subset (\alpha_I/2, \alpha_I)$.

   (ii) For all $\theta \in \Lambda(I)$ and $j \in J$,

$$\sup_{\mathbf{t}\in D_1} d_j(\theta\lambda\Theta(\mathbf{t})) \geq \alpha_I/M.$$

   In this case, let $I_1 = I \cup j_0$, and $\lambda_1 = \lambda\theta_0$. We will now show that conditions (A) and (B) are satisfied for D_1, I_1 and λ_1, with suitable constants a_{I_1},b_{I_1}, and α_{I_1}. Since $d_i(g\theta_0) = d_i(g)$, $\forall i \in I$ and $\forall g \in G$, condition (A) is satisfied with $a_{I_1} = \alpha_I/2$ and $b_{I_1} = \alpha_I$.

   By lemma 1.16, there exists a finite set $E \subset \mathbf{P}(\mathbf{Q})$ (depending only on I and j_0) such that for any $\theta \in \Lambda(I \cup \{j_0\})$, there exists $\theta' \in \Lambda(I)$ and $\delta \in E$ such that $\theta_0\theta = \theta'\delta$. Hence for every $j \in J \setminus \{j_0\}$,

$$\begin{aligned} \sup_{\mathbf{t}\in D_1} d_j(\Theta(\mathbf{t})\lambda_1\theta) &= \sup d_j(\Theta(\mathbf{t})\lambda\theta_0\theta) \\ &= \sup d_j(\Theta(\mathbf{t})\lambda\theta'\delta) \\ &= \sup d_j(\Theta(\mathbf{t})\lambda\theta') \cdot d_j(\delta) \\ &\geq \alpha_I/M \cdot \beta, \end{aligned}$$

where $\beta = \min_{\delta \in E} d_j(\delta) > 0$ depends only on I and j_0. Therefore condition (B) is also satisfied for the family $\mathcal{F}(I_1, \lambda_1)$ and $\alpha_{I_1} = \beta\alpha_I/M > 0$.

This completes the description of our procedure.

To prove the theorem, we start with $I = \emptyset$, $\lambda = e$, and $D = B$. Then condition (A) is vacuously satisfied. We can assume that condition (1) in the statement of the theorem does not hold. Then condition (B) is satisfied for $\mathcal{F}(\emptyset, e)$.

We can repeatedly apply the above procedure till we get I, $\lambda \in \Gamma L_I$, and constants $0 < a_I < b_I$ and $\alpha_I > 0$ such that $d_i(\Theta(B)\lambda\theta) > \alpha_I$ for all $\theta \in \Lambda(I)$; at which step we are through. Since the cardinality of I increases each time we apply the procedure, it must stop after at most r steps. This completes the proof. □

Now in the arithmetic case the theorem 1.14 is obtained by combining theorem 1.12 with theorem 1.15.

1.2.2 Case of semisimple Lie groups of R-rank 1

Let G be a connected semisimple Lie group of $\mathbf{R}$-rank 1, Γ a lattice in G, and $\pi : G \to G/\Gamma$ the quotient map. Let A be a maximal $\mathbf{R}$-split torus in G and P a minimal parabolic subgroup of G containing A. Let U be the unipotent radical of P. Let K be a maximal compact subgroup of G such that the Cartan involution of G associated to K preserves A. Let $M = Z_G(A) \cap K$. Then we have the decompositions $G = KP$ and $P = MAU$.

Let $\underline{\mathbf{g}}$ denote the Lie algebra of G and $\underline{\mathbf{u}}$ the Lie subalgebra of $\underline{\mathbf{g}}$ associated to U. Fix an AdK-invariant norm on $\underline{\mathbf{g}}$ and consider the associated norm on V as described in notation 1.13. The norm on V is now K-invariant. We define a function $d : G \to \mathbf{R}^*$ as

$$d(g) = \|g \cdot \mathbf{p}_U\|^2, \quad \forall g \in G.$$

In view of the decomposition $G = KP$, we have that

$$g \cdot \mathbf{p}_U = \det(\mathrm{Ad}g|_{\underline{\mathbf{u}}}) \cdot \mathbf{p}_U, \quad \forall g \in P,$$

and

$$g \cdot \mathbf{p}_U = \pm \mathbf{p}_U, \quad \forall g \in MU.$$

For $\eta > 0$, define $\mathcal{S}_\eta = \{g \in G : 0 < d(g) < \eta\}$. By [18, Theorems 0.6-0.7] we have the following.

Proposition 1.21 *There exists a finite subset F of G such that the following holds:*

1. *For every $f \in F$, the orbit $U\pi(f) \cong U/(f\Gamma f^{-1} \cap U)$ is compact.*
2. *For any $\eta > 0$, the set $(G/\Gamma) \setminus \pi(\mathcal{S}_\eta F)$ is compact.*
3. *There exists $\eta_0 > 0$ such that for any $f_1, f_2 \in F$ and $g_1, g_2 \in \mathcal{S}_{\eta_0}$, if $\pi(g_1 f_1) = \pi(g_2 f_2)$ then $f_1 = f_2$ and ${g_1}^{-1} g_2 \in MU$.*

   In particular, for any $g \in G$, if there are $\gamma_1, \gamma_2 \in \Gamma$ and $f_1, f_2 \in F$ such that $d(g\gamma_i {f_i}^{-1}) < \eta_0$ for $i = 1, 2$, then $f_1 = f_2$ and $\gamma_1 {f_1}^{-1} \cdot \mathbf{p}_U = \pm\gamma_2 {f_2}^{-1} \mathbf{p}_U$; in particular, $d(g\gamma_1 {f_1}^{-1}) = d(g\gamma_2 {f_2}^{-1})$.

$\square$

The next result is an analogue of theorem 1.15 in the rank 1 case.

Theorem 1.22 *For any connected set $C \subset G$, one of the following conditions is satisfied:*

1. *There exists $\lambda \in F\Gamma$ such that*

$$\|g\lambda^{-1} \cdot \mathbf{p}_U\|^2 < \eta_0/2, \quad \forall g \in C.$$

2. *$\pi(C) \not\subset \pi(S_{\eta_0/2}F)$.*

Proof. Suppose that (1) and (2) do not hold. Then there exist $g_1, g_2 \in C$ and $\lambda_1, \lambda_2 \in F\Gamma$ such that $d(g_1{\lambda_1}^{-1}) < \eta_0/2$, $d(g_2{\lambda_1}^{-1}) = \eta_0/2$, and $d(g_2{\lambda_2}^{-1}) < \eta_0/2$. This contradicts part (2) of proposition 1.21. $\square$

Now in the rank-1 case, the theorem 1.14 is deduced from theorem 1.12 and theorem 1.22. As indicated before this completes the proof. $\square$

Chapter 2

Invariant measures of unipotent flows and behaviour of polynomial trajectories near their supports

In technical terms this chapter is the core the thesis. We describe here a method for investigating dynamics of individual trajectories of unipotent flows, and more generally, 'polynomial trajectories' on homogeneous spaces. This uses crucially the homogeneity of ergodic invariant measures for unipotent flows proved by Ratner. The remaining chapters of the thesis show how the method can be applied in various situations.

2.1 Finite volume, ergodicity and Zariski density

First observe the following.

Lemma 2.1 *Let F be a locally compact Hausdorff second countable group acting continuously on a locally compact Hausdorff second countable space X. For a point $x \in X$ define $F_x = \{g \in F : gx = x\}$. Consider the map $\phi : F/F_x \to X$, defined by $\phi(gF_x) = gx$ for all $g \in F$. Then the orbit Fx is closed if and only if the map ϕ is proper. In particular, Fx is F-equivariantly homeomorphic to the homogeneous space F/F_x.*

Lemma 2.2 *Let G be locally compact Hausdorff second countable group and Γ a discrete subgroup of G. Let F and H be cloded subgroups of G. Let Z_1 and Z_2 be closed orbits of F and H respectively in G/Γ, and put $Z = Z_1 \cap Z_2$. Then every orbit of $F \cap H$ in Z is both open and closed in Z. In particular, for any subgroup L of G and any point $x \in G/\Gamma$, there exists the smallest closed subgroup F of G containing L such that the orbit Fx is closed.*

Proof. Let $z \in Z$. Then $Fz = Z_1$ and $Hz = Z_2$ are closed. Therefore $F/F_z \simeq Fz$ and $H/H_z \simeq Hz$. Also G_z, F_z and H_z are discrete. Therefore there exists a neighbourhood Ω of the identity e in G such that $\Omega\Omega^{-1} \cap G_z = \{e\}$, $(Fz \cap \Omega z) = (F \cap \Omega)z$ and

$(Hz \cap \Omega z) = (H \cap \Omega)z$. This implies that $(Fz \cap Hz \cap \Omega z) = (F \cap H \cap \Omega)z$. Hence $(F \cap H)z$ is open in $Fz \cap Hz = Z$ for every $z \in Z$. Now $(F \cap H)z$ is closed, because its complement in Z is the union of open $F \cap H$ orbits in Z and Z is closed. $\square$

Notation 2.3 Let G be a connected Lie group, Γ a lattice in G, $X = G/\Gamma$, and L a subgroup such that the unipotent one-parameter subgroup of G contained in L generate L.

Our aim in this section is to prove the following:

Theorem 2.4 *For $x \in X$ let F be the smallest subgroup of G such that $L \subset F$ and Fx is closed. Then the following holds.*

1. *The stabilizer F_x is a lattice in F.*
2. *A unipotent one-parameter subgroup of G contained in L acts ergodically on Fx with respect to the F-invariant probability measure.*
3. *Let $\rho : F \to \mathrm{GL}(V)$ be a finite dimensional representation such that $\rho(L)$ is generated by one-parameter groups of unipotent transformations on V. Then $\rho(F_x)$ is Zariski dense in $\rho(F)$.*

We recall some preliminaries and a result due to Margulis before going to the proof of the theorem.

Definition 2.5 *A subgroup H of G is said to have* property-D *if for every locally finite H-invariant measure σ on X, there exist measurable H-invariant subsets X_i, $i \in \mathbf{N}$ such that $\sigma(X_i) < \infty$ for all $i \in \mathbf{N}$ and $X = \bigcup_{i \in \mathbf{N}} X_i$.*

In particular if H has property-D then every locally finite H-ergodic and H-invariant measure on X is finite.

Proposition 2.6 [5, Theorem 4.3]. *Any unipotent subgroup $U \subset G$ has property-D.* $\square$

Definition 2.7 *Let F be a topological group, $H \subset F$ and $L \subset F$. We say that the triple (F, H, L) has the* Mautner property *if the following condition is satisfied: for any continuous unitary representation of F on a Hilbert space $\mathcal{H}$, if a vector $\xi \in \mathcal{H}$ is fixed by L then it is also fixed by H.*

The following Proposition is a slight modification of Theorem 1.1 in [24].

Proposition 2.8 *Let F be a Lie group and L be a subgroup such that the unipotent one-parameter subgroups contained in L generate L. Then there exists a closed normal subgroup H of F such that (i) $L \subset H$ and (ii) the triple (F, H, L) has the Mautner property.*

Proof. Let U be a unipotent one-parameter subgroup contained in L. By Theorem 1.1 of [24], there exists a normal subgroup $H_U \subset F$ such that (a) (F, H_U, U) has the Mautner property and (b) the image of $\mathrm{Ad}(U)$ in the automorphism group of the Lie algebra of F/H_U is relatively compact. For each $u \in U$, $\mathrm{Ad}u$ is a unipotent transformation of the Lie algebra of F, therefore the image of U in F/H_U is in the center. Hence the group UH_U is normal in F and (F, UH_U, U) has the Mautner property.

Let $U_1, \ldots, U_n$ be unipotent one-parameter subgroups of G which generate L. Let $H_1, \ldots, H_n$ be normal subgroups of F such that $U_i \subset H_i$ and the triples (F, H_i, U_i) have the Mautner property for all $1 \leq i \leq n$. Then $H = \overline{H_1 \cdots H_n}$ satisfies the conditions (i) and (ii). □

The proof of theorem 2.4 depends on the following observation by Margulis.

Lemma 2.9 [20, Remarks 3.12]. *Suppose $H \subset G$ admits a Levi decomposition $H = S \cdot N$, where S is a semisimple group without compact factors and N is the unipotent radical of H. Then H has property-D.*

Proof. Let σ be a locally finite H invariant measure on X. We consider the left regular unitary representation on $\mathcal{L}^2(X, \sigma)$.

Let W be a maximal unipotent subgroup of S. Then $W \cdot N$ is a unipotent subgroup of G. By proposition 2.6 there exists a measurable $W \cdot N$ invariant partition $\{X_i\}_{i \in \mathbf{N}}$ of X such that $\sigma(X_i) < \infty$ for all $i \in \mathbf{N}$. If χ_i denotes the characteristic function of X_i then χ_i is a $W \cdot N$ invariant function in $\mathcal{L}^2(X, \sigma)$. By proposition 2.8 there exists a normal subgroup Q of G containing $W \cdot N$ such that χ_i is Q invariant for all $i \in \mathbf{N}$. Since S is semisimple group without compact factors, $S \subset Q$. Hence X_i is H invariant for all $i \in \mathbf{N}$. This completes the proof. □

Now we discuss the group theoretic structure of a closed subgroup generated by unipotent one-parameter subgroups.

Lemma 2.10 *Let $H \subset G$ be a closed subgroup such that the unipotent one-parameter subgroups of G contained in H generate H. Then H admits a Levi decomposition $H = S \cdot N$, where S is a semisimple group with no compact factors and N is the unipotent radical of H.*

Proof. It is enough to prove the lemma for the adjoint group of G. Therefore we may assume that $G \subset \mathrm{GL}(n, \mathbf{R})$ and its unipotent elements are unipotent linear transformations. By Levi decomposition $H = S \cdot R$, where S is a connected semisimple group and R is the radical of H. Suppose H_1 is a normal subgroup of H containing R such that H/H_1 is a compact semisimple group. Note that under a surjective morphism a unipotent element projects to a unipotent element. Since compact semisimple groups contain no nontrivial unipotent elements, by hypothesis $H = H_1$. This shows that S has no compact factors.

To prove the other part we argue as follows; we refer the reader to [26, Preliminaries 2] for the results used in the the argument.

Let $\mathbf{H}$ be the smallest algebraic $\mathbf{R}$-subgroup of $GL(n, \mathbf{C})$ containing H. Let $\mathbf{N}$ be the unipotent radical of $\mathbf{H}$. By Levi decomposition there exists a connected semisimple $\mathbf{R}$-subgroup $\mathbf{S} \subset \mathbf{H}$ such that $\mathbf{S} \cdot \mathbf{N}$ is a normal subgroup of $\mathbf{H}$ and $\mathbf{T} = \mathbf{H}/(\mathbf{S} \cdot \mathbf{N})$ is an algebraic $\mathbf{R}$-torus. Now the projection of any unipotent element of $\mathbf{H}$ in $\mathbf{T}$ is unipotent. But any algebraic torus contains only semisimple elements. Hence by hypothesis $H \subset \mathbf{S} \cdot \mathbf{N}$. By minimality of $\mathbf{H}$, $\mathbf{H} = \mathbf{S} \cdot \mathbf{N}$.

Since H normalizes the Lie subalgebra $\underline{\mathbf{r}}$ corresponding to its radical R, by definition $\mathbf{H}$ normalizes $\underline{\mathbf{r}} \otimes \mathbf{C}$. Hence R is contained in the radical of $\mathbf{H}$. Since the radical of $\mathbf{H}$ is unipotent, R consists of unipotent linear transformations. This completes the proof. □

Lemma 2.11 *Let L be as in notation 2.3. Suppose that L acts ergodically on X with respect to a probability measure ν. Then L contains a unipotent one-parameter subgroup of G acting ergodically on the measure space (X, ν).*

Proof. Let N be the radical of L. Then $L = S \cdot N$, where S is a semisimple group with no compact factors and N is a unipotent subgroup of G. Let U_1 be a unipotent one-parameter subgroup of S such that no proper normal subgroup of S contains U. Then $W = U_1 N$ is a unipotent subgroup of G, and it is not contained in any proper normal subgroup of L. Therefore by Mautner's phenomenon, W acts ergodically with respect to ν, (see [24, Theorem 1.1] and [22]). Now by [8, Proposition 2.2], there exists a one-parameter subgroup of N which acts ergodically on X with respect to ν. □

Proposition 2.12 *Let F be a connected Lie group, Δ be a closed nonconnected subgroup of F and $U = \{u_t\}_{t \in \mathbf{R}}$ be a one-parameter subgroup of F such that $\overline{U\Delta} = F$. Let $\rho : F \to GL(E)$ be a finite dimensional representation of F such that $\rho(U)$ consists of unipotent linear transformations of E. Then every Δ-stable subspace of E is also F-stable.*

Proof. Let W be Δ-stable subspace of E. Passing to a suitable exterior power of ρ, we may assume that $\dim(W) = 1$. For any $v \in E \setminus \{0\}$, let $\bar{v} \in \mathbf{P}^1(E)$ denote the one-dimensional subspace of E containing v. Let $\bar{\rho} : F \to \mathrm{PGL}(E)$ be the projective linear representation of F on the projective space $\mathbf{P}^1(E)$ corresponding to ρ; that is, $\bar{\rho}(\bar{v}) = \overline{\rho(v)}$ for all $v \in E \setminus \{0\}$. Let $w \in W \setminus \{0\}$, and let $\varphi : \mathbf{R} \to E$ be the map given by $\varphi(t) = \rho(u_t)w$ for all $t \in \mathbf{R}$. Fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of E with respect to some inner product. Since $\rho(U)$ consists of unipotent linear transformations, there exist polynomials $\varphi_1, \ldots, \varphi_n$ on $\mathbf{R}$ such that $\varphi(t) = \sum_{i=1}^{n} \varphi_i(t) e_n$ for all $t \in \mathbf{R}$. Now $\varphi_i^2(t)/\sum_{j=1}^{n} \varphi_j^2(t)$ converges as $t \to \infty$ for $1 \le i \le n$. Hence $\lim_{t\to\infty} \varphi(t)/\|\varphi(t)\| = p$ for some $p \in E$ with unit norm.

Let Δ^0 denote the connected component of the identity in Δ. If $F = U\Delta$ then $F = U\Delta^0$ and there exists $t_0 \in \mathbf{R} \setminus \{0\}$ such that $u_{kt_0} \in \Delta$ for all $k \in \mathbf{N}$. Therefore $\rho(\Delta)w = w$ and $\varphi(kt_0) = w$ for all $k \in \mathbf{N}$. Since φ is a polynomial function, it must be constant. Thus $\rho(F)w = w$ in this case.

Suppose $F \setminus U\Delta \neq \emptyset$. For any $f \in F \setminus U\Delta$, there exist sequences $\{t_k\}_{k \in \mathbf{N}} \subset \mathbf{R}$, and $\{\delta_k\}_{k \in \mathbf{N}} \subset \Delta$ such that $t_k \to \infty$ and $u_{t_k}\delta_k \to f$ as $k \to \infty$. For $x \in E \setminus \{0\}$, let $\bar{x}$ denote its image in $\mathbf{P}^1(E)$. Since $\bar{\rho}(\Delta)\bar{w} = \bar{w}$,

$$\bar{\rho}(f)\bar{w} = \lim_{k\to\infty} \bar{\rho}(u_{t_k}\delta_k)\bar{w} = \lim_{k\to\infty} \bar{\rho}(u_{t_k})\bar{w} = \lim_{k\to\infty} \overline{\phi(t_k)} = \bar{p}.$$

Now for any $u \in U$, $uf \in F \setminus U\Delta$. Therefore

$$\bar{p} = \bar{\rho}(uf)\bar{w} = \bar{\rho}(u)(\bar{\rho}(f)\bar{w}) = \bar{\rho}(u)\bar{p}.$$

Thus $\bar{\rho}(U)\bar{p} = \bar{p}$. By putting $u = e$ we get that $\bar{p} = \bar{w}$. Therefore $\bar{\rho}(U\Delta)\bar{w} = \bar{w}$. Hence $\bar{\rho}(F)\bar{w} = \bar{w}$. This completes the proof. $\square$

We also need the following lemma.

Lemma 2.13 *Let F be a Lie group, Λ be a discrete subgroup of F and H be a normal subgroup of F such that $\overline{H\Lambda} = F$. Then H acts ergodically on $(F/\Lambda, \sigma)$, where σ is a locally finite F-semi-invariant measure on F/Λ with the modular function of F as its character (cf. [26, §1.4]).*

Proof. The proof of Lemma 8.2 in [3] goes through as it is, if we replace $\mathcal{L}^2(F/\Lambda, \sigma)$ by the space of locally integrable functions on $(F/\Lambda, \sigma)$. $\square$

Proof of theorem 2.4. By proposition 2.8 there exists a smallest closed normal subgroup H of F containing L such that the triple (F, H, L) has the Mautner property.

Since H is normal in F, HF_x is a subgroup of F. If $H_1 = \overline{HF_x}$ then $H_1 \supset H$ and H_1x is closed in Fx. By minimality of F as in the hypothesis, $H_1 = F$. Hence $\overline{HF_x} = F$.

Let H' be the closure of the group generated by all unipotent one-parameter subgroups of G contained in H. Then $L \subset H'$ and H' in normal in F. Therefore by the hypothesis on H, $H' = H$.

Let σ be a locally finite F-semi-invariant measure on F/F_x with a character Δ_F, where Δ_F is a modular function of F. If $\underline{\mathbf{f}}$ is the Lie subalgebra corresponding to F then $\Delta_F(f) = |\det(\mathrm{Ad}f|_{\underline{\mathbf{f}}})|$ for all $f \in F$.

Since H is the closure of a subgroup generated by unipotent one-parameter subgroups, $\Delta_F(H) = 1$. Therefore σ is H-invariant. By lemma 2.13, H acts ergodically on $(F/F_x, \sigma)$. Since Fx is closed, the natural inclusion $F/F_x \hookrightarrow X$ is proper. Therefore we may treat σ as a locally finite ergodic invariant measure of H on X. By Lemmas 2.9 and 2.10, H has property-D. Hence σ is finite. This proves part (1).

A finite F-semi-invariant measure σ must be F-invariant. Now by the Mautner property of the triple (F, H, L), L also acts ergodically on (Fx, σ). Hence by lemma 2.11, there exists a one-parameter unipotent subgroup U of G contained in L such that U acts ergodically on Fx. This proves part (2).

Ergodicity of the action implies, by Hedlund's lemma, that U has a dense orbit in Fx. Therefore replacing U by one of its conjugates in F, we may assume that $\overline{Ux} = Fx$. To prove part (3), we may assume that $V = \mathbf{R}^n$. Let $d \geq 0$. Let P_d be the finite dimensional vector space of real polynomials of degree $\leq d$ defined on $\mathrm{M}_n(\mathbf{R})$, the space of $n \times n$ matrices with real entries. Consider the representation π of $\mathrm{GL}(n, \mathbf{R})$ on P_d defined as follows: for $g \in \mathrm{GL}_n(\mathbf{R})$, $p \in P_d$ and $x \in M(n, \mathbf{R})$, we have $(\pi(g)p)(x) = p(g^{-1}x)$. Clearly $\pi(g)p \in P_d$. Since $\pi : GL(n, \mathbf{R}) \to GL(P_d)$ is an algebraic morphism, π preserves algebraic unipotent subgroups. Thus $\pi \circ \rho(U)$ consists of unipotent linear transformations of P_d. Define

$$I_d = \{p \in P_d : p(\rho(\delta)) = 0 \text{ for all } \delta \in F_x\}.$$

Then I_d is a subspace of P_d. Since F_x is a group, F_x stabilizes I_d. Therefore by proposition 2.12, for all $f \in F$ and $p \in I_d$ we have $\pi(\rho(f^{-1}))p \in I_d$ and hence $p(\rho(f)) = [\pi(\rho(f^{-1}))p](e) = 0$. Thus $p(\rho(f)) = 0$ for all $f \in F$ and $p \in I_d$. Since this happens for every $d \geq 0$, we conclude that $\rho(F)$ is contained in the Zariski closure of $\rho(F_x)$ in $GL(n, \mathbf{R})$. This proves (3). □

2.2 Finite invariant measures of a unipotent flow

Let G be a Lie group, Γ a discrete subgroup of G, $X = G/\Gamma$, and $\pi : G \to X$ the quotient map. Let L be a subgroup of G generated by unipotent one-parameter subgroups of G contained in L. Let $X_0(L)$ be the set of all $x \in X$ such that there exists a closed subgroup F of G containing L such that the orbit Fx is closed and admits a finite F-invariant measure.

For any $x \in X$, let $F(x, L)$ denote the smallest closed subgroup F of G containing L such that the orbit Fx is closed; such a subgroup exists by lemma 2.2.

Remark 2.14 Let $x \in X_0(L)$. Then by proposition 2.12, $F(x, L)_x$ is a lattice in $F(x, L)$ and $\mathrm{Ad}(F(x, L)_x)$ is Zariski dense in $\mathrm{Ad}(F(x, L))$, where $\mathrm{Ad} : G \to \mathrm{Aut}(\underline{\mathbf{g}})$ is the Adjoint representation.

Notation 2.15 Let $\mathcal{H}$ be the collection of all closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H and $\mathrm{Ad}(H \cap \Gamma)$ is Zariski dense in $\mathrm{Ad}(H)$. Note that by [26, Lemma 1.12], the orbit $H\Gamma/\Gamma$ is closed.

Fix a (positive definite) inner product on $\underline{\mathbf{g}}$, and let σ be the induced Riemannian metric on G which is invariant under all the right translations. Then σ projects to a unique Riemannian metric $\bar{\sigma}$ on X such that the map $\pi : G \to X$ is locally isometric. Now for any $H \in \mathcal{H}$ the restriction of $\bar{\sigma}$ on the submanifold $\pi(H) = H\Gamma/\Gamma$ determines a smooth measure, which is H-invariant. We denote the total measure of this orbit by $\mathrm{vol}_\sigma(\pi(H))$.

It was proved by Ratner [30, Theorem 1] that the collection $\mathcal{H}$ is countable; also compare [35, Lemma 5.2] and [14, Proposition 2.3]. We recall the following result, which provides more precise information in this regard.

Proposition 2.16 ([14, Theorem 5.1]) *For any $c > 0$, the set*

$$\mathcal{H}_c = \{H \in \mathcal{H} : \mathrm{vol}_\sigma(\pi(H)) < c\}$$

is finite. In particular, the collection $\mathcal{H}$ is countable.

Notation 2.17 For any $H \in \mathcal{H}$, define

$$\begin{aligned} N(H, L) &= \{g \in G : L \subset gHg^{-1}\} \text{ and} \\ S(H, L) &= \bigcup_{F \in \mathcal{H},\, F \subset H,\, F \neq H} N(F, L). \end{aligned}$$

By remark 2.14, we have

$$X_0 = \bigcup_{H \in \mathcal{H}} \pi(N(H, L)).$$

Lemma 2.18 *Let $g \in G$, $H \in \mathcal{H}$, and put $x = \pi(g)$. Then*

$$g \in N(H, L) \Leftrightarrow F(x, L) \subset gHg^{-1}$$

and

$$g \in S(H, L) \Leftrightarrow F(x, L) \subset gHg^{-1} \text{ and } \dim F(x, L) < \dim gHg^{-1}.$$

In other words,

$$g \in N(H, L) \setminus S(H, L) \Leftrightarrow F(x, L) = gHg^{-1}.$$

In particular,

$$\pi(N(H, L) \setminus S(H, L)) = \pi(N(H, L)) \setminus \pi(S(H, L)). \tag{2.1}$$

Proof. The assertions easily follow from the definitions and theorem 2.4. □

Notation 2.19 For any $H \in \mathcal{H}$, define

$$\begin{aligned} T_H &= \pi(N(H, L) \setminus S(H, L)), \\ [H] &= \{\gamma H \gamma^{-1} : \gamma \in \Gamma\} \subset \mathcal{H}, \text{ and} \\ [\mathcal{H}] &= \{[H] : H \in \mathcal{H}\}. \end{aligned}$$

Lemma 2.20 *For any $H_1, H_2 \in \mathcal{H}$, the following holds:*

1. $[H_1] = [H_2] \Leftrightarrow T_{H_1} = T_{H_2}$.
2. $[H_1] \neq [H_2] \Leftrightarrow T_{H_1} \cap T_{H_2} = \emptyset$.

In particular, the notation $T_{[H]} := T_H$ is well defined, and X_0 is the disjoint union of $\{T_{[H]} : [H] \in [\mathcal{H}]\}$.

Proof. Suppose that $[H_1] = [H_2]$. Let $\gamma \in \Gamma$ be such that $H_2 = \gamma H_1 \gamma^{-1}$. Then for any $g \in G$ and $x = \pi(g)$, by lemma 2.18,

$$\begin{aligned} g \in N(L, H_2) \setminus S(L, H_2) &\Leftrightarrow F(x, L) = gH_2g^{-1} = (g\gamma)H_1(g\gamma)^{-1} \\ &\Leftrightarrow g\gamma \in N(L, H_1) \setminus S(L, H_1). \end{aligned}$$

Hence $T_{H_1} = T_{H_2}$.

Suppose that there exists $x \in T_{H_1} \cap T_{H_2}$. Then there exist $g_1, g_2 \in \pi^{-1}(x)$ such that $H_1 = {g_1}^{-1} F(x, L) g_1$ and $H_2 = {g_2}^{-1} F(x, L) g_2$. Now $g_1 = g_2\gamma$ for some $\gamma \in \Gamma$. Therefore $H_2 = \gamma H_1 \gamma^{-1}$. Hence $[H_1] = [H_2]$.

From this discussion the statements (1) and (2) follow. □

Now we state the fundamental theorem due to Ratner describing ergodic invariant measures for actions of the subgroups L on G/Γ as above.

Theorem 2.21 (Ratner [30]) *Let μ be an L-invariant and L-ergodic probability measure on X. Let*

$$F = \{g \in G : g \cdot \mu = \mu\}.$$

Then F acts transitively on $\mathrm{supp}(\mu)$*; that is, μ is the unique F-invariant probability measure on a closed F-orbit.*

Using this result we give a description of any finite L-invariant probability measure on X.

Proposition 2.22 *Let μ be a L-invariant probability measure on X. For every $[H] \in [\mathcal{H}]$, let $\mu_{[H]}$ denote the restriction of μ on $T_{[H]}$. Then the following statements hold.*

1. *The measure $\mu_{[H]}$ is L-invariant, and any L-ergodic component of $\mu_{[H]}$ is of the form $g\lambda$, where $g \in N(H, L) \setminus S(H, L)$ and λ is a H-invariant measure on $H\Gamma/\Gamma$.*

2. *For $H_1, H_2 \in \mathcal{H}$, if $[H_1] \neq [H_2]$, then the measures $\mu_{[H_1]}$ and $\mu_{[H_2]}$ are mutually singular.*

3. *For any measurable set $A \subset X$,*

$$\mu(A) = \sum_{[H] \in [\mathcal{H}]} \mu_{[H]}(A).$$

In particular, if $\mu(\pi(S(G, L)) = 0$ then Γ is a lattice in G and μ is the unique G-invariant probability measure on X.

Proof. Since μ and T_H are invariant under the action of L, so is $\mu_{[H]}$. The collection,

$$\{g\pi(H) \cap T_H : g \in N(H, L) \setminus S(H, L)\}$$

forms a measurable partition of $T_{[H]}$ into L-invariant atoms. Hence any ergodic component of $\mu_{[H]}$, say ν, is supported on a closed orbit of the form $g\pi(H)$ for some $g \in N(H, L)$. Now for every $x \in g\pi(H) \cap T_H$, we have that $F(x, L) = gHg^{-1}$. Hence by theorem 2.21, ν is gHg^{-1}-invariant. Hence $\lambda = g^{-1}\nu$ is an H-invariant measure supported on $\pi(H)$. This proves (1).

The statement (2) follows from lemma 2.20(2).

To prove part (3), let $\nu \in \mathcal{P}(X)$ be any ergodic component of μ. Let $x \in \mathrm{supp}(\nu)$ be such that the orbit Lx is dense in $\mathrm{supp}(\nu)$. By Ratner's theorem, $\mathrm{supp}(\nu) = \Lambda(\nu)x$. Clearly, $\Lambda(\nu)^0 = F(x, L)$. If we put $H = g^{-1}F(x, L)g$ for any $g \in \pi^{-1}(x)$, then by theorem 2.4, $H \in \mathcal{H}$. Now by lemma 2.18, we have that $g \in N(H, L) \setminus S(H, L)$. Also note that $\nu(\pi(S(H, L))) = 0$. Therefore $\nu(X \setminus T_{[H]}) = 0$. This shows that ν is an ergodic component of $\mu_{[H]}$. In view of statement (2), this implies the statement (3). $\square$

2.3 Linear presentation of G-actions near singular sets

Let V be the representation of G as described in notation 1.13. For $H \in \mathcal{H}$, let $\eta_H : G \to V$ be the map defined by $\eta_H(g) = g\mathbf{p}_H = (\wedge^d \mathrm{Ad} g)\mathbf{p}_H$ for all $g \in G$. Let $N_G(H)$ denotes the normalizer of H in G. Define

$$N_G^1(H) = {\eta_H}^{-1}(\mathbf{p}_H) = \{g \in N_G(H) : \det(\mathrm{Ad} g|_{\underline{\mathbf{h}}}) = 1\}.$$

Applying proposition 2.16 we deduce the following.

Theorem 2.23 ([14, Theorem 3.4]) *The orbit $\Gamma\mathbf{p}_H$ is closed, and hence discrete. In particular, the following holds.*

1. *The map $\phi : G/\Gamma_H \to G/\Gamma \times V$ defined by*
$$\phi(g\Gamma_H) = (\pi(g), \eta_H(g)), \quad \forall g \in G,$$
is proper.
2. *The orbit $N_G^1(H)\Gamma$ is closed in G/Γ.*
3. *For every $x \in G/\Gamma$, the set $\eta_H(\pi^{-1}(x))$ of representatives of x in V is discrete.*
4. *For any compact set $Z \subset G/\Gamma$, the set $\eta_H(\pi^{-1}(Z))$ is closed in V.*

Proof. First note that for any $g \in G$,

$$\begin{aligned} \|gp\| &= \text{Jacobian of the linear map} : w \in \underline{\mathbf{h}} \mapsto (\mathrm{Ad} g)w \in (\mathrm{Ad} g)\underline{\mathbf{h}} \\ &= \mathrm{vol}_\sigma(g\pi(H))/\mathrm{vol}_\sigma(\pi(H)). \end{aligned} \tag{2.2}$$

For any $\gamma \in \Gamma$, we have $\gamma\pi(H) = \pi(\gamma H \gamma^{-1})$. Therefore, for any $c > 0$,

$$\begin{aligned} & \#\{\eta_H(\gamma) \in V : \gamma \in \Gamma, \|\eta_H(\gamma)\| < c\} \\ \leq\ & \#\{F \in \mathcal{H} : [F] = [H], \mathrm{vol}_\sigma(\pi(F))/\mathrm{vol}_\sigma(\pi(H)) < c\} \\ <\ & \infty, \text{ due to proposition 2.16.} \end{aligned}$$

This shows that $\eta_H(\Gamma)$ is discrete in V, proving the main part of the theorem.

To prove statement (1), let K be a compact subset of G and D a compact subset of V. Define

$$S = \{\gamma \in \Gamma : \gamma\mathbf{p}_H \in K^{-1} \cdot D\}.$$

Since $K^{-1} \cdot D$ is compact and $\eta_H(\Gamma)$ is discrete, the set $S\mathbf{p}_H$ is finite. Since $S \subset \Gamma$ and $\Gamma_{\mathbf{p}_H} = \Gamma_H$, the set $q(S)$ is finite, where $q : G \to G/\Gamma_H$ is the quotient map. Note that

$$\phi^{-1}(\pi(K), D) \subset K\phi^{-1}(\pi(e), K^{-1}D) = Kq(S).$$

Therefore $\phi(K, D)$ is compact in G/Γ_H. This completes the proof of statement (1).

The rest of the statements are easy consequences of statement (1). $\square$

Proposition 2.24 (cf. [14, Prop. 3.2]) *Let $V(H,L)$ be the linear span of $\eta_H(N(H,L))$ in V. Then*

$$\eta_H{}^{-1}(V(H,L)) = N(H,L).$$

Proof. For any $\mathbf{l}$ in the Lie algebra $\underline{\mathbf{l}}$ of L, let a linear map $\phi_{\mathbf{l}} : V \to V$ be defined as $\phi(\mathbf{v}) = \mathbf{l} \wedge \mathbf{v}$ for all $\mathbf{v} \in V$. For any $g \in G$,

$$\begin{aligned} g \in N(H,L) &\Leftrightarrow \underline{\mathbf{l}} \subset \mathrm{Ad}g(\underline{\mathbf{h}}) \\ &\Leftrightarrow \eta(g) \in \ker_{\mathbf{l}\in\underline{\mathbf{l}}} \phi_{\mathbf{l}} \\ &\Leftrightarrow \eta(g) \in \text{ Linear span of } \eta(N(H,L)). \end{aligned}$$

□

Notation 2.25 *Put $\Gamma_H = N_G(H) \cap \Gamma$; then $\gamma\pi(H) = \pi(H)$. Therefore by equation 2.2, $\gamma\mathbf{p}_H = \pm\mathbf{p}_H$. In view of this we define $\bar{V} = V/\{Id, -Id\}$, if $\Gamma_H\mathbf{p}_H = \{\mathbf{p}_H, -\mathbf{p}_H\}$, and define $\bar{V} = V$ if $\Gamma_H\mathbf{p}_H = \mathbf{p}_H$. The action of G factors through the quotient map from V onto $\bar{V}$. Let $\bar{\mathbf{p}}_H$ denote the image of $\mathbf{p}_H$ in $\bar{V}$, and define $\bar{\eta}_H : G \to \bar{V}$ as $\bar{\eta}_H(g) = g\bar{\mathbf{p}}_H$ for all $g \in G$. Now $\Gamma_H = \bar{\eta}_H{}^{-1}(\bar{\mathbf{p}}_H) \cap \Gamma$. Let $\bar{V}(H,L)$ denote the image of $V(H,L)$ in $\bar{V}$. Note that the inverse image of $\bar{V}(H,L)$ in V is $V(H,L)$.*

For any subset Z of G/Γ, define

$$\mathrm{Rep}(Z) := \{g\bar{\mathbf{p}}_H \in \bar{V} : g \in G, \pi(g) \in Z\}.$$

Proposition 2.26 *Let $H \in \mathcal{H}$ and D be a compact subset of $\bar{V}(H,L)$. Let K be a compact subset of G/Γ. Define*

$$\mathcal{S}(K,D) = \{x \in K : \#(\mathrm{Rep}(x) \cap D) > 1\}.$$

Then the following holds.

1. *$\mathcal{S}(K,D))$ is compact.*
2. *There exist $m \in \mathbf{N}$ and $F_i \in \mathcal{H}$, where $F_i \subset H$ and $\dim F_i < \dim H$ for $1 \le i \le m$, such that*
$$\mathcal{S}(K,D) \subset \pi\left(\bigcup_{i=1}^{m} N_G(F_i,L)\right).$$
3. *Given any compact set $K_1 \subset K \setminus \mathcal{S}(K,D)$, there exists a neighbourhood Φ of D in $\bar{V}$ such that for any $x \in K_1$, the set $\mathrm{Rep}(x) \cap \Phi$ contains at most one element.*

Proof. By theorem 2.23, there exists a compact set $C \subset N(H,L)$ such that $q(C) = \phi^{-1}(K,D) \subset G/\Gamma_H$. Put

$$C^* = C \cap \left(\bigcup_{\gamma\in\Gamma\setminus\Gamma_H} C\gamma\right).$$

Then $\mathcal{S}(K,D)=\pi(C^*)$. Put $\Delta=(\Gamma\setminus\Gamma_H)\cap C^{-1}C$. Then

$$C^*=C\cap\left(\bigcup_{\gamma\in\Delta}C\gamma\right).$$

Since Δ is finite, C^* is compact. This proves (1).

Let $x\in S(K,D)$. Then there exist $c\in C^*$ and $\gamma\in\Delta$ such that $x=\pi(c)$ and $c\gamma\in C$. Since $C\subset N(H,L)$, we have $c^{-1}Lc\subset\gamma H\gamma^{-1}$. Let W be the subgroup generated by all unipotent one-parameter subgroups contained in $H\cap\gamma H\gamma^{-1}$. Put $F(\gamma)=F(\pi(e),W)$; clearly $F(\gamma)=F(\Gamma,L)$ in the earlier notation. Then $F(\gamma)\in\mathcal{H}$, $F(\gamma)\subset H\cap\gamma H\gamma^{-1}$, and $c\in N(F(\gamma),L)$. Since $\gamma\notin\Gamma_H$, we have $\dim F(\gamma)<\dim H$. Thus,

$$C^*\subset\cup_{\gamma\in\Delta}N(F(\gamma),L).$$

This proves (2).

Let $\{\Phi_i\}_{i\in\mathbf{N}}$ be a decreasing sequence of relatively compact neighbourhoods of D such that $\cap_{i\in\mathbf{N}}\overline{\Phi}_i=D$. Let C_i be a compact subset of G such that $q(C_i)=\phi^{-1}(K,\overline{\Phi}_i)$. Put $\Delta_i={C_i}^{-1}C_i\cap(\Gamma\setminus\Gamma_H)$ and

$$C_i^*=C_i\cap\left(\cup_{\gamma\in\Gamma\setminus\Gamma_H}C_i\gamma\right)=C_i\cap\left(\cup_{\gamma\in\Delta_i}C_i\gamma\right).$$

Put $\cap_{i\in\mathbf{N}}C_i=C$. Since ϕ is a proper map, we have $q(C)=\phi^{-1}(K,D)$. Therefore $\cap_{i\in\mathbf{N}}\Delta_i=\Delta$. Since Δ_i is finite for each $i\in\mathbf{N}$, there exists $i_1\in\mathbf{N}$ such that $\Delta_i=\Delta$ for all $i\geq i_1$. Therefore, $\cap_{i\in\mathbf{N}}C_i^*=C^*$. Hence given a compact set $K_1\subset K\setminus\pi(C^*)$, there exists $i_0\in\mathbf{N}$ such that $K_1\subset K\setminus\pi(C_i^*)$ for all $i\geq i_0$. Now the statement (3) holds for $\Phi=\Phi_{i_0}$. $\square$

2.4 Dynamics of polynomial trajectories near singular sets

The following growth property of polynomial maps has turned out to be of great significance in the study of polynomial trajectories near affine algebraic varieties.

Proposition 2.27 ([14, Proposition 4.2]) *Let a compact set $C\subset\bar{V}(H,L)$, an $\epsilon>0$ and a $d\in\mathbf{N}$ be given. Then there exists a larger compact set $D\subset\bar{V}(H,L)$ such that the following property holds: for any neighbourhood Φ of D in $\bar{V}$ there exists a neighbourhood Ψ of C in $\bar{V}$ such that for any $\theta\in\mathcal{P}_d(G)$, any $\mathbf{w}\in\bar{V}$ and any bounded interval (a,b) of $\mathbf{R}$, if $\theta(a)\mathbf{w}\notin\Phi$, then*

$$|\{t\in(a,b):\theta(t)\mathbf{w}\in\Psi\}|<\epsilon\cdot|\{t\in(a,b):\theta(t)\mathbf{w}\in\Phi\}|. \tag{2.3}$$

Proof. Let $\mathcal{C}$ be a finite collection of linear functionals on V such that

$$V(H,L)=\cap_{f\in\mathcal{C}}f^{-1}(0).$$

For any $f\in\mathcal{C}$, $\theta\in\mathcal{P}_d(G)$ and $\mathbf{w}\in V$, the maps $t\mapsto f(\theta(t)\cdot\mathbf{w})$ and $t\mapsto\|\theta(t)\cdot\mathbf{w}\|^2$ are in $\mathcal{P}_{d_1}$ for some $d_1\in\mathbf{N}$ depending on d and $\dim G$.

By lemma 1.4, there exists $M > 1$ such that for any bounded interval J and $\psi \in \mathcal{P}_{d_1}$, we have

$$\left| \{ t \in J : |\psi(t)| < (1/M) \sup_{t \in J} |\psi(t)| \} \right| \leq \epsilon \cdot |J|.$$

For $R > 0$, define $B(R) = \{\mathbf{w} \in \bar{V}(H,L) : \|\mathbf{w}\|^2 < R\}$. Let $R > 0$ be such that $C \subset B(R)$. Put

$$D = \bar{V}(H,L) \cap \overline{B(MR)}.$$

For $c > 0$, let $Z_c(\mathcal{C})$ be the image of the set $\{\mathbf{w} \in V : |f(\mathbf{w})| < c, \forall f \in \mathcal{C}\}$ in $\bar{V}(H,L)$. Now given a neighborhood Φ of D, there exists $c > 0$ such that $Z_c(\mathcal{C}) \cap \overline{B(MR)} \subset \Phi$. Put

$$\Psi = Z_{c/M}(\mathcal{C}) \cap B(R).$$

Then Ψ is a neighborhood of C contained in Φ.

Fix any $\mathbf{w} \in V$, let $\bar{\mathbf{w}}$ denote its image in $\bar{V}(H,L)$. Let J be any connected component of I_2. Suppose that $\theta(a) \cdot \bar{\mathbf{w}} \not\subset \Phi$. Then there exists $a_1 \in \overline{J}$ such that $\theta(a_1) \cdot \bar{\mathbf{w}} \notin \Phi$. Therefore either $|f_0(\theta(a_1) \cdot \mathbf{w})| \geq c$ for some $f_0 \in \mathcal{C}$ or $\|\theta(a_1) \cdot \mathbf{w}\|^2 \geq MR$. Hence by the choice of $M > 0$, we have that

$$\left| \{ t \in J : |f(\theta(t) \cdot \mathbf{w})| < c/M \text{ and } \|\theta(t) \cdot \mathbf{w}\|^2 < R \} \right| \leq \epsilon \cdot |J|.$$

From this eq. 2.3 follows.

□

Proposition 2.28 *Given a compact set $C \subset N(H,L) \setminus S(H,L)$, an $\epsilon > 0$, and a $d \in \mathbf{N}$, there exists a neighbourhood Ω of $\pi(C)$ in G/Γ such that for any $\theta \in \mathcal{P}_d(G)$, one of the following conditions is satisfied.*

(i) There exists $\gamma \in \Gamma$ such that

$$\pi(\theta(\mathbf{R})) \subset \theta(0)\gamma\pi(N_G^1(H)).$$

Moreover if $\pi(\theta(0)) \in C$ then γ can be chosen to depend only on $\theta(0)$, rather than the map θ.

(ii) There exists $T_0 \geq 0$ such that for all $T > T_0$,

$$|\{t \in (0,T) : \pi(\theta(t)) \in \Omega\}| \leq \epsilon T.$$

Proof. Let a compact set $D \subset \bar{V}(H,L)$ be as in proposition 2.27, for $\epsilon/2$ in place of ϵ. Let K be any compact neighbourhood of $\pi(C)$ in G/Γ. By proposition 2.26, we have $\pi(C) \cap \mathcal{S}(K,D) = \emptyset$. Let Ω_1 be an open neighbourhood of $\pi(C)$ such that $\overline{\Omega_1} \subset K \setminus \mathcal{S}(K,D)$. Again by proposition 2.26, there exists a neighborhood Φ of D such that for every $x \in \Omega_1$, the set $\mathrm{Rep}(x) \cap \Phi$ contains at most one element.

By the choice of D there exists a neighborhood Ψ of C contained in Φ such that eq. 2.3 holds for $\epsilon/2$ in place of ϵ.

Put

$$\begin{aligned} \Omega &= \{y \in \Omega_1 : \mathrm{Rep}(y) \cap \Psi \neq \emptyset\} \quad \text{and} \\ E &= \{t \in \mathbf{R} : \pi(\theta(t)x) \in \Omega\}. \end{aligned} \tag{2.4}$$

By the choice of Φ, for every $t \in E$, there exists a unique $\mathbf{w}_t \in \Gamma \cdot \bar{\mathbf{p}}_H$ such that $\theta(t) \cdot \mathbf{w}_t \in \Phi$.

Suppose that for some $s \in E$, there exists an unbounded interval J containing $\mathbf{s}$ such that $\theta(J) \cdot \mathbf{w}_s \subset \Phi$. Since $\theta \in \mathcal{P}_d(G)$, we have $\theta(t) \cdot \mathbf{w}_s = \theta(0) \cdot \mathbf{w}_s$ for all $t \in \mathbf{R}$. Let $\gamma \in \Gamma$ such that $\mathbf{w}_s = \gamma \cdot \bar{\mathbf{p}}_H$. Then condition (1) holds for $g = \theta(0)\gamma$. Also $\mathbf{w}_t = \mathbf{w}_s$ for every $t \in E$.

Therefore if condition (1) does not hold, then there exists $T_0 \geq 0$ with the following property: Take any $T > T_0$ and put $I = [0, T]$. Then for any $t \in E \cap I$, there exists a largest open interval $I(t) \subset I$ containing t such that

$$\theta(I(t)) \cdot \mathbf{w}_t \subset \Phi \text{ and } \theta(\overline{I(t)}) \cdot \mathbf{w}_t \not\subset \Phi. \tag{2.5}$$

Put $\mathcal{I} = \{I(t) : t \in E \cap I\}$. Then for any $I_1 \in \mathcal{I}$ and $s \in I_1 \cap E$, we have $I(s) = I_1$. Therefore for any $t_1, t_2 \in E \cap I$, if $t_1 < t_2$ then either $I(t_1) = I(t_2)$ or $I(t_1) \cap I(t_2) \subset (t_1, t_2)$. Hence every $t \in I$ is contained in at most two distinct elements of $\mathcal{I}$. Thus

$$\sum_{I_1 \in \mathcal{I}} |I_1| \leq 2|I|. \tag{2.6}$$

Now by eqs. 2.3 and 2.5, for any $t \in E \cap I$,

$$|\{s \in I(t) : \theta(s) \cdot \mathbf{w}_t \in \Psi\}| < (\epsilon/2) \cdot |I(t)|. \tag{2.7}$$

Therefore by eqs. 2.6 and 2.7, we get

$$|E \cap I| \leq (\epsilon/2) \cdot \sum_{I_1 \in \mathcal{I}} |I_1| \leq \epsilon \cdot |I|.$$

This completes the proof. □

The use of a result like proposition 2.28 is illustrated here by an alternate proof of Ratner's distribution rigidity theorem.

Theorem 2.29 (Ratner [31]) *Let G be a Lie group and Γ a lattice in G. Let $U = \{u(t)\}$ be a unipotent one-parameter subgroup of G and $x \in G/\Gamma$. Then there exists a closed subgroup F of G containing U such that the orbit Fx is closed, it admits a finite F-invariant probability measure, say μ, and the trajectory $\{u(t)x : t > 0\}$ is uniformly distributed with respect to μ.*

Proof. For $T > 0$, let $\nu_T \in \mathcal{P}(G/\Gamma)$ be such that for any bounded continuous function f on G/Γ,

$$\int_{G/\Gamma} f \, d\nu_T = \frac{1}{T} \int_0^T f(u(t)x) \, dt.$$

Let $T_i \to \infty$ be any sequence in $\mathbf{R}$ such that $\nu_{T_i} \to \nu$ for some $\nu \in \mathcal{P}(G/\Gamma \cup \{\infty\})$ in the weak* topology. In view of the theorem due to Dani and Margulis as stated in the introduction, we have that $\nu(\{\infty\}) = 0$. Hence $\nu \in \mathcal{P}(G/\Gamma)$. It is straightforward to verify that ν is U-invariant.

Now in view of lemma 2.20, let $H \in \mathcal{H}$ be of smallest possible dimension such that $\nu_{[H]} \neq 0$. Let C_1 be any compact subset of $N(U,H) \setminus S(U,H)$ such that $\nu_{[H]}(\pi(C_1)) = \epsilon > 0$. Then for any neighbourhood Ω of $\pi(C_1)$, we have that $\nu_{T_i}(\Omega) > \epsilon$ for all large $i \in \mathbf{N}$. Apply proposition 2.28 for $C = C_1$ and the ϵ as above. Then the condition (2) of the proposition fails to hold for $\{u(t)\}$, x, and any neighbourhood Ω of $\pi(C_1)$. Hence there exists $g \in \pi^{-1}(x)$ such that $g \in C$. Hence we have that $g \in N(U,H)$. Therefore $Ux \subset gHg^{-1}x = g(H\Gamma/\Gamma)$.

If $\dim H < \dim G$, then the proof can be completed by an obvious inductive argument. Therefore we may assume that $H = G$. Hence $\nu = \nu_{[H]}$. Now by lemma 2.20, every ergodic component of $\nu_{[H]}$ is G-invariant. Hence ν is G-invariant. Thus we showed that every limit point of the set $\{\nu_T : T > 0\}$ in $\mathcal{P}(G/\Gamma \cup \{\infty\})$ is the unique G-invariant probability measure on G/Γ. This proves that $\nu_T \to \mu$ as $T \to \infty$, where μ is the G-invariant probability measure on G/Γ. This completes the proof. $\square$

Now we generalize proposition 2.28 in the case of $\Theta \in \mathcal{P}_{d,m}(G)$ instead of $\theta \in \mathcal{P}_{d,1}(G)$. Our proof of this generalization requires the following additional condition.

Assumption: G/Γ admits a finite G-invariant measure.

Theorem 2.30 *Given a compact set $C_1 \subset N(H,L) \setminus S(H,L)$, an $\epsilon > 0$, and $d, m \in \mathbf{N}$, there exists a neighbourhood Ω of $\pi(C)$ in G/Γ such that for any $x \in G/\Gamma$ and any sequence $\Theta_i \to \Theta_0$ in $\mathcal{P}_{d,m}(G)$, one of the following conditions is satisfied.*

(I) There exists $g \in \pi^{-1}(\Theta_0(0)x)$ such that

$$\Theta_0(\mathbf{R}^m)x \subset g\pi(N_G^1(H)).$$

(II) There exists a bounded open convex set $B' \subset \mathbf{R}^m$ and $i_0 \in \mathbf{N}$ such that for all bounded open convex subsets B of $\mathbf{R}^m$ containing B' and all $i \geq i_0$,

$$\frac{1}{|B|}|\{\mathbf{t} \in B : \Theta_i(\mathbf{t})x \in \Omega\}| < \epsilon.$$

We shall give a proof by induction on $\dim H$. For this purpose we first need to prove a stronger and technical version of it.

Proposition 2.31 *Let $\epsilon > 0$, $d \in \mathbf{N}$, and a compact set $K \subset G/\Gamma$ be given. Let C_1 be a compact subset of $N(H,L)$. Then there exist compact sets $D \subset \bar{V}(H,L)$ and $S_1 \subset \cup_{i=1}^m N_G(F_i,L)$, where $m \in \mathbf{N}$, and $F_i \in \mathcal{H}$ with $F_i \subset H$ and $\dim F_i < \dim H$ for $1 \leq i \leq m$, such that the following holds: Let a neighbourhood Φ of D in $\bar{V}$ and a compact set $Z \subset K \setminus \pi(S_1)$ be given. Then there exists a neighbourhood Ω of $\pi(C_1)$ in G/Γ such that for any $x \in G/\Gamma$, $\Theta \in \mathcal{P}_{d,m}(G)$, and a bounded open convex set $B \subset \mathbf{R}^m$, one of the following conditions is satisfied:*

(1) $\Theta(B)x \cap Z = \emptyset$.

(2) $\Theta(B)v \subset \Phi$ *for some* $v \in \mathrm{Rep}(x)$.

(3)

$$\frac{1}{|B|}|\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in \Omega\}| < \epsilon.$$

First we derive theorem 2.30 from proposition 2.31.

Proof of theorem 2.30: Let K be the closure of a relatively compact neighbourhood of $\pi(C_1)$. Obtain D and S_1 using proposition 2.31.

Due to eq. 2.1, $\pi(C_1) \cap \pi(S(H,L)) = \emptyset$. Therefore there exists a neighbourhood of $\pi(C_1)$ containded in K with its closure Z such that $Z \cap \pi(S_1) \cup \mathcal{S}(K,D) = \emptyset$. Using proposition 2.26 obtain a neighbourhood Φ of D in $\bar{V}$ such that every $x \in Z$ has at most one representative in Φ. Now using proposition 2.31, obtain a neighbourhood Ω of $\pi(C_1)$ contained in Z such that for any $\Theta \in \mathcal{P}_{d,m}(G)$ and any bounded open convex set $B \subset \mathbf{R}^m$, at least one of the three possibilities of its conclusion holds.

Suppose that the possibility (II) of the present theorem does not hold. Then after passing to a subsequence, there exists a sequence $\{B_i\}_{i\in\mathbf{N}}$ of bounded open convex sets in $\mathbf{R}^m$ such that $B_i \subset B_{i+1}$ for all $i \in \mathbf{N}$, $\cup_{i\in\mathbf{N}} B_i = \mathbf{R}^m$, and for each $i \in \mathbf{N}$, the possibility (3) fails to hold for B_i in place of B and Θ_i in place of Θ. Now since $\Omega \subset Z$, the possibility (1) cannot hold. Therefore the possibility (2) must hold.

Hence for every $i \in \mathbf{N}$, there exists $v_i \in \mathrm{Rep}(x)$ such that $\Theta_i(B_i)v_i \subset \Phi$; in particular, $v_i \subset \Theta_i(B_1)^{-1}\Phi$. Since $\Theta_i \to \Theta_0$ as $i \to \infty$, the set $\left(\cup_{i\in\mathbf{N}}\Theta_i(B_1)^{-1}\right)\Phi$ is relatively compact. Since $\mathrm{Rep}(x)$ is discrete, by passing to a subsequence we may assume that $v_i = v_1$ for all $i \in \mathbf{N}$. Therefore, $\Theta_0(\mathbf{R}^m)v_1 \subset \Phi$. Since the map $\mathbf{R}^m \ni \mathbf{t} \mapsto \Theta_0(\mathbf{t})v_1 \in V$ is polynomial, we have that $\Theta_0(\mathbf{R}^m)v_1 = \Theta(0)v_1$. Let $g_1 \in \pi^{-1}(x)$ such that $v_1 = g_1\mathbf{p}_H$. Since the stabilizer of v_1 in G is $g_1 N_G^1(H) {g_1}^{-1}$, we have $\Theta_0(\mathbf{R}^m)g_1 \subset gN_G^1(H)$, where $g = \Theta_0(\mathbf{t}_0)g_1$. Thus the possibility (I) of the present theorem holds. This completes the proof. □

For the purpose of proving proposition 2.31 by using induction on $\dim H$ we need its following consequence.

Corollary 2.32 *Let* $\epsilon > 0$*,* $d \in \mathbf{N}$*, and a compact set* $K \subset G/\Gamma$ *be given. Let* C_1 *be a compact subset of* $\cup_{i=1}^n N_G(H_i, L)$*, where* $n \in \mathbf{N}$ *and* $H_i \in \mathcal{H}$ *for* $1 \le i \le n$*. Then there exists a compact set* $D_1 \subset \cup_{i=1}^n N_G(H_i, L)$ *such that the following holds: Given a compact set* $Z \subset K \setminus \pi(D_1)$*, there exists a neighbourhood* Ω *of* $\pi(C_1)$ *in* G/Γ *such that for any* $x \in G/\Gamma$*,* $\Theta \in \mathcal{P}_{d,m}(G)$*, and a bounded open convex set* $B \subset \mathbf{R}^m$*, either*

$$\Theta(B)x \cap Z = \emptyset$$

or

$$\frac{1}{|B|}|\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in \Omega\}| < \epsilon. \tag{2.8}$$

Proof. We prove the result for $n = 1$, the general case follows easily from this.

Using proposition 2.31 for $H = H_1$, obtain compact sets $D \subset \bar{V}(H, L)$ and

$$S_1 \subset \cup_{i=1}^m N_G(F_i, L),$$

where $m \in \mathbf{N}$, and $F_i \in \mathcal{H}$ with $F_i \subset H$, and $\dim F_i < \dim H$ for $1 \leq i \leq k$.

By theorem 2.23(4) and proposition 2.24, there exists a compact set $\tilde{D} \subset N(H, L)$ such that

$$\{x \in K : \mathrm{Rep}(x) \in D\} = \pi(\tilde{D}).$$

Now $D_1 = S_1 \cup \tilde{D}$ is a compact subset $N(H, L)$.

Let $Z \subset K \setminus \pi(D_1)$ be a given compact set. Then $\mathrm{Rep}(Z) \cap D = \emptyset$. By theorem 2.23, $\mathrm{Rep}(Z)$ is closed in $\bar{V}$. Therefore there exists a neighbourhood Φ of D in $\bar{V}$ such that

$$\Phi \cap \mathrm{Rep}(Z) = \emptyset. \tag{2.9}$$

Suppose that $\Theta(B)x \cap Z \neq \emptyset$. Then the Possibilities (1) and (2) of proposition 2.31 cannot hold due to eq. 2.9. Therefore the possibility (3) of the proposition must hold, and hence eq. 2.8 holds. This completes the proof. □

Proof of proposition 2.31: Let Ω_1 be a relatively compact neighbourhood of $\pi(C_1)$ in G/Γ. By theorem 1.12, there exists a compact set $K_1 \subset G/\Gamma$ such that for any $y \in \overline{\Omega}_1$, $\theta \in \mathcal{P}_d(G)$, and $T > 0$,

$$\frac{1}{T}\ell(\{t \in [0, T] : \theta(t)x \in K_1\}) > 1 - 1/(4k). \tag{2.10}$$

For the compact set $C = C_1\mathbf{p}_H \subset \bar{V}(H, L)$, obtain a compact set $D \subset \bar{V}(H, L)$ such that the conclusion of proposition 2.27 is satisfied for $\epsilon/(4k)$ in place of ϵ.

As we mentioned earlier, we shall prove this theorem by induction on $\dim H$. Note that when $\dim H$ is small, $S(H, L) = \emptyset$ and hence $\mathcal{S}(K_1, D) = \emptyset$. We begin with some observations for which we first assume that $\mathcal{S}(K_1, D) \neq \emptyset$.

By proposition 2.26, there exists $m \in \mathbf{N}$ and for each $1 \leq i \leq m$ there exists $F_i \in \mathcal{H}$ with $F_i \subset H$ and $\dim F_i < \dim H$ such that

$$\mathcal{S}(K_1, D) \subset \pi\left(\bigcup_{i=1}^m N_G(F_i, L)\right).$$

Therefore there exists a compact set $C_2 \subset \cup_{i=1}^m N_G(F_i, L)$ such that

$$\mathcal{S}(K_1, D) = \pi(C_2).$$

By induction we can assume that the theorem is true for each F_i. Hence the corollary 2.32 is valid for C_2 in place of C_1 and F_i in place of H_i in its statement. Thus we obtain a compact set

$$S_2 \subset \bigcup_{i=1}^m N_G(F_i, L)$$

such that the following holds: Given any compact set $Z_1 \subset \overline{\Omega}_1 \setminus \pi(S_2)$ there exists a neighbourhood Ω_2 of $\mathcal{S}(K_1, D)$ such that for any $y \in Z_1$, $\theta \in \mathcal{P}_d(G)$, and $T > 0$,

$$\frac{1}{T}\ell(\{t \in [0,T] : \theta(t)y \in \Omega_2\}) < 1/(4k). \tag{2.11}$$

Again we apply corollary 2.32 as above for S_2 in place of C_1 and obtain a compact set

$$S_1 \subset \bigcup_{i=1}^{m} N_G(F_i, L)$$

such that the following holds: Given a compact set $Z \subset K \setminus \pi(S_1)$ as in the statement of the proposition, there exists a neighbourhood Ω_3 of $\pi(S_2)$ such that for the given $x \in G/\Gamma$, $\Theta \in \mathcal{P}_{d,m}(G)$, and the bounded open convex set $B \subset \mathbf{R}^m$, either

$$\Theta(B)x \cap Z = \emptyset$$

or

$$\frac{1}{|B|}|\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in \Omega_3\}| < \epsilon/(4k). \tag{2.12}$$

Put

$$Z_1 = \overline{\Omega}_1 \setminus \Omega_3 \tag{2.13}$$

and obtain a neighbourhood Ω_2 of $\mathcal{S}(K_1, D))$ such that eq. 2.11 holds.

Suppose if $\mathcal{S}(K_1, D) = \emptyset$, the above equations are satisfied if we put $S_2 = S_1 = \emptyset$ and $\Omega_3 = \Omega_2 = \emptyset$.

Let Φ be a given neighbourhood of D as in the statement of the present proposition. Using proposition 2.26, we replace Φ by a smaller neighbourhood such that every $y \in K_1 \setminus \Omega_2$ has at most one representative in Φ.

By the choice of D, there exists a neighbourhood Ψ of C in $\bar{V}$ with $\Psi \subset \Phi$ such that for any $\theta \in \mathcal{P}_d(G)$, any $v \in \bar{V}$, and an interval (a, b), if $\theta(a)v \notin \Phi$, then

$$\ell(\{t \in [a,b] : \theta(t)v \in \Psi\}) < \epsilon/(4k) \cdot \ell(\{t \in [a,b] : \theta(t)v \in \Phi\}). \tag{2.14}$$

Put

$$\Omega = \{y \in \Omega_1 : \mathrm{Rep}(y) \cap \Psi \neq \emptyset\}. \tag{2.15}$$

Then Ω is an open neighbourhood of $\pi(C_1)$.

After having made the above constructions, we start analysing the Possibilities (1), (2), and (3) of the conclusion of the present proposition.

First suppose that the possibility (2) does not hold. Take any $v \in \mathrm{Rep}(x)$. Then there exists $\mathbf{t}_v \in B$ such that $\Theta(\mathbf{t}_v)v \notin \Phi$. Let S denote the unit sphere in $\mathbf{R}^m$ centered at the origin. Take any $\mathbf{x} \in S$. Define $\theta_{\mathbf{x}}(t) = \Theta(t\mathbf{x} + \mathbf{t}_v)$ for all $t \in \mathbf{R}$. Thus $\theta_{\mathbf{x}} \in \mathcal{P}_d(G)$. Define

$$\begin{aligned}
\Psi_{\mathbf{x}}(v) &= \{t \in [0,1] : \theta_{\mathbf{x}}(t)v \in \Psi\}, \\
\Phi_{\mathbf{x}}(v) &= \{t \in [0,1] : \theta_{\mathbf{x}}(t)v \in \Phi\}, \\
\Psi^*_{\mathbf{x}}(v) &= \{t \in \Psi_{\mathbf{x}}(v) : \theta_{\mathbf{x}}(t)x \in Z_1\}, \\
\Phi'_{\mathbf{x}}(v) &= \bigcup\{(a,b) \subset \Phi_{\mathbf{x}}(v) : (a,b) \cap \Psi^*_{\mathbf{x}}(v) \neq \emptyset\}, \text{ and} \\
\Phi^*_{\mathbf{x}}(v) &= \{t \in \Phi'_{\mathbf{x}}(v) : \theta_{\mathbf{x}}(t)x \in K_1 \setminus \Omega_2\}
\end{aligned}$$

Let $I = (a, b)$ be any connected component of $\Phi'_{\mathbf{x}}(v)$. Since $0 \notin \Phi_{\mathbf{x}}(v)$, we have $a \notin \Phi_{\mathbf{x}}(v)$. Therefore by eq. 2.14,

$$\ell(\Psi_{\mathbf{x}}(v) \cap I) < \epsilon/(4k)\ell(I). \tag{2.16}$$

Since $I \cap \Psi^*_{\mathbf{x}}(v) \neq \emptyset$, by eqs. 2.10, 2.11, and 2.13,

$$\ell(\Phi^*_{\mathbf{x}}(v) \cap I) > (1 - 1/(2k))\ell(I). \tag{2.17}$$

Define

$$\Psi^*(v) = \{\mathbf{t} \in B : \Theta(\mathbf{t})v \in \Psi,\, \Theta(\mathbf{t})x \in Z_1\} = \bigcup_{\mathbf{x} \in S} \Psi^*_{\mathbf{x}}(v)\mathbf{x} + \mathbf{t}_v, \tag{2.18}$$

$$\Phi'(v) = \bigcup_{\mathbf{x} \in S} \Phi'_{\mathbf{x}}(v)\mathbf{x} + \mathbf{t}_v, \text{ and}$$

$$\Phi^*(v) = \{\mathbf{t} \in \Phi'(v) : \Theta(\mathbf{t})x \in K_1 \setminus \Omega_2\} = \bigcup_{\mathbf{x} \in S} \Phi^*_{\mathbf{x}}(v)\mathbf{x} + \mathbf{t}_v. \tag{2.19}$$

Due to eq. 2.16 and eq. 2.17, we can apply lemma 1.10 for sets $\Psi'(v)$ in place of E, $\Phi^*(v)$ in place of F, and $\Phi^*(v)$ in place of D, and the constants $\epsilon_1 = \epsilon/(4k)$ and $\epsilon_2 = 1/(2k)$. Then

$$|\Psi^*(v)| \leq (\epsilon/2) \cdot |\Phi^*(v)|. \tag{2.20}$$

Observe that by our choice of Φ, for any two distinct $v_1, v_2 \in \mathrm{Rep}(x)$,

$$\Phi^*(v_1) \cap \Phi^*(v_2) = \emptyset. \tag{2.21}$$

Now by eqs.2.13, 2.15, 2.20, 2.21, and 2.18, we get

$$\begin{aligned} |\{\mathbf{t} \in B : \Theta(\mathbf{t})x \in \Omega \cap Z_1\}| &\leq |\cup_{v \in \mathrm{Rep}(x)} \Psi^*(v)| \\ &\leq \sum_{v \in \mathrm{Rep}(x)} |\Psi^*(v)| \\ &\leq (\epsilon/2) \cdot \sum_{v \in \mathrm{Rep}(x)} |\Phi^*(v)| \\ &\leq (\epsilon/2) \cdot |B|. \end{aligned} \tag{2.22}$$

Now suppose that possibility (1) also does not hold. Then eq. 2.12 holds. The possibility (3) follows from eq. 2.12, 2.13, and eq. 2.22. This completes the proof. $\square$

Chapter 3

On the space of ergodic invariant measures of unipotent flows

3.1 Main result and its consequences

Let G be a connected Lie group, Γ a discrete subgroup of G, and $\pi : G \to G/\Gamma$ the natural quotient map. Let X denote the homogeneous space G/Γ on which G acts by left translations.

Let $\mathcal{P}(X)$ denote the set of Borel probability measures on X equipped with the weak* topology. The group G acts on $\mathcal{P}(X)$ such that for every $g \in G$ and $\mu \in \mathcal{P}(X)$, we have $g\mu(A) = \mu(g^{-1}A)$ for all Borel measurable subsets $A \subset X$. The action $(g, \mu) \mapsto g\mu$ is continuous.

For $\mu \in \mathcal{P}(X)$, define

$$\mathrm{supp}(\mu) = \{x \in X : \mu(\Omega) > 0 \text{ for every neighbourhood } \Omega \text{ of } x \text{ in } X\}$$

Then $\mathrm{supp}(\mu)$ is a closed subset of X. Also define the invariance group

$$\Lambda(\mu) = \{g \in G : g\mu = \mu\}.$$

Then $\Lambda(\mu)$ is a closed (and hence a Lie) subgroup of G.

Let $\mathcal{Q}(X) = \{\mu \in \mathcal{P}(X)$: the group generated by all unipotent one-parameter subgroups of G contained in $\Lambda(\mu)$ acts ergodically on X with respect to $\mu\}$.

Recall that by lemma 2.11, every $\mu \in \mathcal{Q}(X)$ is ergodic for the action of a single unipotent one-parameter subgroup of G, say $\{u(t)\}$ contained in $\Lambda(\mu)$. And by the Birkhoff ergodic theorem, for almost all $x \in \mathrm{supp}(\mu)$, the trajectory $\{u(t)x : t > 0\}$ is uniformly distributed with respect to μ.

Now we state the main result of this chapter.

Theorem 3.1 *Let $\{\{u_i(t)\}_{t\in\mathbf{R}}\}$ be a sequence of unipotent one-parameter subgroups of G, and let $\{\mu_i\}$ be a sequence in $\mathcal{P}(X)$ such that for each $i \in \mathbf{N}$, μ_i is an ergodic $\{u_i(t)\}_{t\in\mathbf{R}}$-invariant measure. Suppose that $\mu_i \to \mu$ in $\mathcal{P}(X)$, and let $x \in \mathrm{supp}(\mu)$. Then the following holds:*

1. $\mathrm{supp}(\mu) = \Lambda(\mu)x$.

2. Let $g_i \to e$ be a sequence in G such that for every $i \in \mathbf{N}$, $g_i x \in \operatorname{supp}(\mu_i)$ and the trajectory $\{u_i(t)g_i x : t > 0\}$ is uniformly distributed with respect to μ_i. Then there exists $i_0 \in \mathbf{N}$ such that for all $i \geq i_0$,

$$\operatorname{supp}(\mu_i) \subset g_i \cdot \operatorname{supp}(\mu).$$

3. Let L be the subgroup generated by all the (unipotent one-parameter) subgroups ${g_i}^{-1}\{u_i(t)\}g_i$, $i \geq i_0$. Then μ is invariant and ergodic for the action of L on X.

In particular, $\mathcal{Q}(X)$ is a closed subset of $\mathcal{P}(X)$. Also for any compact set $K \in X$, the set $\mathcal{Q}(K) := \{\mu \in \mathcal{Q}(X) : K \cap \operatorname{supp}(\mu) \neq \emptyset\}$ is closed in $\mathcal{P}(X)$.

In view of a theorem of Dani and Margulis that in a finite volume homogeneous space, unipotent trajectories starting from a fixed compact set visit a (possibly larger) fixed compact set with frequency (of visit) close to 1 (see theorem 1.12), we deduce the some useful consequences of the main theorem under the following additional condition on X.

Assumption: The homogeneous space X admits a finite G-invariant measure.

Let $X \cup \{\infty\}$ denote the one-point compactification of X. Note that $\mathcal{P}(X \cup \{\infty\})$ is compact.

Corollary 3.2 *Let $\{\mu_i\} \subset \mathcal{Q}(X)$ be a sequence of measures converging to a measure $\mu \in \mathcal{P}(X \cup \{\infty\})$. Then either $\mu \in \mathcal{Q}(X)$ or $\mu(\{\infty\}) = 1$.*

Moreover, $\mathcal{Q}(K)$ is compact for any compact set $K \subset X$.

Let $\mathcal{W} = \{U_i = \{u_i(t)\}_{t \in \mathbf{R}} : i \in \mathbf{N}\}$ be a sequence of unipotent one-parameter subgroups of G. We say that a point $x \in X$ is *regular* for $\mathcal{W}$ if there does not exist any proper closed subgroup F of G such that the orbit Fx is closed and $F \supset U_i$ for infinitely many $i \in \mathbf{N}$.

We say that a point $x \in X$ is *generic* for $\mathcal{W}$ if for every bounded continuous function f of X the following holds: There exists a sequence $S_i \to \infty$ in $\mathbf{R}$ such that for any sequence $\{T_i\}$ with each $T_i \geq S_i$, we have

$$\lim_{i \to \infty} \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x)\, dt = \int_X f\, d\mu_G,$$

where μ_G is the G-invariant probability measure on X.

Corollary 3.3 *A point $x \in X$ is generic for $\mathcal{W}$ if and only if it is regular for $\mathcal{W}$.*

3.2 Proofs of the results

Proof of theorem 3.1. Without loss of generality we may assume that $\{u_i(t)\} \neq \{e\}$ for all large $i \in \mathbf{N}$. Then for each large $i \in \mathbf{N}$ there exists $w_i \in \underline{\mathbf{g}}$ such that $\|w_i\| = 1$ and $\{u_i(t) : t \in \mathbf{R}\} = \{\exp(tw_i) : t \in \mathbf{R}\}$, where $\underline{\mathbf{g}}$ is the Lie algebra of G and $\|\cdot\|$ denotes a Euclidean norm on it. By passing to a subsequence, we may assume that $w_i \to w$ for some $w \in \underline{\mathbf{g}}$, $\|w\| = 1$. For any $t \in \mathbf{R}$, we have $\mathrm{Ad}(\exp(tw_i)) \to \mathrm{Ad}(\exp(tw))$ as $i \to \infty$. Therefore $U = \{\exp(tw) : t \in \mathbf{R}\}$ is a (nontrivial) unipotent subgroup of G. Since $\exp tw_i \to \exp tw$ for all $t \in \mathbf{R}$ and $\mu_i \to \mu$, it follows that μ is invariant under the action of U on X.

Let W be the subgroup generated by all unipotent one-parameter subgroups of G contained in $\Lambda(\mu)$. Then $\dim W > 0$.

By proposition 2.22(1), there exists $H \in \mathcal{H}$ such that $\mu(\pi(S(H,W))) = 0$ and $\mu(\pi(N(H,W))) > 0$. Hence there exists a compact set $C \subset N(H,W) \setminus S(H,W)$ such that

$$\mu(\pi(C)) = \alpha > 0. \tag{3.1}$$

Let $g_i \to e$ be a sequence in G such that for every $i \in \mathbf{N}$, $g_i x \in \mathrm{supp}(\mu_i)$ and the trajectory $\{u_i(t)g_i x\}_{t>0}$ is uniformly distributed with respect to μ_i; note that, due to Birkhoff ergodic theorem, such a sequence always exists. Take any $y \in \mathrm{supp}(\mu) \cap \pi(C)$. Then for each $i \in \mathbf{N}$ there exists $y_i \in \{u_i(t)g_i x\}_{t\geq 0}$ such that as $i \to \infty$, $y_i \to y$. Let $h_i \to e$ be a sequence in G such that $h_i y_i = y$ for all $i \in \mathbf{N}$. Put $\mu_i' = h_i \mu_i$ and $u_i'(t) = h_i u_i(t) h_i^{-1}$ for all $t \in \mathbf{R}$ and all $i \in \mathbf{N}$. Then $\mu_i' \to \mu$ as $i \to \infty$. Also $y \in \mathrm{supp}(\mu_i')$ and the trajectory $\{u_i'(t)y : t > 0\}$ is uniformly distributed with respect to μ_i' for each $i \in \mathbf{N}$.

Let $h \in \pi^{-1}(y)$. For each $i \in \mathbf{N}$, apply proposition 2.28 for C as above, $\epsilon = \alpha/2$, and $\theta(t) = u_i(t)h$ ($\forall t \in \mathbf{R}$) to obtain a neighbourhood Ω of $\pi(C)$. By equation 3.1, there exists $k_0 \in \mathbf{N}$ such that $\mu_i'(\Omega) > \epsilon$ for all $i \geq k_0$. Therefore for any $i \geq k_0$ and all large $T > 0$,

$$\frac{1}{T}\ell\left(\{t \in [0,T] : u_i'(t)y \in \Omega\}\right) > \epsilon.$$

This shows that for each $i \geq k_0$, the condition (2) of proposition 2.28 is violated for $\theta(t) = u_i(t)h$, $\forall i \geq k_0$. Therefore according to the condition (1) of proposition 2.28, there exists $\gamma \in \Gamma$ such that for each $i \geq k_0$,

$$\{u_i'(t)y\}_{t\in\mathbf{R}} \subset h\gamma\pi(N_G^1(H)),$$

Put $F = (h\gamma)N_G^1(H)(\gamma^{-1}h^{-1})$. By theorem 2.23, the orbit $F^0 y = h\gamma\pi(N_G^1(H))^0$ is closed in X.

We intend to prove the parts (1) and (2) of theorem 3.1 by induction on $\dim G$.

First suppose that $\dim F^0 < \dim G$. By lemma 2.1, we can treat $F^0 y$ as a homogeneous space of F^0. Also each $\{u_i'(t)\}$ is a unipotent subgroup of F^0 and each μ_i' is supported on $F^0 y$. Therefore by induction hypothesis applied to F^0, we obtain the following: $\mathrm{supp}(\mu) = (\Lambda(\mu) \cap F^0)y$ and there exists $j_0 \in \mathbf{N}$ such that for all $i \geq j_0$, $\mathrm{supp}(\mu_i') \subset \mathrm{supp}(\mu)$.

Next suppose that $\dim F^0 = \dim G$. In this case $F = G$, and hence H is a normal subgroup of G. Let $\bar{G} = G/H$ be the quotient group. Since $\dim H \geq \dim W > 0$, we have $\dim \bar{G} < \dim G$. We will project the measures on the homogeneous space $G/(H\Gamma)$ of $\bar{G}$ and apply induction.

Let $\rho : G \to \bar{G}$ be the quotient homomorphism. Since $H\Gamma$ is closed in G, the subgroup $\bar{\Gamma} = \rho(\Gamma)$ is closed (and hence discrete) in $\bar{G}$. Put $\bar{X} = \bar{G}/\bar{\Gamma}$, and let $\bar{\rho} : X \to \bar{X}$ be the natural quotient map. Define a map $\bar{\rho}_* : \mathcal{P}(X) \to \mathcal{P}(\bar{X})$ such that for any $\nu \in \mathcal{P}(X)$ and any Borel measurable subset $A \subset \bar{X}$, $\bar{\rho}_*(\nu)(A) = \nu(\bar{\rho}^{-1}(A))$. Then $\bar{\rho}_*$ is continuous. Put $\bar{\nu} = \bar{\rho}_*(\nu)$ for any $\nu \in \mathcal{P}(X)$.

Put $\bar{y} = \bar{\rho}(y)$. Observe the following: for each $i \geq k_0$, (1) $\{\rho(u'_i(t))\}$ is a unipotent one-parameter subgroup of $\bar{G}$, (2) $\bar{\mu}_i$ is ergodic $\{\rho(u'_i(t))\}$-invariant, (3) $\bar{y} \in \operatorname{supp}(\bar{\mu}'_i)$, and (4) the trajectory $\{\rho(u'_i(t))\bar{y}\}_{t>0}$ is uniformly distributed with respect to $\bar{\mu}'_i$. Also $\bar{\mu}_i \to \bar{\mu}$ as $i \to \infty$. Therefore by induction hypothesis applied to $\bar{G}$, we obtain the following:

1. $\operatorname{supp}(\bar{\mu}) = \Lambda(\bar{\mu})\bar{y}$.
2. There exists $j_0 \geq k_0$ such that for all $i \geq j_0$,
$$\operatorname{supp}(\bar{\mu}'_i) \subset \operatorname{supp}(\bar{\mu}).$$

Since H is normal in G, by proposition 2.22(2), each ergodic component of μ_H is H-invariant. Since $N(H, W) = G$ and $\mu(\pi(S(H, W))) = 0$, we have $\mu = \mu_H$. Therefore μ is H-invariant.

We claim that
$$\rho^{-1}(\Lambda(\bar{\mu})) = \Lambda(\mu). \tag{3.2}$$

First observe that for any $\nu \in \mathcal{P}(X)$, by Fubini's theorem, there exists a unique $\bar{\nu}$-measurable map
$$\nu_{(\cdot)} : \bar{X} \to \mathcal{P}(X)$$
with the following properties:

1. For almost all $\bar{x} \in (\bar{X}, \bar{\nu})$, we have that $\nu_x \in \mathcal{P}(\bar{\rho}^{-1}(\bar{x}))$.
2. For any bounded continuous function f on X, define a function $\bar{f}$ on $\bar{X}$ as
$$\bar{f}(\bar{x}) = \int_{\bar{\rho}^{-1}(\bar{x})} f \, d\nu_{\bar{x}} \quad \text{for a.e. } \bar{x} \in (\bar{X}, \bar{\nu}).$$
Then $\bar{f}$ is $\bar{\nu}$-measurable, and
$$\int_{\bar{X}} \bar{f} \, d\bar{\nu} = \int_X f \, d\mu.$$

By uniqueness of $\nu_{(\cdot)}$, for any $g \in G$, we have
$$(g \cdot \nu)_{\bar{x}} = g \cdot \nu_{\rho(g^{-1})\bar{x}}, \quad \text{for a.e. } \bar{x} \in (\bar{X}, \rho(g) \cdot \bar{\nu}).$$

Since μ is H-invariant and $H \subset \ker \rho$, for any $h \in H$, we have that

$$\mu_{\bar{x}} = (h \cdot \mu)_{\bar{x}} = h \cdot \mu_{\rho(h^{-1})\bar{x}} = h \cdot \mu_{\bar{x}}, \quad \text{for a.e. } \bar{x} \in (\bar{X}, \bar{\mu}).$$

Therefore the measure $\mu_{\bar{x}}$ is H-invariant for a.e. $\bar{x} \in (\bar{X}, \bar{\mu})$. Since $\bar{\rho}^{-1}(\bar{x})$ is a closed H-orbit in X, any H-invariant probability measure on this set is unique. Hence for any H-invariant probability measures μ_1 and μ_2 on X, if $\bar{\mu}_1 = \bar{\mu}_2$ then $\mu_1 = \mu_2$.

Now suppose that $\rho(g)\bar{\mu} = \bar{\mu}$ for some $g \in G$. Since $\rho(g)\bar{\mu} = \bar{g} \cdot \mu$ and $g \cdot \mu$ is $gHg^{-1} = H$-invariant, we have $g \cdot \mu = \mu$. This completes the proof of the claim as in equation 3.2.

Now for all $i \geq j_0$,

$$\mathrm{supp}(\mu'_i) \subset \bar{\rho}^{-1}(\mathrm{supp}(\bar{\mu})) = \rho^{-1}(\Lambda(\bar{\mu}))y = \Lambda(\mu)y.$$

In particular, $\mathrm{supp}(\mu) \subset \Lambda(\mu)y$. Since $\Lambda(\mu)y$ admits a unique $\Lambda(\mu)$-invariant probability measure, we have that $\mathrm{supp}(\mu) = \Lambda(\mu)y$.

Thus in either of the cases $\dim F^0 < \dim G$ or $\dim F^0 = \dim G$, we have obtained the following conclusions: $\mathrm{supp}(\mu) = \Lambda(\mu)y$, and there exists $j_0 \in \mathbf{N}$ such that for all $i \geq j_0$, $\mathrm{supp}(\mu'_i) \subset \mathrm{supp}(\mu)$. Thus $x \in \Lambda(\mu)y$, and hence

$$\mathrm{supp}(\mu) = \Lambda(\mu)x.$$

Since $h_i(g_i x) \in h_i \cdot \mathrm{supp}(\mu_i) = \mathrm{supp}(\mu'_i)$ for all $i \in \mathbf{N}$, we have $(h_i g_i)x \in \Lambda(\mu)x$ for all $i \geq j_0$. Therefore since $h_i \to e$ and $g_i \to e$, there exists $i_0 \geq j_0$ such that for all $i \geq i_0$, $h_i g_i \in \Lambda(\mu)$. Hence for all $i \geq i_0$,

$$\mathrm{supp}(\mu_i) = h_i^{-1} \cdot \mathrm{supp}(\mu'_i) \subset h_i^{-1} \cdot \Lambda(\mu)x = g_i \cdot \mathrm{supp}(\mu).$$

This proves parts (1) and (2) of theorem 3.1 for G.

Now let L be defined as in part (3) of the theorem. Let M be the smallest closed subgroup of G containing L such that the orbit My is closed. Then $\{u'_i(t)y : t \in \mathbf{R}\} \subset My$ and hence $\mathrm{supp}(\mu'_i) \subset My$. Thus $\Lambda(\mu)y = \mathrm{supp}(\mu) \subset My$. By minimality, $M = \Lambda^0(\mu)$. Now by theorem 2.4, L acts ergodically on $\Lambda(\mu)y$ with respect to μ. This completes the proof of the theorem. □

Proof of corollary 3.2. By lemma 2.11, for each $i \in \mathbf{N}$, there exists a unipotent one-parameter subgroup $\{u(t)\}$ acting ergodically on X with respect to μ_i. Hence by Birkhoff ergodic theorem, there exists a set $X_i \subset X$ such that $\mu_i(X_i) = 1$ and the trajectory $\{u_i(t)x : t > 0\}$ is uniformly distributed with respact to μ_i for each $x \in X_i$. Suppose that $\mu(\infty) < 1$. Then there exists a compact set $K \subset X$ such that $K \cap \mathrm{supp}(\mu_i) \neq \emptyset$ for all $i \in \mathbf{N}$. Since $\overline{X_i} = \mathrm{supp}(\mu)$ for all $i \in \mathbf{N}$, there exists a sequence $x_i \in X_i$ converging to a point $x \in K$. Now by the theorem of Dani and Margulis [14, Theorem 6.1] (see theorem 1.12 for a more general statement), given $\delta > 0$, there exists a compact set $K_1 \subset X$ such that for any $i \in \mathbf{N}$ and $T > 0$,

$$(1/T)|\{t \in [0,T] : u_i(t)x_i \in K_1\}| > 1 - \delta.$$

Hence $\mu(K_1) \geq 1 - \delta$. Now the corollary follows from theorem 3.1. □

Proof of corollary 3.3. First suppose that x is regular for $\mathcal{W}$. By theorem 2.29, for each $i \in \mathbf{N}$, the trajectory $\{u_i(t)x : t > 0\}$ is uniformly distributed with respect to some $\mu_i \in \mathcal{Q}(x)$. By corollary 3.2, there exists a sequence $i_k \to \infty$ such that $\mu_{i_k} \to \mu$ for some $\mu \in \mathcal{Q}(x)$. Then by theorem 3.1, $U_{i_k} \subset \Lambda(\mu)$ for all large $k \in \mathbf{N}$. Since x is regular for $\mathcal{W}$, we have that $\Lambda(\mu) = G$. In particular, $\mu_i \to \mu = \mu_G$ as $i \to \infty$.

Let f be a given bounded continuous function on X. Then for each $i \in \mathbf{N}$, there exists $S_i > 0$ such that for every $T_i > S_i$,

$$\left| \int_X f \, d\mu_i - \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x) \, dt \right| < \epsilon / i.$$

Now since $\mu_i \to \mu_G$, we have

$$\lim_{i \to \infty} \frac{1}{T_i} \int_0^{T_i} f(u_i(t)x) dt = \int f d\mu_G.$$

Thus x is generic for $\mathcal{W}$. The converse implication is obvious. $\square$

Chapter 4

Limit distributions of polynomial trajectories on homogeneous spaces

The basic property of a unipotent one-parameter subgroup used in the work of Ratner is that the map $t \mapsto \mathrm{Ad}u(t)$ is a polynomial function in each co-ordinate of $\mathrm{End}(\mathrm{Lie}(G))$. Therefore it is natural to ask the following question. Suppose $G = \mathrm{SL}_n(\mathbf{R})$, $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, and $\theta : \mathbf{R} \to G$ is a map which is a polynomial function, namely, each matrix co-ordinate is a polynomial. Then is it true that the trajectory $\{\phi(t)\Gamma : t > 0\}$ is uniformly distributed with respect to a measure of the form μ_F as above? In the case when $G = \mathbf{R}^n$ and $\Gamma = \mathbf{Z}^n$ this indeed holds as can be deduced from a classical result due to Weyl. In this paper we answer the question affirmatively in a more general set up.

4.1 Main result and its consequences

A group G is called *real algebraic* if it is an open subgroup of $\mathbf{R}$-points of an algebraic group $\mathbf{G}$ defined over $\mathbf{R}$. A map $\Theta : \mathbf{R}^k \to G$ is called *regular algebraic* if it is the restriction of a morphism $\Theta : \mathbf{C}^k \to \mathbf{G}$ of algebraic varieties defined over $\mathbf{R}$. We caution the reader that a map such as $\phi : \mathbf{R} \to \mathbf{R}^*$ given by $\phi(t) = 1 + t^2$ for all $t \in \mathbf{R}$, is *not* regular algebraic according to our definition, as ϕ does not extend to an algebraic map from $\mathbf{C}$ to $\mathbf{C}^*$.

In this chapter we prove the following result.

Theorem 4.1 *Let G be a real algebraic group and let $\Delta \subset G_1 \subset G$ be closed subgroups and suppose that G_1/Δ admits a finite G_1-invariant measure. Let $k \in \mathbf{N}$ and $\Theta : \mathbf{R}^k \to G$ be a map defined as $\Theta(t_1, \ldots, t_k) = \theta_k(t_k) \cdots \theta_1(t_1)$ for all $(t_1, \ldots, t_k) \in \mathbf{R}^k$, where $\theta_i : \mathbf{R} \to G$ is a regular algebraic map for $i = 1, \ldots, k$. Suppose that $\Theta(0) = e$ and $\Theta(\mathbf{R}^k) \subset G_1$. Let F be the smallest closed subgroup of G containing $\Theta(\mathbf{R}^k)$ such that the orbit $F\Delta$ is closed in G/Δ and admits a unique F-invariant probability measure, say μ_F. Suppose we are given sequences $T_n^{(1)} \to \infty, \ldots, T_n^{(k)} \to \infty$ in $\mathbf{R}$ as $n \to \infty$, and consider the boxes $B_n = [0, T_n^{(1)}] \times \cdots \times [0, T_n^{(k)}]$, $\forall n \in \mathbf{N}$. Then for any* *sequence $g_n \to e$ in F and any $f \in \mathrm{C_c}(G/\Delta)$, we have*

$$\lim_{n\to\infty} \frac{1}{m(B_n)} \int_{\mathbf{t}\in B_n} f(\Theta(\mathbf{t})g_n\Delta)\, dm(\mathbf{t}) = \int_{F\Delta} f\, d\mu_F,$$

where m denotes the Lebesgue measure on $\mathbf{R}^k$.

We will also deduce the following fact using the arguments in our proof of theorem 4.1.

Corollary 4.2 *Let G, Δ, and G_1 be as in theorem 4.1. Let $\theta : \mathbf{R} \to G$ be a regular algebraic map such that $\theta(\mathbf{R}) \subset G_1$ and $\theta(0) = e$. Let F be the smallest closed subgroup of G containing $\theta(\mathbf{R})$ such that the orbit $F\Delta$ is closed and admits a unique F-invariant probability measure, say μ_F. Then for any $k \geq 1$, any sequences $g_n \to e$ in F and $T_n \to \infty$ in $\mathbf{R}$ as $n \to \infty$, and any $f \in \mathrm{C_c}(G/\Delta)$, we have*

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} f(\theta(t^{1/k})g_n\Delta)\, dt = \int_{F\Delta} f\, d\mu_F.$$

Using the above corollary, we obtain a variation of theorem 4.1 which holds for all regular algebraic maps, the averages however being taken over increasing sequences of balls (rather than boxes whose sizes could increase at different rates in different coordinates).

Corollary 4.3 *Let G, Δ, and G_1 be as in theorem 4.1. Let $\Theta : \mathbf{R}^k \to G$ be a regular algebraic map such that $\Theta(0) = e$ and $\Theta(\mathbf{R}^k) \subset G_1$. Let F be the smallest closed subgroup of G containing $\Theta(\mathbf{R}^k)$ such that the orbit $F\Delta$ is closed and admits a unique F-invariant probability measure, say μ_F. Then for any sequences $g_n \to e$ in F and $R_n \to \infty$ in $\mathbf{R}$ as $n \to \infty$, and any function $f \in C_c(G/\Delta)$,*

$$\lim_{n\to\infty} \frac{1}{m(B_{R_n})} \int_{\mathbf{t}\in B_{R_n}} f(\Theta(\mathbf{t})g_n\Delta)\, dm(\mathbf{t}) = \int_{F\Delta} f\, d\mu_F,$$

where B_R denotes the ball of radius R in $\mathbf{R}^k$ around origin.

Using these results we answer affirmatively a question raised by Ratner in [31, 32] regarding the limit distributions of orbits of higher dimensional unipotent subgroups on finite-volume homogeneous spaces of Lie groups. First we recall some notation from [28].

Let N be a simply connected nilpotent group with Lie algebra $\underline{\mathbf{n}}$. Let $B = \{b_1, \ldots, b_k\}$ be a basis in $\underline{\mathbf{n}}$. For $v \in \underline{\mathbf{n}}$ write $v = \sum_{i=1}^k \alpha_i(v) b_i$. We say that the basis B is *triangular* if $\alpha_k([b_i, b_j]) = 0$ for all $k \leq \max\{i, j\}$ and all $i, j = 1, \ldots, k$. Any permutation of a triangular basis is called a *regular* basis.

Corollary 4.4 *Let G be a Lie group, Γ be a closed subgroup of G such that G/Γ admits a finite G-invariant measure, and N be a simply connected unipotent subgroup of G. Let $\{b_1, \ldots, b_k\}$ be a regular basis in $\underline{\mathbf{n}}$. For $s_1, \ldots, s_k > 0$ define*

$$\mathcal{S}(s_1, \ldots, s_k) = \{(\exp t_k b_k) \cdots (\exp t_1 b_1) \in N : 0 \leq t_j \leq s_j,\ j = 1, \ldots, k\}.$$

Then for any $x \in G/\Gamma$, let F be the smallest closed subgroup of G containing N such that the orbit Fx is closed and admits a unique F-invariant probability measure, say μ_F. Then for any $f \in \mathrm{C_c}(G/\Gamma)$,

$$\lim_{s_1\to\infty,\ldots,s_k\to\infty} \frac{1}{\lambda(\mathcal{S}(s_1,\ldots,s_k))} \int_{h\in\mathcal{S}(s_1,\ldots,s_k)} f(hx)\,d\lambda(h) = \int_{Fx} f\,d\mu_F,$$

where λ denotes a haar measure on N.

In view of [14, Theorem 3], we obtain 'uniform versions' of theorem 4.1 and corollary 4.3 in the following results.

Corollary 4.5 *Let G, Δ, and G_1 be as in theorem 4.1. Let $\Theta : \mathbf{R}^k \to G$ be a regular algebraic map with $\Theta(0) = 0$ and $\Theta(\mathbf{R}^k) \subset G_1$. Let a compact set $K \subset G/\Delta$, a function $f \in \mathrm{C_c}(G/\Delta)$, and an $\epsilon > 0$ be given. Then there exist finitely many closed subgroups $H_1, \ldots, H_r$ of G, with each orbit $H_j\Gamma$ being homogeneous in X, and compact sets*

$$C_j \subset \{g \in G : \Theta(\mathbf{R}^k)g \subset gH_j\}, \quad j = 1, \ldots, r,$$

such that the following holds: For any compact set $K_1 \subset K \setminus \bigcup_{j=1}^r C_j\Gamma$ there exists $T_0 > 0$ such that for any $x \in K_1$ and any ball B in $\mathbf{R}^k$ around origin with radius at least T_0,

$$\left| \frac{1}{m(B)} \int_{\mathbf{t}\in B} f(\Theta(\mathbf{t})x)\,d\mathbf{t} - \int f\,d\mu_{G_1} \right| < \epsilon,$$

where μ_{G_1} is the G_1-invariant probability measrue on G_1/Δ.

Further if there exist regular algebraic maps $\theta_l : \mathbf{R} \to G$ for $l = 1, \ldots, k$ such that $\Theta(t_1, \ldots, t_k) = \theta_k(t_k)\cdots\theta_1(t_1)$ for all $(t_1, \ldots, t_k) \in \mathbf{R}^k$, then the above result holds for any box $B = [0, s_1] \times \cdots \times [0, s_k]$ with each $s_l > T_0$.

We conclude this section with a natural question. Let θ be a map as in the corollary 4.2. Then does there exist a closed subgroup F of G containing $\theta(\mathbf{Z})$ such that the orbit $F\Delta$ is closed and admits a unique F-invariant probability measure, say μ_F, and for any $f \in \mathrm{C_c}(G/\Delta)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(\theta(n)\Delta) = \int_{F\Delta} f\,d\mu_F?$$

In the case when G is a nilpotent group, this question can be answered affirmatively (cf. [2, Chap. 7], for abelian case). But even in the simplest semisimple case of $G = G_1 = \mathrm{SL}_2(\mathbf{R})$, $\Delta = \mathrm{SL}_2(\mathbf{Z})$, and the regular algebraic map $\theta(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}$ for all $t \in \mathbf{R}$, the question is unanswered.

4.2 Some reductions

We begin by noting the following.

Proposition 4.6 *Let G be a real algebraic group, and $\Theta : \mathbf{R}^k \to G$ be a regular algebraic map such that $\Theta(0) = e$.*

1. *Let L be the smallest closed subgroup of G containing $\Theta(\mathbf{R}^k)$. Then L is generated by algebraic unipotent one-parameter subgroups of G.*

2. *Suppose further that there exist closed subgroups $\Delta \subset G_1 \subset G$ such that G_1/Δ admits a finite G_1-invariant measure and $\Theta(\mathbf{R}^k) \subset G_1$. Let F be the smallest closed subgroup containing $\Theta(\mathbf{R}^k)$ such that the orbit $F\Delta$ is closed. Then $F/(F \cap \Delta)$ admits a finite F-invariant measure, and the Zariski closure of $F \cap \Delta$ contains F.*

Proof. Let $\tilde{L}$ denote the Zariski closure of L in G. Recall that any regular algebraic map from $\mathbf{R}^k$ to $\mathbf{R}^*$ or to a compact algebraic group is constant. Therefore by the definition of L, we have that $\tilde{L}$ has no nontrivial toral or compact factors. Hence $\tilde{L}$ is generated by algebraic unipotent one-parameter subgroups of G. In particular the radical of L is unipotent, and hence $L = \tilde{L}$. This proves 1).

Let $\tilde{\Delta}$ denote the Zariski closure of Δ in G. By a version of Borel's density theorem as in [9, Theorem 4.1], all unipotent one-parameter subgroups of G_1 are contained in $\tilde{\Delta}$, in particular $L \subset \tilde{\Delta}$. Now replacing G by $\tilde{\Delta}$ and G_1 by $G_1 \cap \tilde{\Delta}$, we can assume that Δ is Zariski dense in G and in particular Δ^0, the connected component of e in Δ, is a normal subgroup of G.

Now let $\phi : G \to G/\Delta^0$ denote the quotient homomorphism. Then $\phi(\Delta)$ is a discrete subgroup of $\phi(G)$. And the map $\bar{\phi} : G/\Delta \to \phi(G)/\phi(\Delta)$, defined as $\bar{\phi}(g\Delta) = \phi(g)\phi(\Delta)$ for all $g \in G$, is an equivariant isomorphism. Now $\phi(F)$ is the smallest closed subgroup of $\phi(G_1)$ containing $\phi(L)$ such that $\phi(F)\phi(\Delta)$ is closed. By [35, Theorem 2.1], $F/(F \cap \Delta) \cong \phi(F)/(\phi(F) \cap \phi(\Delta))$ admits a finite F-invariant measure.

Now by the above Borel density argument L is contained in the Zariski closure of $F \cap \Delta$. Therefore due to the definition of F, the Zariski closure of $F \cap \Delta$ also contains F. This proves (2). □

We now list some simplifying assumptions that can be made without loss of generality in proving theorem 4.1.

Note 4.7 In view of proposition 4.6 by replacing Δ by $F \cap \Delta$, G_1 by F, and G by the Zariski closure of F we may assume that there is no proper algebraic subgroup A of G such that $\Theta(\mathbf{R}^k) \subset A$ and $A\Delta$ is closed. Moreover we may also assume that $F = G_1$.

Note 4.8 Let W be the closed subgroup generated by all algebraic unipotent one-parameter subgroups of G contained in Δ^0. Then W is a normal subgroup of G. Let $q : G \to G/W$ be the natural quotient map. By [35, Lemma 2.9], W is a real algebraic group. Therefore G/W is a real algebraic group and q is a regular algebraic map. Note that $G/\Delta \cong q(G)/q(\Delta)$ equivariantly. Therefore without loss of genrality we can replace Δ by $q(\Delta)$, G_1 by $q(G_1)$, G by $q(G)$, and Θ by $q \circ \Theta$. In view of this we can assume that Δ contains no nontrivial algebraic unipotent one-parameter subgroups of G.

Note 4.9 Also, without loss of generality we may assume that θ_k is nonconstant.

Note 4.10 Let $\bar{G}_1 = G_1/\Delta^0$, $\rho : G_1 \to \bar{G}_1$ denote the natural quotient homomorphism, $\bar{\Delta} = \rho(\Delta)$, and $\bar{\rho} : G_1/\Delta \to \bar{G}_1/\bar{\Delta}$ be the natural quotient map. Consider the action of G_1 on $\bar{G}_1/\bar{\Delta}$ via the map ρ. Then $\bar{\rho}$ is a G_1-equivariant isomorphism. Also for any $d, m \in \mathbf{N}$, and $\Theta \in \mathcal{P}_{d,m}(G_1)$ we have that $\rho \circ \Theta \in \mathcal{P}_{d,m}$. Also $\bar{\Delta}$ is a discrete subgroup of $\bar{G}_1$.

Therefore the results of Chapter 1 and Chapter 2 hold (with obvious necessary changes) for the closed subgroup $\Delta \subset G_1$ and the quotient space G_1/Δ in place of the discrete subgroup Γ of G and the quotient space G/Γ.

Limit distributions of polynomial trajectories on G/Δ

Corollary 4.11 *Let the notation be as in theorem 4.1. Let $\{B_n\}_{n\in\mathbf{N}}$ be a sequence of bounded open convex subsets of $\mathbf{R}^k$ containing 0. For each $n \in \mathbf{N}$, let μ_n be the probability measure on G/Δ such that for any $f \in \mathrm{C_c}(G/\Delta)$, we have*

$$\int_{G/\Delta} f \, d\mu_n = \frac{1}{m(B_n)} \int_{\mathbf{t}\in B_n} f(\Theta(\mathbf{t})g_n) \, dm(\mathbf{t}).$$

Then there exist a strictly increasing sequence $\{n_i\}_{i\in\mathbf{N}} \subset \mathbf{N}$ and a measure $\mu \in \mathcal{P}(G/\Delta)$ such that $\mu_{n_i} \to \mu$ as $i \to \infty$.

Proof. Using the existence of limits in the space of probability measures on the one-point compactification of G/Δ, say X^*, we obtain a subsequence $\{\mu_{n_i}\}$ converging to a probability measure μ on X^*. Since $\Theta(\mathbf{R}^k)g_n\Delta \subset G_1/\Delta$ $(\forall n \in \mathbf{N})$, due to theorem 1.12, for any $\epsilon > 0$ there exists a compact set $K \subset G/\Delta$ such that $\mu_n(K) \geq 1 - \epsilon$ for all $n \in \mathbf{N}$. Therefore $\mu(K) \geq 1 - \epsilon$, and hence $\mu(G/\Delta) = 1$. $\square$

Note 4.12 *In view of corollary 4.11, to prove the theorem 4.1, it is enough to show the following: For $i = 1, \ldots, k$, let sequence $T_n^{(i)} \to \infty$ as $n \to \infty$ be given. Put $B_n = [0, T_n^{(1)}] \times \cdots \times [0, T_n^{(k)}]$ for all $n \in \mathbf{N}$. Suppose that $\mu_n \to \mu$ in $\mathcal{P}(G/\Delta)$ as $n \to \infty$. Then μ is F-invariant and $\mu(F\Delta) = 1$.*

4.3 Invariance under a unipotent flow

In this section we show that the limiting distribution μ as in note 4.12 is invariant under the action of a nontrivial unipotent one-parameter subgroup of G. This result allows us to apply Ratner's measure rigidity theorem in our study.

We use the following observation to associate nontrivial unipotent one-parameter subgroups to nonconstant regular algebraic maps into algebraic groups (cf. [11, Proposition 2.4]).

Lemma 4.13 *Let G be a real algebraic group and $\theta : \mathbf{R} \to G$ be a non-constant regular algebraic map. Then there exists a $q \geq 0$ and a nontrivial algebraic unipotent one-parameter subgroup $\rho : \mathbf{R} \to G$ such that for any $s \in \mathbf{R}$,*

$$\lim_{t\to\infty} \theta(t + st^{-q})\theta(t)^{-1} = \rho(s).$$

Proof. Let $\mathrm{M}(n, \mathbf{R})$ denote the affine space of $n \times n$ real matrices which contains G as an affine subvariety. There exist polynomials $\theta_{ij}(t)$ for $i, j = 1, \ldots, n$, such that $\theta(t) = (\theta_{ij}(t))_{n\times n}$. Put

$$d = \deg(\theta(t)) := \max_{i,j=1,\ldots,n} \deg(\theta_{ij}(t)).$$

For $\xi \in \mathbf{R}$, we have

$$\theta(t + \xi) = \theta(t) + \sum_{l=1}^{d} \theta^{(l)}(t)\frac{\xi^l}{l!}.$$

Note that the l-th derivative $\theta^{(l)}(t)$ is a regular algebraic map of degree $d - l$. Since the map $g \to g^{-1}$ is regular algebraic, we have that the map $t \mapsto \theta(t)^{-1}$ is also regular algebraic. Put

$$q = \max_{1\leq l\leq d} (1/l) \deg(\theta^{(l)}(t)\theta(t)^{-1}) \geq 0.$$

Then for every $1 \leq l \leq d$,

$$\lim_{t\to\infty} \theta^{(l)}(t)\theta(t)^{-1}t^{-ql} = \lambda_l \in \mathrm{M}(n, \mathbf{R}), \tag{4.1}$$

and $(\lambda_1, \ldots, \lambda_d) \neq 0$. Put

$$\rho(s) = I + \sum_{l=1}^{d} \lambda_l \frac{s^l}{l!}$$

for all $s \in \mathbf{R}$. Then for any $s \in \mathbf{R}$ and any map $t \mapsto s_t$ with $s_t \to s$ as $t \to \infty$, we have

$$\lim_{t\to\infty} \theta(t + s_t t^{-q})\theta(t)^{-1} = \rho(s). \tag{4.2}$$

Now for $s_1, s_2 \in \mathbf{R}$,

$$\begin{aligned}
\rho(s_1 + s_2)\rho(s_2)^{-1} &= \lim_{t\to\infty} \left(\theta(t + (s_1 + s_2)t^{-q})\theta(t)^{-1}\right) \cdot \left(\theta(t)\theta(t + s_2 t^{-q})^{-1}\right) \\
&= \lim_{t\to\infty} \theta(y_t + s_t y_t^{-q})\theta(y_t)^{-1}, \\
&\quad \text{where } y_t = t + s_2 t^{-q} \text{ and } s_t = s_1(y_t/t)^q. \\
&= \rho(s_1). \qquad (4.3)
\end{aligned}$$

Thus by eqs. 4.1, 4.2, and 4.3, $\rho : \mathbf{R} \to G$ is a nontrivial algebraic group homomorphism. Therefore ρ is a nontrivial algebraic unipotent one-parameter subgroup of G. This completes the proof. $\square$

We digress to modify the above result in the next lemma for its use in proving corollary 4.2 later.

Lemma 4.14 *Let the notation be as in lemma 4.13. Take $k \geq 1$. Put $\psi(t) = \theta(t^{1/k})$ for all $t > 0$. Then for $q_1 = (1/k)(q+1) - 1 > -1$ and every $s \in \mathbf{R}$,*

$$\lim_{t\to\infty} \psi(t + st^{-q_1})\psi(t)^{-1} = \rho(s/k),$$

where ρ is the unipotent one-parameter subgroup as in lemma 4.13.

Proof. Using Taylor's expansion, we get

$$(t + st^{-q_1})^{1/k} = t^{1/k} + s_t t^{-q_1-1+1/k} = y_t + s_t y_t^{-k(q_1+1-1/k)},$$

where $y_t = t^{1/k}$, and $s_t \to s/k$ as $t \to \infty$. Now the result follows from eq. 4.2. □

We need the following elementary fact.

Lemma 4.15 *For any bounded continuous function f on $\mathbf{R}$, any $q > -1$, and $s \in \mathbf{R}$,*

$$\lim_{T\to\infty} \frac{1}{T}\int_1^T f(t + st^{-q}) - f(t)\, dt = 0,$$

where the rate of convergence depends only on s, q, and $\sup|f|$, rather than f itself.
□

The next result is the first main step in the proof of theorem 4.1. Note that this is the only place where we make use of the fact that Θ is of the product form, rather than of the general form as in corollary 4.3.

Proposition 4.16 *Let the measure μ be as constructed in note 4.12. Then μ is invariant under a nontrivial unipotent one-parameter subgroup of G.*

Proof. By note 4.9, θ_k is nonconstant. Obtain $q > -1$ and a nontrivial unipotent one-parameter subgroup $\rho : \mathbf{R} \to G$ as in lemma 4.13 for θ_k in place of θ. To show that μ is invariant under the action of ρ, take any $s \in \mathbf{R}$, and any $f \in \mathrm{C_c}(G/\Delta)$. Then

$$\begin{aligned}
&\int_{G/\Delta} f(\rho(s)x)\, d\mu(x) \\
= \ &\lim_{n\to\infty} \frac{1}{m(B_n)} \int_{\mathbf{t}=(t_1,\dots,t_k)\in B_n} f(\rho(s)\theta(t_k)\cdots\theta(t_1))g_n\Delta)\, dm(\mathbf{t}) \\
= \ &\lim_{n\to\infty} \frac{1}{m(B_n)} \int_{(t_1,\dots,t_{k-1})\in[0,T_n^{(1)}]\times\cdots\times[0,T_n^{(k-1)}]} dt_1 \cdots dt_{k-1}\, \cdot \\
&\cdot \left(\int_{t_k\in[0,T^{(k)}]} f(\theta_k(t_k + st_k^{-q})\theta_{k-1}(t_{k-1})\cdots\theta_1(t_1))g_n\Delta)\, dt_k \right) \\
= \ &\lim_{n\to\infty} \frac{1}{m(B_n)} \int_{\mathbf{t}\in B_n} f(\Theta(t_1,\dots,t_k)g_n\Delta) d\mathbf{t} \\
= \ &\int_{G/\Delta} f(x) d\mu(x),
\end{aligned}$$

where the second equality follows from the choice of ρ and the uniform continuity of f, and the third equality follows from lemma 4.15 applied to the integration in the variable t_k. This completes the proof. □

4.4 Proofs of the results

Proof of theorem 4.1. We shall prove the theorem by induction on $\dim G/\Delta$. Let μ be a limiting distribution as in the note 4.12. Let W be the subgroup generated by all the unipotent one-parameter subgroups of G preserving μ. By proposition 4.16, $\dim W > 0$. By proposition 2.22, there exists $H \in \mathcal{H}$ such that $\mu(\pi(S(H,W))) = 0$ and $\mu(\pi(N(H,W))) > 0$ (see note 4.10). Let $C_1 \subset N(H,W) \setminus S(H,W)$ be a compact set such that $\mu(\pi(C_1)) > \epsilon$ for some $\epsilon > 0$. Let Θ be as in the hypothesis, then there exists $l \in \mathbf{N}$ such that $\Theta \in \mathcal{P}_l(\mathbf{R}^k, G)$. Since $\Theta(\mathbf{R}^k) \subset G_1$ and G_1/Δ admits a finite G_1-invariant measure, applying theorem 2.30 to G_1 in place of G, we deduce the following: There exists a neighbourhood Ω of $\pi(C_1)$ such that either (I) $\Theta(\mathbf{R}^k)\Delta \subset gN^1(H)\Delta$ for some $g \in G$, or (II) $\mu_n(\Omega) < \epsilon$ for all large $n \in \mathbf{N}$. Now if (II) holds then $\mu(\pi(C_1)) \leq \epsilon$, which is a contradiction. Therefore (I) must hold.

Since $gN^1(H)g^{-1} \supset \Delta^0$, we have $\Theta(\mathbf{R}^k) \subset gN^1(H)g^{-1}$. By theorem 2.23, the orbit $gN^1(H)\Delta$ is closed. Also $gN^1(H)g^{-1}$ is an algebraic group. Therefore by the note 4.7, $G = gN^1(H)g^{-1}$; that is $G = N(H)$. Since $\mu(\pi(N(H,W))) > 0$, we have that $W \subset H$ and $G = N(H,W)$. Thus $\mu(\pi(N(H,W))) = 1$. Now by proposition 2.22 (2), μ is H-invariant.

Put $\Lambda = H\Delta$. Since $H \in \mathcal{H}$ and $N(H) = G$, we have that Λ is a closed subgroup of G. Consider the G-equivariant quotient map $q : G/\Delta \to G/\Lambda$. Let $q_* : \mathcal{P}(G/\Delta) \to \mathcal{P}(G/\Lambda)$ be the map defined as $q_*\nu(E) = \nu(q^{-1}(E))$ for all Borel sets $E \subset G/\Lambda$ and all $\nu \in \mathcal{P}(G/\Delta)$. The map q_* is continuous. Since $\mu_n \to \mu$, we have that $q_*(\mu_n) \to q_*(\mu)$. Note that for any $\bar{f} \in \mathrm{C_c}(G/\Lambda)$,

$$\begin{aligned}
\int_{G/\Lambda} \bar{f}\, dq_*(\mu_n) &= \int_{G/\Delta} \bar{f} \circ q \, d\mu_n \\
&= \frac{1}{m(B_n)} \int_{B_n} \bar{f}(q \circ \Theta(\mathbf{t}) g_n \Delta)\, dm(\mathbf{t}) \\
&= \frac{1}{m(B_n)} \int_{B_n} \bar{f}(\Theta(\mathbf{t}) g_n \Lambda)\, d\mathbf{t}. \qquad (4.4)
\end{aligned}$$

Since Λ contains a nontrivial unipotent one-parameter subgroup of G, due to note 4.8 we have $\dim \Lambda^0 > \dim \Delta^0$. Thus $\dim(G/\Lambda) < \dim(G/\Delta)$. Using the induction hypothesis we can assume that theorem 4.1 is valid for the Λ in place of Δ. Hence due to eq. 4.4, $q_*(\mu)$ has the following property: There exists a closed subgroup F of G containing H such that $\Theta(\mathbf{R}^k) \subset F$, $q_*(\mu)$ is F-invariant, the orbit $F\Lambda$ is closed, and $q_*(\mu)(F\Lambda) = 1$. Now since the fibres of q are closed H-orbits and μ is H-invariant, we have that μ is F-invariant; (see the proof of the claim made in equation 3.2). Since $F \supset H$, we have that $\mu(F\Delta) = q_*\mu(F\Lambda) = 1$. Let L be any closed subgroup of G containing $\Theta(\mathbf{R}^k)$ such that the orbit $L\Delta$ is closed. Then $\mu(L\Delta) = 1$. Hence $F \subset L\Delta^0$. Thus in view of note 4.12 the proof is complete. □

Proof of corollary 4.2. Let $\psi : \mathbf{R}_+ \to G$ be the function defined as $\psi(t) = \phi(t^{1/k})$ for all $t \geq 0$.

Now argue just as in the proof of theorem 4.1 replacing Θ by ψ. There are exactly three places where we use that Θ is a regular algebraic function: (1) theorem 1.12, (2) proposition 4.16, and (3) proposition 2.27. Therefore if the corresponding statements are shown to hold for ψ in place of Θ, we would get a proof of the corollary.

First we have lemma 4.14, which replaces θ by ψ in lemma 4.13. Therefore in the proof of proposition 4.16, we can use lemma 4.14 in place of lemma 4.13 and obtain the same conclusion for ψ in place of Θ. Also due to lemma 1.9, it is clear that theorem 1.12 and proposition 2.27 are valid for ψ in place of Θ or θ. This completes the proof. □

Proof of corollary 4.3. Let S denote the unit sphere in $\mathbf{R}^k$ and σ denote the rotation invariant probability measure on S. Using polar decomposition, for any $f \in \mathrm{C_c}(G/\Delta)$ and any $T > 0$,

$$\frac{1}{m(B_T)} \int_{\mathbf{t}\in B_T} f(\Theta(\mathbf{t})\Delta)\, dm(\mathbf{t}) = \int_{\mathbf{x}\in S} d\sigma(\mathbf{x}) \left(\frac{1}{T^k} \int_0^{T^k} f(\Theta(t^{1/k}\mathbf{x})\Delta)\, dt \right). \tag{4.5}$$

For every $\mathbf{x} \in S$ define $\theta_{\mathbf{x}}(t) = \Theta(t\mathbf{x})$ for all $t \in \mathbf{R}$. Let $F_{\mathbf{x}}$ be the smallest closed subgroup L of G such that $L \supset \Delta^0$, $L \supset \theta_{\mathbf{x}}(\mathbf{R})$, and the orbit $L\Delta$ is closed.

By proposition 4.6, we have $F_{\mathbf{x}} \in \mathcal{H}$. The set $\Theta^{-1}(F_{\mathbf{x}}) \cap S$ is an analytic submanifold of S. Hence if $\dim(\Theta^{-1}(F_{\mathbf{x}}) \cap S) = \dim S$ then $F_{\mathbf{x}} \supset \Theta(\mathbf{R}^k)$. Note that if $\mathbf{y} \in \Theta^{-1}(F_{\mathbf{x}}) \cap S$ then $F_{\mathbf{y}} \subset F_{\mathbf{x}}$. Put

$$E = \{\mathbf{x} \in S : \dim(\Theta^{-1}(F_{\mathbf{x}}) \cap S) < \dim(S)\}.$$

Now $\sigma(\Theta^{-1}(F_{\mathbf{x}}) \cap S) = 0$ for every $\mathbf{x} \in E$. Since $\mathcal{H}$ is a countable collection, we have that

$$E = \bigcup_{\mathbf{x}\in E} \left(\Theta^{-1}(F_{\mathbf{x}}) \cap S\right)$$

is a countable union. Therefore

$$\sigma(E) = 0. \tag{4.6}$$

Now let F denote the smallest closed subgroup L of G such that $L \supset \Delta^0$, $L \supset \Theta(\mathbf{R}^k)$, and the orbit $L\Delta$ is closed. Then $F_{\mathbf{x}} = F$ for all $\mathbf{x} \in S \setminus E$. Let μ_F denote the unique F-invariant probability measure on $F\Delta$.

By corollary 4.2, for any $\mathbf{x} \in S \setminus E$, any sequences $T_i \to \infty$ in $\mathbf{R}$ and $g_i \to e$ in F, and any $f \in \mathrm{C_c}(G/\Delta)$,

$$\lim_{i\to\infty} \frac{1}{T_i} \int_0^{T_i} f(\theta_{\mathbf{x}}(t^{1/k})g_i\Delta)\, dt = \int_{F_{\mathbf{x}}\Delta} f\, d\mu_{\mathbf{x}}, \tag{4.7}$$

where $\mu_{\mathbf{x}}$ denotes the unique $F_{\mathbf{x}}$-invariant probability measure supported on the closed orbit $F_{\mathbf{x}}\Delta$.

Let $\epsilon > 0$ and $f \in \mathrm{C_c}(G/\Delta)$ be given. Take any sequence $R_i \to \infty$ in $\mathbf{R}$. Take $i \in \mathbf{N}$, define W_i to be the set of all $\mathbf{x} \in S$ such that

$$\left| \frac{1}{R_j^k} \int_0^{R_j^k} f(\theta_{\mathbf{x}}(t^{1/k})g_j\Delta)\, dt - \int_{F\Delta} f\, d\mu_F \right| < \epsilon/2 \tag{4.8}$$

for all $j \geq i$. One can easily verify that W_i is Borel measurable. Also note that $W_{i_1} \subset W_{i_2}$ for all $i_1 \leq i_2$. Now by eq. 4.7, we have

$$S \setminus E = \cup_{i \in \mathbf{N}} W_i.$$

Therefore by eq. 4.6, there exists $i_0 \in \mathbf{N}$ such that

$$\sigma(S \setminus W_{i_0}) < \epsilon/(4 \cdot \sup |f|). \tag{4.9}$$

Now by eqs. 4.5, 4.8, and 4.9, for every $i \geq i_0$, we get

$$\begin{aligned}
&\left| \frac{1}{B_{R_i}} \int_{\mathbf{t} \in B_{R_i}} f(\Theta(\mathbf{t}) g_i \Delta)\, d\mathbf{t} - \int_{F\Delta} f\, d\mu_F \right| \\
\leq\ & \int_{\mathbf{x} \in W_{i_0} \cup (S \setminus W_{i_0})} \left| \frac{1}{{R_i}^k} \int_0^{{R_i}^k} f(\theta_{\mathbf{x}}(t^{1/k}\mathbf{x}) g_i \Delta)\, dt - \int_{F\Delta} f\, d\mu_F \right| d\sigma(\mathbf{x}) \\
<\ & \epsilon.
\end{aligned}$$

This completes the proof. □

Proof of corollary 4.4. For $i = 1, \ldots, k$, define $\theta_i(t) = \exp(tb_i)$ for all $t \in \mathbf{R}$. Define $\Theta(t_1, \ldots, t_k) = \theta_k(t_k) \cdots \theta_1(t_1)$ for all $(t_1, \ldots, t_k) \in \mathbf{R}^k$.

Note that due to [28, Lemma 1.4], the Lebesgue measure on $\mathbf{R}^k$ projects under Θ to a haar measure on N.

Note that if G a real algebraic group and N is an algebraic unipotent subgroup, we have that θ_i is a regular algebraic map for each $i = 1, \ldots, k$. And the corollary immediately follows from theorem 4.1.

Now in the general case we argue just as in the proof of theorem 4.1. First note that $\Theta \in \mathcal{P}_l(\mathbf{R}^k, G)$ for $l = \dim G - 1$. Therefore theorem 1.12 and proposition 2.27 are applicable to Θ. Since each θ_i is a nontrivial unipotent one-parameter subgroup of G, the proposition 4.16 holds in this case. Now in view of the remarks made in the proof of corollary 4.2, the proof of theorem 4.1 yields the validity of the corollary. □

Proof of corollary 4.5. In view of note 4.10, for any $H \in \mathcal{H}$ for G_1 and Δ, define

$$N(H, \Theta) = \{g \in G_1 : g^{-1}\Theta(\mathbf{R}^k) g \subset H\}.$$

Put

$$S = \bigcup_{H \in \mathcal{H}, \dim H < \dim G_1} N(H, \Theta).$$

For each $i \in \mathbf{N}$, let $H_i \in \mathcal{H}$ with $\dim H_i < \dim G_1$ and let C_i be a compact subset of $N(H_i, \Theta)$ such that $\cup_{i \in \mathbf{N}} C_i = S$.

Suppose that the result is not true. Then there exists a function $f \in \mathrm{C_c}(G/\Delta)$ and for every $i \in \mathbf{N}$ there exist $x_i \in K \setminus \cup_{j \geq i} C_j$ and $T_i > 0$ such that $T_i \to \infty$, as $i \to \infty$, and if B_i denotes the ball in $\mathbf{R}^k$ around the origin with radius T_i then,

$$\left| (1/m(B_i)) \int_{\mathbf{t} \in B_i} f(\Theta(\mathbf{t}) x_i)\, dm(\mathbf{t}) - \int f\, d\mu_{G_1} \right| \geq \epsilon, \quad \forall i \in \mathbf{N}. \tag{4.10}$$

By passing to a subsequence, without loss of generality, we may assume that $x_i \to x$. In particular, $x \notin \pi(S)$. Therefore G_1 is the smallest closed subgroup F of G such that $\Theta(\mathbf{R}^k) \subset F$ and Fx is admits a finite F-invariant measure. But then equation 4.10 contradicts corollary 4.3 (or theorem 4.1 in the case when Θ is of product type and B_i's are boxes), and the proof of the present corollary is complete. $\square$

Chapter 5

Limit distributions of translates of orbits of horospherical subgroups of semisimple subgroups

In this chapter we generalize certain results for actions of certain subgroups of a semisimple group G acting on homogeneous spaces of Lie groups containing G. These results were known earlier only for the actions of G on its own homogeneous spaces.

5.1 Statement of results

Let L be a connected Lie group, Λ a lattice in L, $\pi : L \to L/\Lambda$ the natural quotient map, and μ_L the (unique) L-invariant probability measure on L/Λ.

Theorem 5.1 *Let G be a connected semisimple real algebraic group and A a maximal $\mathbf{R}$-split torus in G. Fix an order on the set of real roots on A for G, and let Δ be the corresponding system of simple roots. Let $\overline{A}^+$ be the closure of the positive Weyl chamber in A. Let $\{a_i\}_{i\in\mathbf{N}}$ be a sequence in $\overline{A}^+$ such that for any $\alpha \in \Delta$, either $\sup_{i\in\mathbf{N}} \alpha(a_i) < \infty$ or $\alpha(a_i) \to \infty$ as $i \to \infty$. Put*

$$U^+ = \{g \in G : {a_i}^{-1} g a_i \to e \text{ as } i \to \infty\}.$$

Assume that no proper normal subgroup of G contains U^+.

Suppose that G is immersed in L as a subgroup and $\pi(G)$ is dense in L/Λ. Let Ω be a relatively compact open subset of U^+ such that π is injective on Ω. Let μ_Ω be a probability measure on $\pi(\Omega)$ which is the image of the restriction of a Haar measure on U^+ to Ω. Then the sequence of measures $a_i\mu_\Omega$ converges weakly to μ_L. In particular, $\{a_i : i \in \mathbf{N}\} \cdot \pi(\Omega)$ is dense in L/Λ.

Using this theorem we obtain the several applications. To state the first result we need a definition.

Definition 5.2 Let G be a semisimple Lie group. A subgroup S of G is said to be *symmetric* if there exists an automorphism σ of G such that $\sigma^2 = 1$ and $S = \{g \in G : \sigma(g) = g\}$.

Corollary 5.3 *Let G be a connected real algebraic semisimple Lie group immersed in L, S the component of the identity of a symmetric subgroup of G, and $\{g_i\}_{i\in\mathbf{N}}$ a sequence contained in G. Suppose that $\pi(S)$ is closed and admits an S-invariant probability measure, say μ_S, and that $\pi(G_1)$ is dense in L/Λ for any closed normal subgroup G_1 of G such that the image of $\{g_i\}$ in $G/(G_1H)$ has no convergent subsequence. Then the sequence of measures $g_i\mu_S$ converges weakly to μ_L.*

The above result generalizes a theorem of Duke, Rudnik and Sarnak [16] (see also Eskin and McMullen [17]) proved in the case of $L = G$.

In view of a result due to Dani [7], and certain other considerations, Stuck and Zimmer [39, Question C] asked the following question:

Question: *Let G be a simple Lie group with finite center and $\mathbf{R}$-rank ≥ 2. Suppose G acts minimally and locally freely on a compact Hausdorff space X. Let P be a proper parabolic subgroup of G and suppose there are equivariant continuous surjective maps $X \times G/P \to Y \to X$ such that the composite map is the projection to X. Is it true that Y is equivariantly homeomorphic to $X \times G/P'$ for some parabolic $P' \supset P$ and the given map is the quotient map?*

In the following result, we show in particular that this question has an affirmative answer under the additional assumption that $X = L/\Lambda$ for a Lie group L containing G and the action of G is via translations.

Theorem 5.4 *Let G be a semisimple Lie group of $\mathbf{R}$-rank ≥ 2 with finite center, realized as a subgroup of L such that the G-action is ergodic with respect to μ_L. Further assume that $\overline{G_1x} = \overline{Gx}$ for any $x \in L/\Lambda$ and any closed normal connected subgroup G_1 of G such that $\mathbf{R}$-rank$(G/G_1) \leq 1$. Let P be a parabolic subgroup of G and consider the diagonal action of G on $L/\Lambda \times G/P$. Let Y be a Hausdorff space with a continuous G-action and $\phi : L/\Lambda \times G/P \to Y$ be a continuous G-equivariant map. Then there exist a parabolic subgroup Q of G containing P, a topological G-space X with a continuous surjective G-equivariant map $\phi_1 : L/\Lambda \to X$, and a continuous G-equivariant map $\psi : X \times G/Q \to Y$ such that the following holds:*

1. *Define the G-equivariant map $\rho : L/\Lambda \times G/P \to X \times G/Q$ as $\rho(x, gP) = (\phi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$. Then $\phi = \psi \circ \rho$.*

2. *For any $x \in L/\Lambda$, define*

$$\mathcal{Q}(x) = \{h \in G : \phi(x, ghP) = \phi(x, gP), \forall g \in G\}.$$

Put

$$X_Q = \{x \in L/\Lambda : \mathcal{Q}(x) = Q\}.$$

Then X_Q is a nonempty open G-invariant subset of L/Λ, $\phi^{-1}(\phi(X_Q \times G/P)) = X_Q \times G/P$, and ψ restricted to $\rho(X_Q) \times G/Q$ is injective.

Further if Y is locally compact and ϕ is surjective then $\phi(X_Q)$ is open in Y.

This result generalizes a result of Dani [7] proved in the case of $L = G$. Its proof is based on the techniques of Dani's proof and uses theorem 5.1 in addition.

To obtain more detailed information about the G-equivariant factores of L/Λ, we prove the following result using certain results from earlier chapters. First we need a definition.

Definition 5.5 Let L be a Lie group and Λ a closed subgroup of L. A map $\tau : L/\Lambda \to L/\Lambda$ is called an *affine automorphism* of L/Λ if there exists $\sigma \in \mathrm{Aut}(L)$ such that $\tau(gx) = \sigma(g)\tau(x)$ for all $x \in L/\Lambda$.

Put $\mathrm{Aut}(L)_\Lambda = \{\sigma \in \mathrm{Aut}(L) : \sigma(\Lambda) = \Lambda\}$. Define a map $\pi : L \cdot \mathrm{Aut}(L)_\Lambda \to \mathrm{Aff}(L/\Lambda)$ by $\pi(h, \sigma)(g\Lambda) = h\sigma(g)\Lambda$ for all $g \in L$. Observe that π is a surjective map. Hence $\mathrm{Aff}(L/\Lambda)$ has the structure of a Lie group acting differentiably on L/Λ.

Theorem 5.6 *Let L be a Lie group, Λ a lattice in L, and G a subgroup of L generated by unipotent one-parameter subgroups contained in it. Suppose that G acts ergodically on L/Λ. Let X be a Hausdorff locally compact space with a continuous G action and $\phi : L/\Lambda \to X$ a continuous surjective G-equivariant map. Then there exists a closed subgroup Λ_1 containing Λ, a compact group $\mathcal{K}$ contained in the centralizer of the image of G in $\mathrm{Aff}(L/\Lambda_1)$, and a G-equivariant continuous surjective map $\psi : \mathcal{K}\backslash L/\Lambda_1 \to X$ such that the following holds:*

1. *Define the G-equivariant map $\rho : L/\Lambda \to \mathcal{K}L/\Lambda_1$ as $\rho(g\Lambda) = \mathcal{K}\backslash g\Lambda_1$ fo all $g \in L$. Then $\phi = \psi \circ \rho$.*

2. *Given a neighbourhood Ω of e in $Z_L(G)$, there exists an open dense G-invariant subset X_0 of L/Λ_1 such that for any $x \in X_0$ and $y \in L/\Lambda_1$ if $\psi(\mathcal{K}(x)) = \psi(\mathcal{K}(y))$ then $y \in \mathcal{K}(\Omega x)$. In this situation, if $\overline{Gx} = L/\Lambda_1$, then $\mathcal{K}(y) = \mathcal{K}(x)$.*

From theorem 5.4 and theorem 5.6 the following result is immediate.

Corollary 5.7 *Let L be a Lie group, Λ a lattice in L, and G a connected semisimple Lie group with finite center, realized as a subgroup of L. Suppose that the action of G_1 on L/Λ is minimal for any closed normal subgroup G_1 of G such that $\mathbf{R}$-$\mathrm{rank}(G/G_1) \leq 1$. Let Y be a locally compact Hausdorff space with continuous G-action, P a parabolic subgroup of G, and $\phi : L/\Lambda \times G/P \to Y$ a continuous surjective G-equivariant map, where G-acts diagonally on $L/\Lambda \times G/P$. Then there exist a parabolic subgroup Q of G containing P, a closed subgroup Λ_1 of L containing Λ, and a compact group $\mathcal{K}$ contained in the centralizer of the image of G in $\mathrm{Aff}(L/\Lambda_1)$ such that Y is G-equivariantly homeomorphic to $\mathcal{K}\backslash L/\Lambda_1 \times G/Q$ and ϕ is the natural quotient map.*

5.2 Some results in Linear algebra

Lemma 5.8 *Let V be a finite dimensional real vector space equipped with a Euclidean norm. Let $\underline{\mathbf{n}}$ be a nilpotent Lie subalgebra of $\mathrm{End}(V)$. Let N be the associated* *unipotent subgroup of* $\operatorname{Aut}(V)$*. Let* $\{\mathbf{b}_1, \ldots, \mathbf{b}_q\}$ *be a basis of* $\underline{\mathbf{n}}$*. Consider the map* $\Theta : \mathbf{R}^q \to N$ *defined as*

$$\Theta(t_1, \ldots, t_q) = \exp(t_q \mathbf{b}_q) \cdots \exp(t_1 \mathbf{b}_1), \quad \forall (t_1, \ldots, t_q) \in \mathbf{R}^q.$$

For $\rho > 0$*, define*

$$B_\rho = \{\Theta(t_1, \ldots, t_q) \in N : 0 \leq t_k < \rho,\, k = 1, \ldots, q\}.$$

Put

$$W = V^N := \{\mathbf{v} \in V : n \cdot \mathbf{v} = \mathbf{v} \textit{ for all } n \in N\}.$$

Let p_W *denote the orthogonal projection on* W*. Then for any* $\rho > 0$*, there exists* $c > 0$ *such that for every* $\mathbf{v} \in V$*,*

$$\|\mathbf{v}\| \leq c \cdot \sup_{\mathbf{t} \in B_\rho} \|p_W(\Theta(\mathbf{t}) \cdot \mathbf{v})\|.$$

Proof. For $k = 1, \ldots, q$, let $n_k \in \mathbf{N}$ be such that $\mathbf{b}_k^{n_k} = 0$. Let

$$\mathcal{I} = \{I = (i_1, \ldots, i_q) : 0 \leq i_k \leq n_k - 1,\, k = 1, \ldots, q\}.$$

For $\mathbf{t} = (t_1, \ldots, t_q) \in \mathbf{R}^q$ and $I = (i_1, \ldots, i_q) \in \mathcal{I}$, define

$$\mathbf{t}^I = t_q^{i_q} \cdots t_1^{i_1} \quad \text{and} \quad \mathbf{b}^I = \frac{\mathbf{b}_q^{i_q} \cdots \mathbf{b}_1^{i_1}}{i_q! \cdots i_1!}.$$

Then for all $\mathbf{v} \in V$ and $\mathbf{t} \in \mathbf{R}^q$, we have

$$\Theta(\mathbf{t}) \cdot \mathbf{v} = \sum_{I \in \mathcal{I}} \mathbf{t}^I \cdot (\mathbf{b}^I \mathbf{v}). \tag{5.1}$$

We define a transformation $T : V \to \oplus_{I \in \mathcal{I}} W$ by

$$T(\mathbf{v}) = \left(p_W(\mathbf{b}^I \cdot \mathbf{v})\right)_{I \in \mathcal{I}}, \quad \forall \mathbf{v} \in V. \tag{5.2}$$

We claim that T is injective. To see this, suppose there exists $\mathbf{v} \in V \setminus \{0\}$ such that $T(\mathbf{v}) = 0$. Then $N \cdot \mathbf{v} \subset W^\perp$, the orthogonal complement of W. Hence $W^\perp$ contains a nontrivial N-invariant subspace. Then by Lie-Kolchin theorem, $W^\perp$ contains a nonzero vector fixed by N. Then $W \cap W^\perp \neq \{0\}$, which is a contradiction.

We consider $\oplus_{I \in \mathcal{I}} V$ equipped with a box norm; that is

$$\|(v_I)_{I \in \mathcal{I}}\| = \sup_{I \in \mathcal{I}} \|v_I\|, \quad \text{where } v_I \in V,\, \forall I \in \mathcal{I}.$$

Then there exists a constant $c_1 > 0$ such that

$$\|\mathbf{v}\| \leq c_1 \cdot \|T(\mathbf{v})\|, \quad \forall \mathbf{v} \in V.$$

For all $k = 1, \ldots, q$, and $j_k = 1, \ldots, n_k$, fix $0 < t_{k,1} < \cdots < t_{k,n_k} < \rho$ and put $M_k = \left(t_{k,j_k}^{i_k}\right)_{0 \leq i_k \leq n_k - 1,\, 1 \leq j_k \leq n_k}$ for $k = 1, \ldots, q$. Then $\det M_k$ is a Vandermonde determinant and hence M_k is invertible.

Let
$$\mathcal{J} = \{J = (j_1, \ldots, j_q) : 1 \le j_k \le n_k,\ k = 1, \ldots, q\}.$$
For $J = (j_1, \ldots, j_q) \in \mathcal{J}$, put
$$\mathbf{t}_J = (t_{1,j_1}, \ldots, t_{q,j_q}) \quad \text{and} \quad M = \left(\mathbf{t}_J^I\right)_{(I,J)\in\mathcal{I}\times\mathcal{J}}.$$

Take $\mathbf{v} \in V$. Put
$$X_{\mathcal{I}} = T(\mathbf{v}) \quad \text{and} \quad Y_{\mathcal{J}} = \left(p_W(\Theta(\mathbf{t}_J)\mathbf{v})\right)_{J\in\mathcal{J}}.$$
Then by equations 5.1 and 5.2,
$$M \cdot X_{\mathcal{I}} = Y_{\mathcal{J}}.$$
Since $M = M_1 \otimes \cdots \otimes M_q$ and each M_k is invertible, we have that M is invertible. Hence
$$X_{\mathcal{I}} = M^{-1} \cdot Y_{\mathcal{J}}.$$
Put $c_2 = \|M^{-1}\|$ and $c = c_1c_2$. Then
$$\|\mathbf{v}\| \le c_1\|T(\mathbf{v})\| = c_1\|X_{\mathcal{I}}\| \le c_1c_2\|Y_{\mathcal{J}}\| = c \cdot \sup_{J\in\mathcal{J}} \|p_W(\Theta(\mathbf{t}_J)\mathbf{v})\|.$$
This completes the proof. □

Remark 5.9 Let the notation be as in theorem 5.1. Put
$$\Phi = \{\alpha \in \Delta : \alpha(a_i) \to \infty \text{ as } i \to \infty\}.$$
Let P^+ be the standard parabolic subgroup associated to the set of roots $\Delta \setminus \Phi$. Then U^+ as defined before is the unipotent radical of P^+. Let P^- denote the standard opposite parabolic subgroup for P^+ and let U^- be the unipotent radical of P^-. Note that
$$P^- = \{g \in G : \overline{\{a_i g a_i{}^{-1} : i \in \mathbf{N}\}} \text{ is compact}\}.$$
Also note that the Lie algebra of G is the direct sum of the Lie subalgebras associated to U^-, $P^- \cap P^+$ and U^+.

Lemma 5.10 *Let the notation be as in theorem 5.1. Consider a representation of G on a finite dimensional normed linear space V. Suppose that the action of G is nontrivial and irreducible. Then for any sequence $\{\mathbf{v}_i\} \subset V^{U^+}$ which is bounded away from the origin,*
$$\|a_i \cdot \mathbf{v}_i\| \to \infty \quad \textit{as } i \to \infty.$$

Proof. Since A is $\mathbf{R}$-split, there is a finite set Λ of real characters on A such that for each $\lambda \in \Lambda$, if we define
$$V_\lambda = \{\mathbf{v} \in V : a \cdot \mathbf{v} = \lambda(a)\mathbf{v},\ \forall a \in A\},$$

then $V = \oplus_{\lambda\in\Lambda} V_\lambda$. After passing to an appropriate subsequence, if we define

$$\begin{aligned}
\Lambda_+ &= \{\lambda \in \Lambda : \lambda(a_i) \to \infty \text{ as } i \to \infty\} \\
\Lambda_- &= \{\lambda \in \Lambda : \lambda(a_i) \to 0 \text{ as } i \to \infty\}, \quad \text{and} \\
\Lambda_0 &= \{\lambda \in \Lambda : \lambda(a_i) \to c \text{ for some } c > 0 \text{ as } i \to \infty\},
\end{aligned}$$

then $\Lambda = \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$.

Put $W = V^{U^+}$. Since U^+ is normalized by A, we have that W is invariant under the action of A. Therefore

$$W = \oplus_{\lambda\in\Lambda}(W \cap V_\lambda).$$

Suppose that there exists $\mathbf{w} \in W \cap V_\lambda \setminus \{0\}$ for some $\lambda \in \Lambda_0 \cup \Lambda_-$. For any $g \in P^-$, we have $a_i g {a_i}^{-1} \to g_0$ for some $g_0 \in P^-$. Therefore as $i \to \infty$,

$$a_i(g\mathbf{w}) = a_i g {a_i}^{-1}(a_i\mathbf{w}) \to c(g_0\mathbf{w}) \quad \text{for some } c \geq 0.$$

Hence $P^-\mathbf{w} \subset \oplus_{\lambda\in\Lambda_0\cup\Lambda_-} V_\lambda$. Now $U^+\mathbf{w} = \mathbf{w}$ and by remark 5.9 P^-U^+ is open in G. Therefore $G \cdot \mathbf{w} \subset \oplus_{\lambda\in\Lambda_0\cup\Lambda_-} V_\lambda$. Since G is semisimple and V is irreducible, $\Lambda = \Lambda_0$. Therefore there exists $M > 0$ such that $\sup_{i\in\mathbf{N}} \|a_i\mathbf{v}\| \leq M \cdot \|\mathbf{v}\|$ for all $\mathbf{v} \in V$.

Now for any relatively compact neighbourhood Ω of U^+ and any $\mathbf{v} \in V_\lambda$, there exists a compact ball $B \subset V$ such that for all $i \in \mathbf{N}$,

$$B \supset a_i\Omega \cdot \mathbf{v} = (a_i\Omega {a_i}^{-1})a_i \cdot \mathbf{v} = \lambda(a_i)(a_i\Omega {a_i}^{-1})\mathbf{v}.$$

Since $\lambda(a_i) \to c$ for some $c > 0$ and $\cup_{i\in\mathbf{N}} a_i\Omega {a_i}^{-1} = U^+$, we have $U^+ \cdot \mathbf{v} \subset c^{-1}B$. Since U^+ acts on V by unipotent linear transformations, we obtain that $U^+ \cdot \mathbf{v} = \mathbf{v}$. Thus U^+ acts trivially on V. Since the kernel of G action on V is a normal subgroup of G containing U^+, it is equal to G by our assumption. This contradicts our hypothesis in the lemma that the action of G is nontrivial. This proves that $W \subset \sum_{\lambda\in\Lambda_+} V_\lambda$ and the conclusion of the lemma follows. □

Corollary 5.11 *Let the notation be as in theorem 5.1. Consider a representation of G on a finite dimensional real vector space V equipped with a Euclidean norm. Suppose that the action of G is nontrivial and irreducible. Let $\{\mathbf{v}_i\} \subset V$ be a sequence which is bounded away from the origin. Then for any neighbourhood Ω of e in U^+,*

$$\sup_{\omega\in\Omega} \|a_i\omega \cdot \mathbf{v}_i\| \to \infty \quad \textit{as } i \to \infty.$$

Proof. Let $W = V^{U^+}$. By lemma 5.8, there exists $c > 0$ such that for all $i \in \mathbf{N}$,

$$\sup_{\omega\in\Omega} \|p_W(\omega \cdot \mathbf{v}_i)\| \geq c\|\mathbf{v}_i\| \geq c \cdot \inf_{i\in\mathbf{N}} \|\mathbf{v}_i\|.$$

Since $\inf_{i\in\mathbf{N}} \|\mathbf{v}_i\| > 0$, by lemma 5.10,

$$\sup_{\omega\in\Omega} \|a_i \cdot \omega\mathbf{v}_i\| \geq \sup_{\omega\in\Omega} \|a_i \cdot p_W(\omega \cdot \mathbf{v}_i)\| \to \infty \quad \text{as } i \to \infty.$$

□

5.3 Proofs of the main results

Translates of horospherical orbits

Proof of theorem 5.1. First note that there is no loss of generality in passing to a subsequence of $\{a_i\}$, whenever convenient.

Let $\underline{\mathbf{u}}^+$ denote the Lie algebra of U^+. Since $\underline{\mathbf{u}}^+$ is a nilpotent Lie algebra, we can choose a basis $\{\mathbf{b}_1, \dots, \mathbf{b}_q\}$ for it such that $[\mathbf{b}_i, \mathbf{b}_j] \in \mathbf{R}\text{-span}\{\mathbf{b}_k : k > i, j\}$. Consider the map $\Theta : \mathbf{R}^q \to U^+$ defined by $\Theta(t_1, \dots, t_q) = \exp(t_q \mathbf{b}_q) \cdots \exp(t_1 \mathbf{b}_1)$ for all $(t_1, \dots, t_q) \in \mathbf{R}^q$.

The result for general open set $\Omega \subset U^+$ follows if we prove it for $\Omega = \Theta(B)$ for every bounded open convex set $B \subset \mathbf{R}^q$. Due to our choice of the basis of $\underline{\mathbf{u}}^+$, there is a constant multiple of the Lebesgue measure on $\mathbf{R}^q$ such that if λ is its restriction to B then $\mu_\Omega = \pi_*(\Theta_*(\lambda))$.

We first claim that given $\epsilon > 0$ there exists a compact set $K \subset L/\Lambda$ such that

$$(a_i \mu_\Omega)(K) > 1 - \epsilon \quad \text{for all } i \in \mathbf{N}.$$

Put $\Theta_i = a_i \cdot \Theta : \mathbf{R}^q \to G$ for each $i \in \mathbf{N}$. Then there exists $d \in \mathbf{N}$ such that $\Theta_i \in \mathcal{P}_{d,q}(G)$ for all $i \in \mathbf{N}$. If the claim fails to hold, then by theorem 1.14, after passing to a subsequence, there exists a representation of G on a finite dimensional normed linear space V of G and a nonzero vector $\mathbf{p} \in V$ such that,

$$\sup_{\omega \in \Omega} \|a_i \omega \cdot \mathbf{p}\| \to 0 \quad \text{as } i \to \infty.$$

Decomposing V into G-irreducible components and noting that the projection of $\Omega \mathbf{p}$ is zero on the space of G-fixed vectors, we see that this contradicts corollary 5.11. Thus the claim is proved.

Now by passing to a subseqnence we may assume that the sequence $a_i \cdot \mu_\Omega$ converges to a probability measure μ on L/Λ.

Clearly μ is U^+ invariant. Therefore by proposition 2.22, there exists a closed subgroup H in the collection $\mathcal{H}$ associated to L/Λ in place of G/Γ there, such that

$$\mu(\pi(S(H, U^+)) = 0 \quad \text{and} \quad \mu(\pi(N(H, U^+))) > 0.$$

Let a compact set $C \subset N(H, U^+) \setminus S(H, U^+)$ be such that $\epsilon := \mu(\pi(C)) > 0$.

Let the finite dimensional vector space V with a Euclidean norm and a unit vector $\mathbf{p}_H \in V$ be as described in notation 1.13, for L in place of G there. Let K be a compact neighbourhood of $\pi(C)$ in L/Λ. We apply proposition 2.31 for $\epsilon > 0$ and $d \in \mathbf{N}$ as above, and obtain compact sets $D \subset V(H, U^+)$ and $S_1 \subset S(H, U^+)$ as in that proposition. Since $\pi(C) \cap \pi(S(H, U^+)) = \emptyset$, there exists a compact neighbourhood Z of $\pi(C)$ contained in $(L/\Lambda) \setminus \pi(S_1)$. Let Φ be a relatively compact neighbourhood of D in V. Now for each $i \in \mathbf{N}$, applying the proposition to Θ_i in place of Θ and $x = \pi(e)$, we can easily see that the first and the last possibilities of its conclusion do not hold. Therefore we conclue that for each $i \in \mathbf{N}$, there exists $\mathbf{v}_i \in \Lambda \cdot \mathbf{p}_H$ such that

$$a_i \Theta(B) \cdot \mathbf{v}_i \subset \Phi.$$

Since the orbit $\Lambda \cdot \mathbf{p}_H$ is discrete, by corollary 5.11 applied to each irreducible component of V for the G action, the set $\{\mathbf{v}_i\}_{i \in \mathbf{N}}$ is finite and it consists of fixed points. Moreover, by passing to a subsequence, there exists $\gamma \in \Lambda$ such that $\mathbf{v}_i = \gamma \cdot \mathbf{p}_H$ for all $i \in \mathbf{N}$, and

$$G \cdot \gamma \cdot \mathbf{p}_H = \gamma \cdot \mathbf{p}_H.$$

Therefore, $G \subset \gamma N_L^1(H)\gamma^{-1}$. But $\pi(N_L^1(H))$ is closed in L/Λ and $\pi(G)$ is dense in L/Λ. Therefore we conclude that H is a normal subgroup of L. Since $N(H, U^+) \supset C \neq \emptyset$, this implies in particular that U^+ is contained in H. Thus $U^+ \subset G \cap H$ and $G \cap H$ is normal in G. Therefore by our hypothesis in the theorem $G \cap H = G$, or in other words $G \subset H$. Again since $\pi(G)$ is dense in L/Λ, we have $H = L$. Hence $\mu(\pi(S(L, U^+))) = 0$. Therefore by proposition 2.22, we have that μ is L-invariant. $\square$

Translates of orbits of symmetric subgroups

Proof of corollary 5.3. Note that without loss of generality we may replace the sequence $\{g_i\}$ by a subsequence, whenever necessary. Let G_1 be a minimal normal connected subgroup of G such that the image of a subsequece of $\{g_i\}$ in $G/(G_1S)$ is contained in a compact subset. Therefore replacing the sequence $\{g_i\}$ by a subsequence, then modifying it from left by multiplications with elements from a compact set, and from right by multiplications with elements of S, without loss of generality we may assume that the sequence $\{g_i\}$ is contained in G_1.

Let σ be an involution of G such that $S = \{g \in G : \sigma(g) = g\}^0$. Note that G_1 is stable under σ. Put $S_1 = (S \cap G_1)^0$. Then S_1 is the component of identity of the symmetric subgroup of G_1 associated to the restriction of σ to G_1.

Using the results in [34, Section 7.1] one can deduce the following. There exits a Cartan involution θ of G_1 commuting with σ, a maximal $\mathbf{R}$-split torus A of G_1, a subtorus B of A and a system of potitive roots on A for G_1, such that the torus A is invariant under θ and σ, $\theta(a) = a^{-1}$ for any $a \in A$ and $B = \{b \in A : \sigma(b) = b^{-1}\}$. Let Δ denote the system of simple roots on A. Put

$$\Delta^B = \{\alpha \in \Delta : \alpha|_B \neq 1\}.$$

Then Δ^B is a system simple roots on B for G_1. Let $\overline{A}^+$ (respectively $\overline{B}^+$) denote the closure of the positive Weyl chember in A (respectively B). Let K_1 be the maximal compact subgroup of G_1 associated to θ. We have a decomposition $G_1 = K_1 \overline{B}^+ S_1$.

In view of this decomposition, in proving the corollary, without loss of generality we may assume that $g_i = b_i \in \overline{B}^+$ for all $i \in \mathbf{N}$. By passing to a subsequence, we may assume that $\{b_i\}_{i \in \mathbf{N}}$ has no convergent subsequence and for any $\alpha \in \Delta$, either $\sup_{i \in \mathbf{N}} \alpha(b_i) < \infty$ or $\alpha(b_i) \to \infty$ as $i \to \infty$. Let

$$\Phi = \{\alpha \in \Delta : \alpha(b_i) \to \infty\}.$$

Then $\Phi \neq \emptyset$. Let P^+ be the standard parabolic subgroup associated to the set of roots $\Delta \setminus \Phi$. Then the unipotent radical of P^+ is given by

$$U^+ = \{g \in G_1 : b_i{}^{-1} g b_i \to e \text{ as } i \to \infty\}.$$

Let P^- be the standard opposite parabolic subgroup for P^+ and U^- be the unipotent radical of P^-.

Let F be the smallest closed normal subgroup of G_1 containing U^+. By the minimality of G_1, if $F \neq G_1$, the image of the sequence $\{b_i\}$ in G_1/F has no convergent subsequence. But then the image of U^+ in G_1/F is noncompact, which is a contradiction. Therefore $F = G_1$.

Note that $\sigma(P^+) = P^-$ and $\sigma(U^+) = U^-$. Let $Z = P^+ \cap P^-$. Let $\underline{\mathbf{g}}_1$, $\underline{\mathbf{s}}_1$, $\underline{\mathbf{u}}^+$, $\underline{\mathbf{u}}^-$, and $\underline{\mathbf{z}}$ denote the Lie algebras associated to the Lie groups G_1, S_1, U^+, U^-, and Z, respectively. We have that

$$\underline{\mathbf{g}}_1 = \underline{\mathbf{u}}^- \oplus \underline{\mathbf{z}} \oplus \underline{\mathbf{u}}^+.$$

Note that for any $X \in \underline{\mathbf{u}}^+$, we have $\sigma(X) \in \underline{\mathbf{u}}^-$ and $X + \sigma(X) \in \underline{\mathbf{s}}_1$. Therefore

$$\underline{\mathbf{g}}_1 = \underline{\mathbf{u}}^- + \underline{\mathbf{z}} + \underline{\mathbf{s}}_1.$$

Let Ω^-, Ω_0, Ω^+, and Φ_1 be neighbourhoods of identity respectively in U^-, Z, U^+, and S_1. We may assume that these neighbourhoods are small enough so that π is injective on $\Omega^-\Omega^0\Omega^+$ and $\Omega^-\Omega^0\Phi_1$. Let ν^-, ν^0, ν^+, and μ_1 be the probability measures obtained by restricting the corresponding Haar measures for the groups U^-, Z, U^+, and S_1 to the sets Ω^-, Ω^0, Ω^+, and Φ_1, respectively.

Let $m : G \times G \times G \to G$ be the map given by $m(g_1, g_2, g_3) = g_1 g_2 g_3$ for all $g_1, g_2, g_3 \in G$. Put $\Omega_1 = m(\Omega^- \times \Omega^0 \times \Phi_1)$ and consider on it the probability measure $\lambda_1 = m_*(\nu^- \times \nu^0 \times \mu_1)$.

On Ω_1 we define a probability measure λ_2 such that for any $f \in \mathrm{C_c}(\Omega_1)$, we have

$$\begin{aligned} &\int_{\omega \in \Omega_1} f(\omega)\, d\lambda_2(\omega) \\ = \quad & c \cdot \int_{(v_1,v_2) \in \Omega^- \times \Omega^0} d(\nu^- \times \nu^0)(v_1, v_2) \int_{v_3 \in ((v_1 v_2)^{-1}\Omega_1 \cap U^+)} f(v_1 v_2 v_3)\, d\nu^+(v_3), \end{aligned}$$

where $c > 0$ is a constant independent of f.

We note that that λ_1 and λ_2 are absolutely continuous with respect to each other and there are constants $c_1 > 0$ and $c_2 > 1$ such that for all Borel measurable subset $E \subset \Omega_1$, we have

$$c_1 \lambda_1(E) \leq \lambda_2(E) \leq c_2 \lambda_1(E). \tag{5.3}$$

First we claim that

$$b_i \cdot \pi_*(\lambda_2) \to \mu_L \quad \text{ as } i \to \infty. \tag{5.4}$$

To prove this, let $f \in \mathrm{C_c}(L/\Lambda)$. Then

$$\int_{(v_1,v_2) \in \Omega^- \times \Omega^0} d(\nu^- \times \nu^0)(v_1, v_2) \cdot$$

$$
\begin{aligned}
&\cdot \int_{v_3\in((v_1v_2)^{-1}\Omega_1\cap U^+)} f(b_i\pi(v_1v_2v_3))\,d\nu^+(v_3)\\
= &\int_{(v_1,v_2)\in\Omega^-\times\Omega^0} d(\nu^-\times\nu^0)(v_1,v_2)\cdot\\
&\cdot \int_{v_3\in((v_1v_2)^{-1}\Omega_1\cap U^+)} f((b_i(v_1v_2)b_i{}^{-1})\cdot b_i\pi(v_3))\,d\nu^+(v_3).
\end{aligned}
$$

For any $v_1, v_2 \in \Omega^-\times\Omega^0$, as $i\to\infty$, $b_i(v_1v_2)g_i{}^{-1}\to v'$ for some $v'\in\Omega^0$. By theorem 5.1, for any open set Ψ in U^+,

$$
\int_{v\in\Psi} f(b_i\pi(v_3))\,d\nu^+(v)\to\int f\,d\mu_L \quad \text{as } i\to\infty.
$$

Therefore

$$
\begin{aligned}
&\lim_{i\to\infty}\int_{v\in\Omega_1} f(b_i\pi(v))\,d\lambda_2(v)\\
= \;& c(\int f\,d\mu_L)\int_{(v_1,v_2)\in\Omega^-\times\Omega^+}\nu^+((v_1v_2)^{-1}\Omega_1\cap U^+)\,d(\nu^-\times\nu^0)(v_1,v_2)\\
= \;& \int f\,d\mu_L.
\end{aligned}
$$

This proves the claim.

By passing to a subsequence, we may assume that

$$
b_i\cdot\pi_*(\lambda_1)\to\lambda', \quad \text{as } i\to\infty, \tag{5.5}
$$

in the space of probability measures on the one-point compactification of L/Λ.

By eq. 5.3, for any continuous function f on the one-point compactification of L/Λ, we have that

$$
c_1\int_{L/\Lambda} f\,d(b_i\cdot\pi_*(\lambda_1))\le\int_{L/\Lambda} f\,d(b_i\cdot\pi_*(\lambda_2))\le c_2\int_{L/\Lambda} f\,d(b_i\cdot\pi_*(\lambda_1))
$$

for all $i\in\mathbf{N}$. And hence as $i\to\infty$, by equations 5.4 and 5.5, we get

$$
c_1\int_{L/\Lambda} f\,d\lambda'\le\int_{L/\Lambda} f\,d\mu_L\le c_2\int_{L/\Lambda} f\,d\lambda'.
$$

Thus λ' is a probability measure on L/Λ and it is absolutely continuous with respect to μ_L. Also there exists a relatively compact neighbourhood Ψ_1 of e in Z and a probability measure ν_1^0 on Ψ_1 which is the restriction of a haar measure on Z such that as $i\to\infty$,

$$
I_{b_i}(\Omega^0)\to\Psi_1 \quad\text{and}\quad (I_{b_i})_*(\nu^0)\to\nu_1^0,
$$

where I_b denotes the inner conjugation by an element b on G_1. Also as $i\to\infty$,

$$
I_{b_i}(\Omega^-)\to\{e\} \quad\text{and}\quad (I_{b_i})_*(\nu^-)\to\delta_{\{e\}},
$$

where $\delta_{\{e\}}$ denotes the probability measure supported at the identity. Therefore, by passing to a subsequence, we have that

$$b_i \cdot \pi_*(\mu_1) \to \eta \quad \text{as } i \to \infty,$$

where η is a probability measure on L/Λ such that for any $f \in \mathrm{C_c}(L/\Lambda)$,

$$\int f \, d\lambda' = \int_{v \in \Psi_1} d\nu_1^0(v) \int f(vx) \, d\eta(x).$$

Next we claim that η is U^+-invariant. To see this, let $u = \exp(X)$ for some $X \in \underline{\mathbf{u}}^+$. Put $X_i = \mathrm{Ad}(b_i{}^{-1})X$ and $h_i = \exp(X_i + \sigma(X_i)) \in S_1$ for all $i \in \mathbf{N}$. Then $h_i \to e$ as $i \to \infty$ and $u = b_i h_i b_i{}^{-1}$ for all $i \in \mathbf{N}$. For any $f \in \mathrm{C_c}(L/\Lambda)$, we have

$$\left| \int_{L/\Lambda} f(x) \, d\pi_*(\mu_1)(x) - \int_{L/\Lambda} f(h_i x) \, d\pi_*(\mu_1)(x) \right| \leq \epsilon_i \cdot \sup |f|,$$

where ϵ_i depends only on h_i, and $\epsilon \to 0$ as $h_i \to e$. Let $i \in \mathbf{N}$. Applying this equation for $f_i(x) := f(b_i x)$ for all $x \in L/\Lambda$, we get

$$\left| \int_{L/\Lambda} f(b_i x) \, d\pi_*(\mu_1)(x) - \int_{L/\Lambda} f((b_i h_i b_i{}^{-1})(b_i x)) \, d\pi_*(\mu_1)(x) \right| \leq \epsilon_i \cdot \sup |f|.$$

We have that $b_i \cdot \pi_*(\mu_1) \to \eta$ weakly as $i \to \infty$, $b_i h_i b_i{}^{-1} = u$ for all $i \in \mathbf{N}$, and f is uniformly continuous. Therefore

$$\int_{L/\Lambda} f(x) \, \eta(x) = \int_{L/\Lambda} f(ux) \, d\eta(x).$$

Thus η is invariant under the action of U^+.

Since $\pi(G_1)$ is dense in L/Λ, by the Mautner's phenomenon described as in Section 2.1, we have that U^+ acts ergodically on L/Λ with respect to μ_L.

Since Z normalilzes U^+, we have that $v \cdot \eta$ is U^+-invariant for all $v \in Z$. Hence λ' is U^+-invariant. Since λ' is absolutely continuous with respect to μ_L, we have that U^+ acts ergodically with respect to λ'. Therefore $v \cdot \eta = \mu_L$ for almost all $v \in \Psi_1$. But μ_L being Z-invariant, we have that $\eta = \mu_L$. That is,

$$b_i \cdot \pi_*(\mu_1) \to \mu_L \quad \text{as } i \to \infty.$$

There exists a normal subgroup S_2 of S such that $S = S_2 S_1$ and $\{b_i : i \in \mathbf{N}\} \subset Z_G(S_2)$. Let Ψ_2 be a relatively compact open neighbourhood of e in S_2 such that π is injective on $\Psi = \Psi_2 \Psi_1$. Let μ_2 denote the probability measure which is the restriction of a Haar measure on S_2 to Ψ_2. Then $\mu = \mu_2 \times \mu_1$ is the restriction of a Haar measure on S to Ψ. Also we have that

$$b_i \cdot \pi_*(\mu) \to \mu_L \quad \text{as } i \to \infty.$$

Considering small neighbourhoods like Ψ associated to different points in $\pi(S)$, we see that $b_i \cdot \mu_S \to \mu_L$ as $i \to \infty$. Since we had seen that we may assume $g_i = b_i \in \overline{B^+}$, this shows that $g_i \cdot \mu_S \to \mu_L$ as $i \to \infty$. $\square$

Continuous G-equivariant factors

First we extract the following result from [7, Section 2].

Proposition 5.12 (Dani) *Let G be a semisimple group with finite center and $\mathbf{R}$-rank$(G) \geq 2$. Let P be a parabolic subgroup of G. Then given $g \in G \setminus P$, there exist $k \in \mathbf{N}$ ($k < \mathbf{R}$-rank(G)), elements $g_1, \ldots, g_{k+1}$ in G, and unipotent one-parameter subgroups $\{u_1(t)\}, \ldots, \{u_k(t)\}$ of G contained in P such that the following holds:*

1. *$g_1 = g$, $g_k \notin P$, and $g_{k+1} = e$.*
2. *For each $i = 1, \ldots, k$, we have*
$$u_i(t)g_iP \to g_{i+1}P \text{ in } G/P \text{ as } t \to \infty.$$
3. *For an Ad-semisimple element $a \in G$, and the associated expanding horospherical subgroup*
$$U^+ := \{u \in G : a^{-n}ua^n \to e \text{ as } n \to \infty\},$$
we have
$$< a > \cdot U^+ \subset g_kPg_k{}^{-1} \cap P.$$
Moreover if G_1 is the smallest normal subgroup of G containing U^+, then $\mathbf{R}$-rank$(G/G_1) \leq 1$.

Proof. Apply [7, Corollary 2.3] iteratively. Also use the proofs of [7, Corollary 2.6 and Lemma 2.7]. □

Proposition 5.13 *Let the notation and assumptions be as in theorem 5.4. Let $x, y \in L/\Lambda$ and $g \in G$. If $\phi(x, gP) = \phi(y, P)$, then the following holds:*

1. *$\phi(x, P) = \phi(x, gP)$.*
2. *If $g \notin P$ then there exists a parabolic subgroup Q containing $\{g\} \cup P$ such that $\phi(z, P) = \phi(z, qP)$ for all $z \in \overline{Gx}$ and $q \in Q$.*

Proof. Let $k \in \mathbf{N}$, elements $g_1, \ldots, g_{k+1}$ in G, unipotent one-parameter subgroups $\{u_1(t)\}, \ldots, \{u_k(t)\}$ contained in P, and an Ad-semisimple element a of G and the associated expanding horospherical subgroup U^+ be as in proposition 5.12. For each $i = 1, \ldots, k$, Ratner's theorem applied to the diagonal action of $\{u_i(t)\}$ on $L/\Lambda \times L/\Lambda$, there exists a sequence $t_n \to \infty$ such that $(u_i(t_n)x, u_i(t_n)y) \to (x, y)$ as $n \to \infty$. Therefore, for any $i \in \{1, \ldots, k\}$, if $\phi(x, g_iP) = \phi(y, P)$ then

$$\phi(u_i(t_n)x, u_i(t_n)g_iP) = \phi(u_i(t_n)y, P), \quad \forall n \in \mathbf{N}.$$

Taking the limit as $n \to \infty$, we get that $\phi(x, g_{i+1}P) = \phi(y, P)$. Since $g_1 = g$, by induction on i, we get that $\phi(x, g_iP) = \phi(y, P)$ for all $1 \leq i \leq k + 1$. Since $g_{k+1} = e$, we have that condition (1) of the present proposition holds.

In particular, we have

$$\phi(x, g_k P) = \phi(y, P) = \phi(x, P).$$

Since $F =< a > \cdot U^+ \subset g_k P {g_k}^{-1} \cap P$, we have that

$$\phi(fx, g_k P) = \phi(fx, P), \; \forall f \in F.$$

Let G_1 be the smallest closed normal subgroup of G containing U^+. Then **R**-rank$(G/G_1) \leq 1$. Therefore by our hypothesis in theorem 5.4, $\overline{G_1 x} = \overline{Gx}$. By Ratner's theorem $\overline{Gx}$ is an orbit of a closed subgroup of L containing G. Applying theorem 5.1 to that subgroup in place of L there, we see that we have $\overline{Fx} = \overline{Gx}$. Thus

$$\phi(z, g_k P) = \phi(z, P), \quad \forall z \in \overline{G_1 x} = \overline{Gx}.$$

Put

$$Q = \{h \in G : \phi(z, fhP) = \phi(z, fP), \; \forall z \in \overline{Gx}, \forall f \in G\}. \tag{5.6}$$

Then Q is a closed subgroup of G containing $P \cup \{g_k\}$. Since $g_k \notin P$,

$$Q \neq P. \tag{5.7}$$

Now if $g \notin Q$, then replacing P by Q and L/Λ by $\overline{Gx}$, we repeat the above argument. Note that by definition the new set given by equation 5.6 still turns out to be same as Q. This fact contradicts the new equation 5.7. This completes the proof of (2). □

Proof of theorem 5.4. Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x, gP) = \phi(y, gP) \text{ for some } g \in G\}$$

on L/Λ. Clearly R is a closed G-invariant subset of $L/\Lambda \times L/\Lambda$. Let X be the space of equivalence classes of R and let $\phi_1 : L/\Lambda \to X$ be the map taking any element of L/Λ to its equivalence class. Equip X with the quotient topology. Then X is a locally compact Hausdorff space.

For any $x \in L/\Lambda$, put

$$\mathcal{Q}(x) = \{h \in G : \phi(x, gP) = \phi(x, ghP), \; \forall g \in G\}.$$

Observe that $\mathcal{Q}(x)$ is a closed subgroup of G containing P and for any $y \in \overline{Gx}$, we have $\mathcal{Q}(y) \supset \mathcal{Q}(x)$. Let $x_0 \in L/\Lambda$ such that $\overline{Gx} = L/\Lambda$ and put $Q = \mathcal{Q}(x_0)$. Then $\mathcal{Q}(y) \supset Q$ for all $y \in L/\Lambda$. Since Q is a parabolic subgroup of G, there are only finitely many closed subgroups of G containing Q. Therefore the set $X_Q := \{x \in L/\Lambda : \mathcal{Q}(x) = Q\}$ is open in L/Λ. Also X_Q is nonempty and G-invariant. Now since G acts ergodically on L/Λ, the set $L/\Lambda \setminus X_Q$ is closed and nowhere dense.

Note that for any $x, y \in L/\Lambda$, if $\phi_1(x) = \phi_1(y)$ then by proposition 5.13, we have that $\mathcal{Q}(x) = \mathcal{Q}(y)$. Let $\rho : L/\Lambda \times G/P \to X \times G/Q$ be the (G-equivariant) map defined by $\rho(x, gP) = (\phi_1(x), gQ)$ for all $x \in L/\Lambda$ and $g \in G$. Then there exists a uniquely defined map $\psi : X \times G/Q \to Y$ such that $\phi = \psi \circ \rho$. It is straightforward to verify that ψ is continuous and G-equivariant.

Take any $x \in X_Q$, $y \in L/\Lambda$, and $g, h \in G$ such that $\phi(x, ghP) = \phi(y, gP)$. Then $\phi_1(y) = \phi_1(x)$, and hence $h \in \mathcal{Q}(y) = \mathcal{Q}(x) = Q$. This proves that ψ restricted to $\phi_1(X_Q) \times G/Q$ is injective and $y \in X_Q$.

Now if Y is locally compact and ϕ is surjective, then using Baire's catetory theorem for Hausdorff locally compact spaces, one can show that ϕ is an open map. This completes the proof of the theorem. $\square$

Proof of theorem 5.6. Define $\Lambda_1 = \{h \in L : \phi(gh\Lambda) = \phi(g\Lambda), \forall g \in L\}$. Then Λ_1 is a closed subgroup of L containing Λ. Since G-acts ergodically on L/Λ, by theorem 2.4, $\mathrm{Ad}(\Lambda)$ is Zariski dense in $\mathrm{Ad}(L)$. Therefore Λ_1^0 is a normal subgroup of L. Let Λ_1' be the largest subgroup of Λ_1 which is normal in L. Therefore replaning L by L/Λ_1' and Λ by Λ_1/Λ_1', without loss of generality we may assume that $\Lambda_1 = \Lambda$.

Define the equivalence relation

$$R = \{(x, y) \in L/\Lambda \times L/\Lambda : \phi(x) = \phi(y)\}$$

on L/Λ. Then R is a closed and $\Delta(G)$-invariant, where $\Delta : L \to L \times L$ denotes the diagonal imbedding of L in $L \times L$.

Let

$$\mathcal{K} = \{\tau \in \mathrm{Aff}(L/\Lambda : (z, \tau(z)) \in R \text{ and } \tau(gz) = g\tau(z) \, \forall z \in L/\Lambda \text{ and } \forall g \in L\}$$

Let $X_1 = \{x \in L/\Lambda : \overline{Gx} = L/\Lambda\}$. Then X_1 is dense in L/Λ.

Claim 5.13.1 *Let $(x, y) \in R$. If $x \in X_1$ then $y \in X_1$ and there exists $\tau \in \mathcal{K}$ such that $y = \tau(x)$ and*

$$\overline{\Delta(G)(x, y)} = \{(z, \tau(z)) : z \in L/\Lambda\} \subset R.$$

To prove the claim, we apply Ratner's theorem and obtain a closed subgroup F of $L \times L$ containing $\Delta(G)$ such that $\overline{\Delta(G) \cdot (x, y)} = F \cdot (x, y)$. Let $p_i : L \times L \to L$ denote the projection on the i-th coordinate, where $i = 1, 2$. Since $\overline{Gx} = L/\Lambda$, we have that $p_1(F) = L$. Let $N_1 = p_1(F \cap \ker(p_2))$. Then N_1 is a normal subgroup of $p_1(F) = L$ and $(N_1 z, w) \subset R$ for all $(z, w) \in F \cdot (x, y)$. Therefore $N_1 \subset \Lambda_1$, and hence by our assumption in the first paragraph we have that $N_1 = \{e\}$. Thus $F \cap \ker(p_2) = N_1 \times \{e\} = \{e\}$. In other words $F \cong p_2(F)$. Since $p_1(F) = L$, we have $\dim(p_2(F)) = \dim(L)$. Now since L is connected, we have that $p_2(F) = L$. Thus $\overline{Gy} = L/\Lambda$. Now interchanging the roles of x and y in the above argument, we conclude that $F \cap \ker(p_1) = \{e\}$. Hence there exists $\sigma \in \mathrm{Aut}(L)$ such that $F = \{(g, \sigma(g)) \in L \times L : g \in L\}$.

Thus $(gx, \sigma(g)y) \in R$ for all $g \in L$. Now for any $\delta \in L$, if $\delta x = x$, then $(gx, \sigma(g)\sigma(\delta)y) \in R$ for all $g \in L$. Let $h \in L$ such that $y = h\Lambda$. Then $\phi(\sigma(g)h\Lambda) = \phi(\sigma(g)\sigma(\delta)h)\Lambda)$ for all $g \in L$. Since $\sigma(L)h = L$, we conclude that $h^{-1}\sigma(\delta)h \in \Lambda_1$. Now since $\Lambda_1 = \Lambda$, we have that $\sigma(\delta)y = y$. Therefore the map $\tau : L/\Lambda \to L/\Lambda$, given by $\tau(gx) = \sigma(g)y$ for all $g \in L$, is well defined. It is straightforward to verify that $\tau \in \mathrm{Aff}(L/\Lambda)$. Thus, $F(x,y) = \{(z, \tau(z)) : z \in L/\Lambda\}$.

Since $\Delta(G) \subset F$, we have that $\sigma(g) = g$ for all $g \in G$, and hence $\tau(gz) = gz$ for all $g \in G$. Therefore, $\tau \in \mathcal{K}$ and the proof of the claim is complete.

Claim 5.13.2 *The group $\mathcal{K}$ is compact.*

We prove the claim as follows. Let μ_L denote the L-invariant probability measure on L/Λ. Then $\mu_L(X_1) = 1$. For any $x \in X_1$, if $y \in \overline{\mathcal{K}(x)}$ then $(x,y) \in R$, and by claim 5.13.1 there exists $\tau \in \mathcal{K}$ such that $y = \tau(x)$. Thus $\mathcal{K}(x)$ is closed for all $x \in X_1$. Therefore by Hedlund's Lemma and the ergodic decomposition of μ_L with respect to the action of $\mathcal{K}$ on L/Λ, we have that almost all $\mathcal{K}$-ergodic components are suppoted on closed $\mathcal{K}$-orbits. Thus for almost all $x \in L/\Lambda$, the orbit $\mathcal{K}x$ supports a $\mathcal{K}$-invariant probability measure.

For any $x \in L/\Lambda$, put $\mathcal{K}_x = \{\tau \in \mathcal{K} : \tau(x) = x\}$. Let $\xi : \mathcal{K}/\mathcal{K}_x \to L/\Lambda$ be the map defined by $\xi(\tau\mathcal{K}_x) = \tau(x)$ for all $\tau \in \mathcal{K}$. Then ξ is a continuous injective $\mathcal{K}$-equivariant map. Let $x \in X_1$ be such that $\mathcal{K}x$ supports a $\mathcal{K}$-invariant probability measure. Since ξ is injective, the measure can be lifted to a $\mathcal{K}$-invariant probability measure on $\mathcal{K}/\mathcal{K}_x$. Let $\tau \in \mathcal{K}_x$. Then for any $g \in G$, we have $\tau(gx) = g\tau(x) = gx$. Now since $\overline{Gx} = L/\Lambda$, we have that $\tau(y) = y$ for all $y \in L/\Lambda$. Hence $\mathcal{K}_x$ is the trivial subgroup of $\mathrm{Aff}(L/\Lambda)$. Thus $\mathcal{K}$ admits a finite haar measure. Hence $\mathcal{K}$ is a compact group and the claim is proved.

Let Ω be any neighbourhood of e in $Z_L(G)$. Put

$$R' = \{(x,y) \in R : y \notin \mathcal{K}(\Omega x)\}.$$

Let X_c be the closure of the projection of R' on the first factor of $L/\Lambda \times L/\Lambda$. Put $X_0 = (L/\Lambda) \setminus X_c$.

Claim 5.13.3 $X_1 \subset X_0$.

Suppose the claim does not hold. Then there exists a sequence $\{(x_i, y_i)\} \subset R'$ converging to $(x,y) \in R$ with $x \in X_1$. By claim 5.13.1, there exists $\tau \in \mathcal{K}$ such that $y = \tau(x)$. Therefore, after passing to a subsequence, there exists a sequence $g_i \to e$ in L such that $y_i = \tau(g_i x_i)$ for all $i \in \mathbf{N}$. By the definition of R', $g_i \notin \Omega \subset Z_L(G)$ for all $i \in \mathbf{N}$. Also $(x_i, g_i x_i) \in R$ for all $i \in \mathbf{N}$. By Ratner's theorem, there exits a $\Delta(G)$-invariant $\Delta(G)$-ergodic probability measure μ_i on $L/\Lambda/ \times L/\Lambda$ such that $\overline{\Delta(G)(x_i, g_i x_i)} = \mathrm{supp}\mu_i$. Let $h_i \to e$ be a sequence in L such that $x_i = g_i x$ for all $i \in \mathbf{N}$. By corollary 3.2 and theorem 3.1, after passing to a subsequence, we may assume that $\mu_i \to \mu$ in $\mathcal{P}(L/\Lambda \times L/\Lambda)$ as $i \to \infty$ such that $\mathrm{supp}(\mu) = F(x,x)$, where F is a closed subgroup of $L \times L$, and

$$(h_i{}^{-1}, h_i{}^{-1} g_i{}^{-1})\Delta(G)(h_i, g_i h_i) \subset F, \quad \forall i \in \mathbf{N}. \tag{5.8}$$

In particular, $F(x,x) \subset R$ and $\Delta(G) \subset F$. Since $x \in X_1$, we have that $F \supset \Delta(L)$. By an argument as in the proof of claim 5.13.1, we conclude that $F \cap \ker(p_i) = \{e\}$ for $i = 1, 2$. Therefore $F = \Delta(L)$. Hence by equation 5.8, we conclude that $g_i \in Z_L(G)$, which is a contradiction. This completes the proof of the claim, and the proof of the theorem. $\square$

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