ASYMPTOTIC EVOLUTION OF SMOOTH CURVES UNDER GEODESIC FLOW ON HYPERBOLIC MANIFOLDS

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Abstract

Extending earlier results for analytic curve segments, in this article we describe the asymptotic behavior of evolution of a finite segment of a C^n-smooth curve under the geodesic flow on the unit tangent bundle of a hyperbolic n-manifold of finite volume. In particular, we show that if the curve satisfies certain natural geometric conditions, then the pushforward of the parameter measure on the curve under the geodesic flow converges to the normalized canonical Riemannian measure on the tangent bundle in the limit. We also study the limits of geodesic evolution of shrinking segments.

We use Ratner's classification of ergodic invariant measures for unipotent flows on homogeneous spaces of $\mathrm{SO}(n, 1)$ *and an observation relating local growth properties of smooth curves and dynamics of linear* $\mathrm{SL}(2, \mathbb{R})$*-actions.*

1. Introduction

Let M be a hyperbolic n-dimensional manifold of finite volume, let $p : T^1(M) \to M$ be the unit tangent bundle over M, and let $\{g_t\}_{t\in\mathbb{R}}$ denote the geodesic flow on $T^1(M)$. Let $\pi : \mathbb{H}^n \to M$ be a locally isometric universal cover of M, and let $D\pi : T^1(\mathbb{H}^n) \to T^1(M)$ be the corresponding covering map. If $\{\tilde{g}_t\}$ denotes the geodesic flow on $T^1(\mathbb{H}^n)$, then $p(\tilde{g}_t(v)) \xrightarrow{t\to\infty} \mathrm{Vis}(v)$ for all $v \in T^1(\mathbb{H}^n)$, where $\mathrm{Vis} : T^1(\mathbb{H}^n) \to \partial\mathbb{H}^n \cong \mathbb{S}^{n-1}$ denotes the visual map. We define

$$\begin{aligned}\mathcal{S} = \{\partial\mathbb{H}^m \subset \mathbb{S}^{n-1} : \mathbb{H}^m \hookrightarrow \mathbb{H}^n \text{ is an isometric embedding} \\ \text{such that } \pi(\mathbb{H}^m) \text{ is closed in } M, \text{ where } 2 \le m \le n-1\}.\end{aligned} \tag{1}$$

Then $\mathcal{S}$ is a countable collection of proper closed subspheres of $\mathbb{S}^{n-1}$ (see [11], [12, Lemma 5.2]).

Let I be a compact interval with nonempty interior. Let $\psi : I \to T^1(M)$ be a continuous map with the following property: if $\tilde{\psi} : I \to T^1(\mathbb{H}^n)$ is any continuous

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lift of ψ under $D\pi$, then

(a) $\mathrm{Vis} \circ \tilde{\psi} \in C^n(I, \mathbb{S}^{n-1})$;

(b) the first derivative $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(s) \neq 0$ for Lebesgue a.e. $s \in I$; and

(c) for any $S \in \mathscr{S}$, $\mathrm{Vis}(\tilde{\psi}(s)) \notin S$ for Lebesgue a.e. $s \in I$.

THEOREM 1.1

Let the notation be as above. Then for any $f \in \mathrm{C_c}(T^1(M))$, we have

$$\frac{1}{|I|} \int_I f\big(g_t \psi(s)\big)\, \mathrm{d}s \xrightarrow{t \to \infty} \int_{T^1(M)} f \, \mathrm{d}\mu, \tag{2}$$

where $|I|$ denotes the Lebesgue measure of I and where μ denotes the normalized measure associated to the canonical Riemannian volume form on $T^1(M)$.

For the motivation behind this result, the reader is referred to the discussion preceding [13, Theorem 1.1], where the same conclusion is obtained for the special case of analytic curve segments ψ. That proof involves the use of "(C, α)-growth properties" (in the sense of Kleinbock and Margulis [6, (3.1)]) of finite-dimensional spaces of analytic functions. As these could not be extended to smooth functions, the analogous result could not be proved by the techniques in [13] for smooth curve segments, although the conclusion could be expected to hold in that generality, as was especially emphasized to the author by Peter Sarnak in response to [13, Theorem 1.1].

In this article, we overcome this difficulty by making a new observation of a linear dynamical nature. It implies that if we approximate an arbitrarily short piece of a C^n-curve by a polynomial curve of degree at most n, then the geodesic flow expands both the approximating curves into long curves, while still keeping them sufficiently close. This observation allows us to use the growth properties of polynomial curves of bounded degrees for linearization method (see [1], [2], [7], [12]).

On the space $C^n(I, T^1(M))$, we consider the topology of uniform convergence up to n derivatives. We now state a more robust form of Theorem 1.1.

THEOREM 1.2

Let the map ψ be as above. Then, given $f \in \mathrm{C_c}(T^1(M))$ and $\epsilon > 0$, there exists a neighborhood Ω of ψ in $C^n(I, T^1(M))$ and $T > 0$ such that

$$\Big| \frac{1}{|I|} \int_I f\big(g_t \psi_1(s)\big)\, \mathrm{d}s - \int_{T^1(M)} f \, \mathrm{d}\mu \Big| < \epsilon, \quad \forall \psi_1 \in \Omega,\ \forall t > T. \tag{3}$$

It may be noted that even for analytic maps, the above uniform version could not be proved in [13] as we do not have (C, α)-growth property for the linear span of a neighborhood of an analytic function.

1.1. Evolution of general C^n-curves

Let $\bar{\mathscr{S}}$ denote the collection of all closed totally geodesic immersed hyperbolic submanifolds of M (including M itself). Given $M_1 \in \mathscr{S}$, let $\mathscr{S}(M_1) \subset \mathscr{S} \cup \{\mathbb{S}^{n-1}\}$ be the collection of the boundaries of all possible lifts of M_1 in $\mathbb{H}^n$.

Let $\tilde{\psi} \in C^n(I, \mathbb{H}^n)$, and let $\psi = D\pi \circ \tilde{\psi}$. We define

$$I_\psi(M_1) = \bigcup_{S_1 \in \mathscr{S}(M_1)} \Big\{ s \in I : \mathrm{Vis}\big(\tilde{\psi}(s)\big) \in S_1 \setminus \bigcup_{S \subsetneq S_1,\, S \in \mathscr{S}} S \Big\}. \tag{4}$$

In particular,

$$I_\psi(M) = \big\{ s \in I : \mathrm{Vis}(\tilde{\psi}(s)) \notin S,\ S \in \mathscr{S} \big\}.$$

THEOREM 1.3

Suppose that $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(s) \neq 0$ *for almost all* $s \in I$. *Then, given any* $f \in \mathrm{C_c}(T^1(M))$,

$$\lim_{t \to \infty} \int_I f\big(a_t \psi(s)\big)\, \mathrm{d}s = \sum_{M_1 \in \bar{\mathscr{S}}} |I_\psi(M_1)| \int_{T^1(M)} f\, \mathrm{d}\mu_{M_1}, \tag{5}$$

where μ_{M_1} *is the normalized measure associated to the canonical volume form on* $T^1(M_1)$.

To prove this, we describe limiting distributions of the evolution of shrinking curves under the geodesic flow (see Section 7).

1.2. Flows on homogeneous spaces

Theorems 1.2 and 1.3 are derived from their analogues in terms of the dynamics of flows on homogeneous space of Lie groups.

Let $G = \mathrm{SO}(n, 1) = \mathrm{SO}(Q_n)$, where Q_n is a quadratic form in $(n + 1)$ real variables, defined as

$$Q_n(x_0, x_1, \ldots, x_{n-1}, x_n) = 2x_0x_n - (x_1^2 + \cdots + x_{n-1}^2). \tag{6}$$

Let Γ be a lattice in G. For $t \in \mathbb{R}$ and for $\boldsymbol{x} = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, we define

$$a_t = \begin{bmatrix} e^t & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & e^{-t} \end{bmatrix} \in G \qquad \text{and} \qquad u(\boldsymbol{x}) = \begin{bmatrix} 1 & x_1 & \cdots & x_{n-1} & \|\boldsymbol{x}\|^2/2 \\ & 1 & & & x_1 \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1} \\ & & & & 1 \end{bmatrix} \in G. \tag{7}$$

THEOREM 1.4

Let I be a compact interval with nonempty interior, and let $\varphi : I \to \mathbb{R}^{n-1}$ be a C^n-map such that $\varphi^{(1)}(s) \neq 0$ for all $s \in I$ and for any sphere or a proper affine

subspace S in $\mathbb{R}^{n-1}$, we have

$$\left|\left\{s \in I : \varphi(s) \in S\right\}\right| = 0. \tag{8}$$

Let $\varphi_k \xrightarrow{k\to\infty} \varphi$ be a convergent sequence in $C^n(I, \mathbb{R}^{n-1})$, let $x_k \to x_0$ be a convergent sequence in G/Γ, and let $t_k \to \infty$ in $\mathbb{R}$. Then for any $f \in \mathrm{C_c}(G/\Gamma)$, we have

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f\left(a_{t_k} u(\varphi_k(s)) x_k\right) \mathrm{d}s = \int_{G/\Gamma} f \, \mathrm{d}\mu_G, \tag{9}$$

where μ_G is the unique G-invariant probability measure on G/Γ.

In fact, we obtain the following more general version. Let $P^- = \{g \in G : \overline{\{a_t g a_t^{-1} : t > 0\}} \text{ is compact}\}$. Then P^- is a proper parabolic subgroup of G, and $P^-\backslash G$ naturally identifies with $\mathrm{SO}(n-1)\backslash \mathrm{SO}(n) \cong \mathbb{S}^{n-1}$. Let $\mathcal{I} : G \to P^-\backslash G \cong \mathbb{S}^{n-1}$ be the corresponding map. We note that under this identification, G acts on $\mathbb{S}^{n-1}$ by conformal transformations. For $m \geq 2$ and for $g \in G$, $\mathcal{I}(\mathrm{SO}(m,1)g)$ is a subsphere of $\mathbb{S}^{n-1}$ of dimension $m-1$. We note that $\mathcal{I}\big(N_G(\mathrm{SO}(m,1))\big) = \mathcal{I}(\mathrm{SO}(m,1))$. Let

$$\mathscr{S} = \left\{\mathcal{I}(\mathrm{SO}(m,1)g) : N_G(\mathrm{SO}(m,1))g\Gamma \text{ is closed},\ 2 \leq m \leq n-1,\ g \in G\right\}. \tag{10}$$

Then $\mathscr{S}$ is a countable collection of proper subspheres of $\mathbb{S}^{n-1}$ (see [12, Section (5.2)], [10, Corollary A]). It can be shown that

$$\mathscr{S} = \left\{\mathcal{I}(Fg) : N_G(F)g\Gamma \text{ is closed}, F \text{ semisimple}, A \subset F \neq G\right\}. \tag{11}$$

THEOREM 1.5
Let $\psi \in C(I, G)$ be such that $\mathcal{I} \circ \psi \in C^n(I, \mathbb{S}^{n-1})$, and we have

$$\left|\left\{s \in I : (\mathcal{I} \circ \psi)^{(1)}(s) = 0\right\}\right| = 0 \qquad \textit{and} \qquad \left|\left\{s \in I : \mathcal{I} \circ \psi(s) \in \bigcup_{s \in \mathscr{S}} S\right\}\right| = 0. \tag{12}$$

Then, given a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset C(I, G)$ such that

$$\mathcal{I} \circ \psi_k \xrightarrow{k\to\infty} \mathcal{I} \circ \psi \quad \textit{in } C^n(I, \mathbb{S}^{n-1}), \tag{13}$$

and sequences $t_k \to \infty$ in $\mathbb{R}$ and $x_k \to x_0 = e\Gamma$ in G/Γ, we have

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f\left(a_{t_k} \psi_k(s) x_k\right) \mathrm{d}s = \int_{G/\Gamma} f \, \mathrm{d}\mu_G, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{14}$$

From this statement, we obtain the following uniform version. Let $\ell_h : G \to G$ denote the left multiplication by $h \in G$.

THEOREM 1.6
Let $\psi : I \to G$ be a C^n-map such that $(\mathcal{I} \circ \ell_h \circ \psi)^{(1)}(s) \neq 0$ for Lebesgue a.e. $s \in I$, and for every $h \in G$,

$$\left|\left\{s \in I : (\mathcal{I} \circ \ell_h \circ \psi)(s) \in S\right\}\right| = 0, \quad \forall\, S \in \mathscr{S}. \tag{15}$$

Then, given $f \in \mathrm{C_c}(G/\Gamma)$, a compact set $\mathcal{K} \subset G/\Gamma$, and $\epsilon > 0$, there exist a neighborhood Ω of ψ in $C^n(I, G)$ and a compact set $\mathcal{C}$ in G such that

$$\left|\frac{1}{|I|}\int_I f\big(g\psi_1(s)x\big)\,\mathrm{d}s - \int_{G/\Gamma} f\,\mathrm{d}\mu_G\right| < \epsilon, \quad \forall\, \psi_1 \in \Omega,\ x \in \mathcal{K},\ g \in G \smallsetminus \mathcal{C}.$$

2. Linear dynamics and growth properties of functions

Let $V = \bigoplus_{d=1}^{\dim \mathfrak{g}} \bigwedge^d \mathfrak{g}$, and consider the $\bigoplus_{d=1}^{\dim \mathfrak{g}} \bigwedge^d \mathrm{Ad}$ representation of G on V. For $\mu \in \mathbb{R}$, we define

$$V_\mu = \{v \in V : a_t v = e^{\mu t} v,\ t \in \mathbb{R}\}. \tag{16}$$

Fix an inner product $\langle \cdot, \cdot \rangle$ on V such that the V_μ's are orthogonal. Let $\|\cdot\|$ denote the associated norm.

If $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{\dim \mathfrak{g}}\}$ is a basis of $\mathfrak{g}$ consisting of eigenvectors of $\{a_t\}$, then there is a basis of V_μ consisting of elements of the form $\boldsymbol{x}_{i_1} \wedge \cdots \wedge \boldsymbol{x}_{i_d}$. The eigenvalues of a_t on $\mathfrak{g}$ other than 1 are e^t with multiplicity $n-1$ and e^{-t} with multiplicity $n-1$. Therefore, for $t > 0$, the smallest eigenvalue of a_t on V is $e^{-(n-1)t}$ and the largest one is $e^{(n-1)t}$. Therefore,

$$V = \bigoplus_{\mu=-(n-1)}^{n-1} V_\mu. \tag{17}$$

Let $q_\mu : V \to V_\mu$ be the projection associated to this decomposition.

Notation 2.1
Let $I = [a, b]$. Let $\varphi_k \to \varphi$ be a convergent sequence in $C^n(I, \mathbb{R}^{n-1})$ such that

$$\inf_{s \in I} \|\varphi^{(1)}(s)\| > 0. \tag{18}$$

Let $M = Z_G(A) \cap \mathrm{SO}(n)$. Then $M \cong \mathrm{O}(n-1)$ and $Z_G(A) = AM$. We define the action of any $z \in Z_G(A)$ on $\mathbb{R}^{n-1}$ by the relation $u(z \cdot \boldsymbol{v}) := zu(\boldsymbol{v})z^{-1}$ for all $\boldsymbol{v} \in \mathbb{R}^{n-1}$. Then M acts on $\mathbb{R}^{n-1}$ via its identification with $\mathrm{O}(n-1)$, and $a_t \cdot \boldsymbol{v} = e^t \boldsymbol{v}$.

PROPOSITION 2.1 (Basic lemma, I)
Given $C > 0$, there exists $R_0 > 0$ such that, for any sequence $t_k \to \infty$ in $\mathbb{R}$, there exists $k_0 \in \mathbb{N}$ such that for any $x \in I$ there exists an interval $[s_k, s_k'] \subset I$ containing

x such that for any $k \geq k_0$ and for any $v \in V$, the following conditions are satisfied:

$$e^{t_k}(s_k' - s_k)^n < C, \tag{19}$$

$$\left\|a_{t_k} u(\varphi_k(s_k))v\right\| \geq \|v\|/R_0 \quad \text{if } s_k > a, \tag{20}$$

$$\left\|a_{t_k} u(\varphi_k(s_k'))v\right\| \geq \|v\|/R_0 \quad \text{if } s_k' < b. \tag{21}$$

Proof

If for every $R_0 > 0$ the above conditions are not satisfied, then after passing to a subsequence, there exist sequences $t_k \to \infty$ and $R_k \to \infty$ in $\mathbb{R}$, $[r_k, r_k'] \subset I$ with $r_k \to r_0$, $r_k' \to r_0$, and $v_k \to v_0$ in V with $\|v_0\| = 1$ such that the following holds:

$$\sup_{r_k \leq s \leq r_k'} \left\|a_{t_k} u(\varphi_k(s))v_k\right\| \leq R_k^{-1}, \tag{22}$$

$$e^{t_k}\delta_k^n \geq C, \quad \text{where } \delta_k = r_k' - r_k. \tag{23}$$

For any $k \in \mathbb{N}$, let $w_k = \varphi_k(r_k)v_k$, and let

$$\varphi_{k,r_k}(s) := \varphi_k(r_k + s) - \varphi_k(r_k), \quad \forall s \in [a - r_k, b - r_k].$$

Then

$$\sup_{s \in [0,\delta_k]} \left\|a_{t_k} u(\varphi_{k,r_k}(s))w_k\right\| \leq R_k^{-1}. \tag{24}$$

Therefore, for any $0 \leq \mu \leq n - 1$, we have

$$\sup_{s \in [0,\delta_k]} \left\|q_\mu\big(u(\varphi_{k,r_k}(s))w_k\big)\right\| \leq R_k^{-1} e^{-\mu t_k}. \tag{25}$$

Then by (23), we get

$$\sup_{s \in [0,\delta_k]} \left\|q_\mu\big(u(\varphi_{k,r_k}(s))w_k\big)\right\| \leq R_k^{-1} C^{-\mu} \delta_k^{n\mu}. \tag{26}$$

Putting $\mu = 1$ in (26), for any $v \in V_1$ with $\|v\| = 1$, we get

$$\sup_{s \in [0,\delta_k]} \left|\big\langle u(\varphi_{k,r_k}(s)w_k)v\big\rangle\right| \leq R_k^{-1} C^{-1} \delta_k^n. \tag{27}$$

We define

$$\varphi_{0,r_0}(s) = \varphi(r_0 + s) - \varphi(s), \quad \forall s \in [a - r_0, b - r_0].$$

As $k \to \infty$, we have $R_k^{-1} \to 0$, $w_k \to w_0 = u(\varphi(r_0))v_0$, and $\delta_k \to 0$. Therefore by (26),

$$q_\mu\big(u(\varphi_{0,r_0}(0))w_0\big) = q_\mu(w_0) = 0, \quad \forall\, 0 \le \mu \le n-1. \tag{28}$$

There exists an interval $J \supset [0, \delta_k]$ for all large k such that we can define

$$\psi_0(s) = \big\langle u(\varphi_{0,r_0}(s)w_0),\ v\big\rangle \qquad \text{and} \qquad \psi_k(s) = \big\langle u(\varphi_{k,r_k}(s)w_k),\ v\big\rangle, \quad \forall s \in J. \tag{29}$$

By (27), we have

$$\sup_{s\in[0,\delta_k]} |\psi_k(s)| \le C^{-1} R_k^{-1} \delta_k^n. \tag{30}$$

For each $0 \le m \le n$, we have $\psi_k^{(m)} \to \psi_0^{(m)}$ as $k \to \infty$ uniformly on J. Since $R_k^{-1} \to 0$ and $\delta_k \to 0$ as $k \to \infty$, using Taylor's formula we deduce that

$$\psi_0^{(m)}(0) = 0, \quad 0 \le m \le n, \tag{31}$$

and hence $\lim_{s\to 0} \psi_0(s)/s^n = 0$. In other words,

$$\lim_{s\to 0} \big\| q_1\big(u(\varphi_{0,r_0}(s))w_0\big)\big\| / s^n = \lim_{s\to 0} \psi_0(s)/s^n = 0. \tag{32}$$

Next, using finite-dimensional representations of $\mathrm{SL}(2,\mathbb{R})$, we show that (32) leads to a contradiction.

In view of (6), let $H = \mathrm{SO}(Q_2) = \mathrm{SO}(2,1) \hookrightarrow \mathrm{SO}(n,1)$. Then H is generated by $\{u(se_1)\}_{s\in\mathbb{R}}$, by $A = \{a_t\}_{t\in\mathbb{R}}$, and by $\{{}^{\mathrm{t}}u(te_1)\}_{t\in\mathbb{R}}$, where $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$. We realize H as the image of $\mathrm{SL}(2,\mathbb{R})$ under the adjoint representation on its Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ such that $\mathrm{diag}(e^t, e^{-t}) \in \mathrm{SL}(2,\mathbb{R})$ maps to $a_{2t} \in H$.

Let $\mathcal{W}$ be a finite collection of irreducible H-submodules of V such that

$$V = \bigoplus_{W\in\mathcal{W}} W. \tag{33}$$

For any $W \in \mathcal{W}$, let $P_W : V \to W$ denote the projection with respect to the decomposition (33). In view of Notation 2.1, for any $s \in [a - r_0, b - r_0]$, there exists $\theta(s) \in M \subset \mathrm{O}(n-1)$ such that

$$\theta(s) \cdot \varphi_{0,r_0}(s) = \big\|\varphi_{0,r_0}(s)\big\| e_1. \tag{34}$$

Let

$$\mu_0 = \max\big\{\mu : q_\mu(w_0) \ne 0\big\}. \tag{35}$$

Then $\mu_0 = \max\{\mu : q_\mu(zw_0) \neq 0\}$ for any $z \in M$. Let $W_0 \in \mathcal{W}$ be such that

$$P_{W_0}\big(q_{\mu_0}(\theta(0) \cdot w_0)\big) \neq 0. \tag{36}$$

Therefore, there exists $a_1 < b_1$ such that $0 \in [a_1, b_1] \subset [a - r_0, b - r_0]$ and

$$\eta_0 := \inf_{s \in [a_1, b_1]} \big\| P_{W_0}\big(q_{\mu_0}(\theta(s) \cdot w_0)\big) \big\| > 0. \tag{37}$$

By (17) and (28), we have $-1 \geq \mu_0 \geq -(n-1)$. Recall that $\varphi_{0,r_0}(0) = 0$, and recall that, by (18), there exists $\rho_0 > 0$ such that $|\varphi^{(1)}(r)| \geq \rho_0$ for all $r \in I$. Let $s \in [a_1, b_1]$, and let

$$h = \|\varphi_{0,r_0}(s)\| = \|\varphi^{(1)}(r)\| s \geq \rho_0 s \quad \text{for some } r \in [a_1, b_1]. \tag{38}$$

Then

$$\begin{aligned} \theta(s) q_1\big(u(\varphi_{0,r_0}(s))w_0\big) &= q_1\big(\theta(s) u(\varphi_{0,r_0}(s))w_0\big) \\ &= q_1\big(u(he_1)\theta(s)w_0\big). \end{aligned} \tag{39}$$

By the standard description of an irreducible representation of $\mathrm{SL}(2, \mathbb{R})$, we have

$$\begin{aligned} P_{W_0}\big(q_1(u(he_1)\theta(s)w_0)\big) &= q_1\big(u(he_1) P_{W_0}(\theta(s)w_0)\big) \\ &= h^{1-\mu_0} q_{\mu_0}\big(P_{W_0}(\theta(s)w_0)\big) \\ &\quad + \sum_{\mu \leq \mu_0 - 1} h^{1-\mu} q_\mu\big(P_{W_0}(\theta(s)w_0)\big). \end{aligned} \tag{40}$$

Since P_{W_0} is norm decreasing and since $\theta(s) \in \mathrm{O}(n-1)$, by (37) and (38) we conclude that

$$\lim_{s \to 0} \big\| q_1\big(u(\varphi_{0,r_0}(s))w_0\big) \big\| / s^{1-\mu_0} \geq \eta_0 \rho_0^{1-\mu_0} > 0. \tag{41}$$

Since $0 < 1 - \mu_0 \leq n$, this contradicts (32). □

Notation 2.2
For any $x \in I = [a, b]$, we define

$$P_{k,x}(s) = \varphi_k(x) + \sum_{i=1}^{n} \varphi_k^{(i)}(x) s^i / i!, \quad \forall s \in \mathbb{R}. \tag{42}$$

COROLLARY 2.2
Let $R_0 > 0$ be as in Proposition 2.1 for $C = 1$. Then, given a sequence $t_k \to \infty$ and a constant $c > 0$, there exists $k_1 \in \mathbb{N}$ such that for any $k \geq k_1$ and for any $x \in I$,

there exist $s_k, s_k' \in I$ with $x \in [s_k, s_k']$ such that for any $v \in V$, we have

$$e^{t_k}(s_k' - s_k)^n < 1, \tag{43}$$

$$\left\| a_{t_k} u(\varphi_k(s))v - a_{t_k} u(P_{k,x}(s))v \right\| \le c \left\| a_{t_k} u(\varphi_k(s))v \right\|, \quad \forall s \in [s_k, s_k'], \tag{44}$$

$$\left\| a_{t_k} u(\varphi_k(s_k))v \right\| \ge \|v\|/R_0 \quad \text{if } s_k > a, \tag{45}$$

$$\left\| a_{t_k} u(\varphi_k(s_k'))v \right\| \ge \|v\|/R_0 \quad \text{if } s_k' < b. \tag{46}$$

Proof

Given a sequence $t_k \to \infty$, let $k_0 \in \mathbb{N}$ be as in Proposition 2.1. Let $x \in I$. By Proposition 2.1, for any $k \ge k_0$ there exists a subinterval $J_k = [s_k, s_k']$ containing x such that

$$|J_k|^n \le e^{-t_k}, \tag{47}$$

and (45) and (46) hold for all $v \in V$.

For the operator norm $\|\cdot\|_V$, let $\delta > 0$ be such that

$$\left\| u(y_1)u(y_2)^{-1} - I \right\|_V \le c, \quad \forall\, |y_1 - y_2| \le \delta,\ y_1, y_2 \in \mathbb{R}^{n-1}. \tag{48}$$

By (47) and equicontinuity of the family $\{\varphi_k^{(n)}\}$, there exists $k_1 \ge k_0$ such that

$$\|\varphi_k^{(n)}(x_1) - \varphi_k^{(n)}(x_2)\| \le \delta, \quad \forall x_1, x_2 \in J_k,\ \forall k \ge k_1. \tag{49}$$

Let $k \ge k_1$, and let $s \in J_k$. By Taylor's formula and (49), we have

$$|\varphi_k(s) - P_{k,x}(s)| \le \delta |J_k|^n \le \delta e^{-t_k}. \tag{50}$$

By (48) and (50), for all $s \in J_k$,

$$\begin{aligned} &\left\| a_{t_k} u(P_{k,x}(s))v - a_{t_k} u(\varphi_k(s))v \right\| \\ &\quad \le \left\| u\big(e^{t_k}(P_{k,x}(s) - \varphi_k(x))\big) - I \right\|_V \left\| a_{t_k} u(\varphi_k(s))v \right\| \le c \left\| a_{t_k} u(\varphi_k(s))v \right\|. \qquad \square \end{aligned}$$

Notation 2.3

Let $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$. Then (by Notation 2.1) there exists a continuous map $z : I \to Z_G(A)$ such that

$$z(s) \cdot \varphi^{(1)}(s) = e_1, \quad \forall s \in I. \tag{51}$$

Let $R_1 := \sup_{s \in I} \|z(s)\|_V$.

PROPOSITION 2.3
Let $R_0 > 0$ be as in Proposition 2.1 for $C = 1$. Let $\mathcal{A}$ be a linear subspace of V, and let $\mathcal{C}$ be a compact subset of $\mathcal{A}$. Then, given $\epsilon > 0$, there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that the following holds: given any neighborhood Φ of $\mathcal{D}$ in V, there exist a neighborhood Ψ of $\mathcal{C}$ and $k_2 \in \mathbb{N}$ such that, for any $v \in V$ with

$$\|v\| \geq R_0 R_1 \sup\left\{\|w\| : w \in \Phi\right\}, \tag{52}$$

a subinterval $J \subset I$, and any $k \geq k_2$ with $e^{-t_k} < |J|^n$, if we define

$$E = \left\{s \in J : z(s) a_{t_k} u(\varphi_k(s)) v \in \Psi\right\}, \tag{53}$$

$$F = \left\{s \in J : z(s) a_{t_k} u(\varphi_k(s)) v \in \Phi\right\}, \tag{54}$$

then for any maximal interval F_1 contained in F, we have

$$|F_1 \cap E| \leq \epsilon |F_1|. \tag{55}$$

Note that if $\mathcal{A} = 0$, we can choose Φ and Ψ to be balls centered at zero.

Proof
There exists $n_1 \in \mathbb{N}$ such that if $P : \mathbb{R} \to \mathbb{R}^{n-1}$ is a polynomial map of degree at most n and if $v \in V$, then $s \mapsto u(P(s))v$ is a polynomial map of degree at most n_1.

As in [2, Proposition 4.2], there exists a compact set $\mathcal{D} \subset \mathcal{A}$ containing $\mathcal{C}$ such that, given an open neighborhood Φ_1 of $\mathcal{D}$ in V, there exists an open neighborhood Ψ_1 of $\mathcal{C}$ in V contained in Φ_1 such that for any polynomial map $\zeta : \mathbb{R} \to V$ of degree at most n_1 and for any bounded interval $J \subset \mathbb{R}$, if $\overline{\zeta(J)} \not\subset \Phi_1$, then

$$\left|\left\{s \in J : \zeta(s) \in \Psi_1\right\}\right| \leq \epsilon \left|\left\{s \in J : \zeta(s) \in \Phi_1\right\}\right|. \tag{56}$$

Given a bounded open neighborhood Φ of $\mathcal{D}$ in V, choose an open neighborhood Φ_1 of $\mathcal{D}$ in Φ and $\delta_1 > 0$ such that Φ contains the $2\delta_1$-tubular neighborhood of $\overline{\Phi_1}$. We obtain an open neighborhood Ψ_1 of $\mathcal{C}$ contained in Φ_1 so that (56) holds. Let Ψ be a neighborhood of $\mathcal{C}$, and let $0 < \delta < \delta_1$ be such that Ψ_1 contains the 2δ-tubular neighborhood of $\overline{\Psi}$.

Let $R = \sup_{w \in \Phi} \|w\|$. For $c = \delta/(R_1 R) > 0$, let $k_1 \in \mathbb{N}$ be as in the Corollary 2.2. Let $k_2 \geq k_1$ be such that

$$\|z(r) - z(r')\|_V \leq \delta/R, \quad \forall |r' - r| \leq e^{-t_k},\ r, r' \in I,\ k \geq k_2. \tag{57}$$

Let J, k, and F_1 be as in the statement of Proposition 2.3. To show that (55) holds, suppose that $x \in F_1 \cap E$. By Corollary 2.2 there exists $J_k = [s_k, s_k'] \subset I$ containing x such that (43) – (46) hold. Then by (45) and (46), by definitions of R_1, R, and δ, and by

(57), we have

$$F_1 \cap E \subset \big\{ s \in J_k \cap F_1 : z(s_k) a_{t_k} u(P_{k,x}(s)) v \in \Psi_1 \big\}. \tag{58}$$

Since $e^{t_k}|J_k|^n < 1 < e^{t_k}|J|^n$ and since $x \in J \cap J_k$, we have $\{s_k, s_k'\} \cap J \setminus \{a, b\} \neq \emptyset$. Therefore by (45) and (46), and by the maximality of the interval F_1, we have

$$\overline{\Phi} \supset \overline{a_{t_k} u\big(\varphi_k(J_k \cap F_1)\big) v} \not\subset \Phi. \tag{59}$$

Hence by (44) and the choices of c and R_1,

$$\overline{z(s_k) a_{t_k} u\big(P_{k,x}(J_k \cap F_1)\big) v} \not\subset \Phi_1. \tag{60}$$

Since $z \mapsto \zeta(s) := z(s_k) a_{t_k} u(P_{k,x}(s)) v$ is a polynomial map of degree at most n_1, we apply (56) to the interval $J_k \cap F_1$ in place of J. Then in view of (53), (58), and (60), we deduce that

$$|F_1 \cap E| \leq \big|\big\{ s \in J_k \cap F_1 : z(s_k) a_{t_k} u(P_{k,x}(s)) v \in \Psi_1 \big\}\big| \leq \epsilon |F_1|. \tag{61}$$

□

3. Limiting measure and invariance under unipotent flow

Let $\varphi_k \to \varphi$ be a convergent sequence in $C^n(I, \mathbb{R}^{n-1})$ as in Notation 2.1. Let $z : I \to Z_G(A)$ be the continuous function as in Notation 2.3 such that $z \cdot \varphi^{(1)}(s) = e_1$ for all $s \in I$. Let $g_k \to g_0$ be a convergent sequence in G. Then $x_k = g_k\Gamma \to x_0 = g_0\Gamma$ in G/Γ.

PROPOSITION 3.1

Given that $\epsilon > 0$, there exists a compact set $\mathcal{K} \subset G/\Gamma$ such that for any sequence $t_k \to \infty$, we have

$$\frac{1}{|I|}\big|\big\{ s \in I : z(s) a_{t_k} u(\varphi_k(s)) x_i \in \mathcal{K} \big\}\big| \geq 1 - \epsilon \quad \textit{for all large } k \in \mathbb{N}. \tag{62}$$

Proof

Let N be a maximal unipotent subgroup of G such that $N/(N \cap \Gamma)$ is compact. Let $\mathfrak{n}$ denote the Lie algebra of N. Fix $p_N \in V \setminus \{0\}$ such that $p_N \in \bigwedge^{\dim \mathfrak{n}} \mathfrak{n}$. Then Γp_N is discrete (see [1]). Let

$$r_3 = \inf_{\gamma \in \Gamma,\, k \in \mathbb{N}} \|g_k \gamma p_N\| > 0. \tag{63}$$

By Proposition 2.3 applied to $\mathcal{A} = \{0\}$, and given that $0 < R \leq r_3/R_0R_1$, there exists $r > 0$ such that the following holds: given any sequence $t_k \to \infty$, there exists $k_2 \in \mathbb{N}$

such that, for any interval $J \subset I$, $k \geq k_2$ with $e^{-t_k} < |J|^n$, and $\gamma \in \Gamma$, we have

$$\begin{aligned} &\left|\left\{s \in J : \|z(s)a_{t_k}u(\varphi_k(s))g_k\gamma p_N\| < r\right\}\right| \\ &\qquad \leq \epsilon \cdot \left|\left\{s \in J : \|z(s)a_{t_k}u(\varphi(s))g_k\gamma p_N\| < R\right\}\right|. \end{aligned} \tag{64}$$

In view of this inequality, Dani's nondivergence criterion [1] for homogeneous spaces of rank one semisimple groups implies the existence of a compact set $\mathcal{K}$ such that (62) holds. (To determine $\mathcal{K}$ one needs to consider finitely many choices of N as above, each corresponding to a distinct cusp in the Siegel fundamental domain for Γ; see [4]). □

Fix a sequence $t_k \to \infty$ in $\mathbb{R}$. Let λ_k be the probability measure on G/Γ defined by

$$\int_{G/\Gamma} f \, \mathrm{d}\lambda_k = \frac{1}{|I|}\int_I f\left(z(s)a_{t_k}u(\varphi_i(s))x_k\right) \mathrm{d}s, \quad \forall f \in \mathrm{C_c}(G/\Gamma). \tag{65}$$

Then Proposition 3.1 implies the following.

THEOREM 3.2
After passing to a subsequence, $\lambda_k \overset{k\to\infty}{\longrightarrow} \lambda$ in the space of probability measures on G/Γ with respect to the weak- topology.*

THEOREM 3.3
The limit measure λ is invariant under the action of $W := \{u(re_1) : r \in \mathbb{R}\}$.

Proof
The proof follows from the same argument as in the proof of [13, Theorem 3.1]. □

The next result says that the limit measure is null on the parabolic cylinders embedded in the cusps. To prove this, we use an idea from the proof of [10, Lemma 2.1].

PROPOSITION 3.4
Let U be any maximal unipotent subgroup of G containing W and $x \in G$ such that Ux is compact. Then $\lambda(N_G(U)x) = 0$.

Proof
Since G is a rank one group, W is contained in a unique maximal unipotent subgroup (see [8, Section 12.17]). Therefore, $U = U^+$, the expanding horospherical subgroup for $\{a_t : t > 0\}$, and $N_G(U) = Z_G(A)U^+$. Let C be any compact subset of $N_G(U^+)$. Then

$$C_1 := \overline{\{a_{-t}Ca_t : t > 0\}}$$

is compact. Given that $\epsilon > 0$, let $\mathcal{K}$ be as in Proposition 3.1. Let $\mathcal{K}_1 = C_1^{-1}\mathcal{K}$.

Since Ux is compact, there exists $u \in U \setminus \{e\}$ such that $ux = x$. Then $a_{-t}ua_t \to e$ as $i \to \infty$. Therefore, by [8, Section 1.12], $a_{-t}x \notin \mathcal{K}_1$ for all $t \geq T_0$ for some $T_0 > 0$. Therefore, $Cx \cap a_{T_0}\mathcal{K} = \emptyset$.

Let $t_k' = t_k - T_0$. Then by Proposition 3.1, for all large $k \in \mathbb{N}$, we have

$$\left|\left\{s \in I : a_{t_k'}u(\varphi_k(s))x_k \in \mathcal{K}\right\}\right| \geq (1-\epsilon)|I|,$$

and hence $|\{s \in I : a_{t_k}u(\varphi_k(s))x_k \in a_{T_0}\mathcal{K}\}| \geq (1-\epsilon)|I|$. Since $(G/\Gamma) \setminus a_{T_0}\mathcal{K}$ is a neighborhood of Cx, we conclude that $\lambda(Cx) \leq \epsilon$. □

4. Ratner's theorem and linearization method

Since the limit measure λ is invariant under a nontrivial unipotent one-parameter subgroup $W = u(\mathbb{R}e_1)$, we want to use Ratner's theorem describing the W-ergodic components of λ. We use linearization techniques in combination with the linear dynamical results proved in Section 2 to derive the following result.

THEOREM 4.1

Let the measure λ be as in Theorem 3.3. Suppose further that the limit function φ satisfies the following condition: for any $(n-2)$-sphere or a proper affine subspace S_1 contained in R^{n-1}, we have

$$\left|\left\{s \in I : \varphi(s) \in S_1\right\}\right| = 0. \tag{66}$$

Then measure λ is G-invariant.

4.1. Positive limit measure on singular sets

Let $\mathcal{H}$ be the collection of all closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H and a nontrivial unipotent one-parameter subgroup of H acts ergodically on $H/H \cap \Gamma$. Then $\mathcal{H}$ is countable (see [12, Section 5.1], [9, Theorem 1.1]). For $H \in \mathcal{H}$ and a nontrivial one-parameter unipotent subgroup W, we define

$$N(H, W) = \{g \in G : W \subset gHg^{-1}\}, \tag{67}$$

$$S(H, W) = \bigcup_{\substack{H' \subset H,\ \dim H' < \dim H \\ H' \in \mathcal{H}}} N(H', W). \tag{68}$$

Suppose that λ is not G-invariant. Since λ is W-invariant and since $\mathcal{H}$ is countable, by Ratner's theorem [9, Theorem 1] there exists $H \in \mathcal{H}$ such that $\dim H < \dim G$ and

$$\lambda\big(\pi(N(H, W))\big) > 0 \qquad \text{and} \qquad \lambda\big(\pi(S(H, W))\big) = 0, \tag{69}$$

where $\pi : G \to G/\Gamma$ is the natural quotient map.

4.2. Algebraic consequence of positive limit measure on singular sets
Since G is a semisimple group of real rank one and since $H \in \mathcal{H}$, if H is not reductive, then H is contained in a unique maximal unipotent subgroup intersecting Γ in a cocompact lattice. Hence for any $g \in N(H, W)$, we have that gHg^{-1} is contained in a maximal unipotent subgroup U of G containing W such that $U\pi(g)$ is compact, and $\pi(N(H, W)) \subset N_G(U)\pi(g)$. By Proposition 3.4, we have $\lambda\big(\pi(N(H, W))\big) = 0$. Thus in view of (69), we conclude that H is a reductive subgroup of G.

Let H^{nc} denote the subgroup of H generated by all one-parameter unipotent subgroups of G contained in it. Since H is a proper reductive subgroup of $G = \mathrm{SO}(n, 1)$, we have $H^{\mathrm{nc}} \cong \mathrm{SO}(k, 1)$ for some $2 \le k \le n-1$, and we have $H = Z_1 H^{\mathrm{nc}}$, where Z_1 is a compact normal subgroup of H. Let $\mathfrak{h}^{\mathrm{nc}}$ denote the Lie algebra associated to H^{nc}, and let $p_0 := p_{H^{\mathrm{nc}}} \in \bigwedge^{\dim H^{\mathrm{nc}}} \mathfrak{h}^{\mathrm{nc}} \smallsetminus \{0\}$. Then the orbit Gp_0 is closed and the orbit Γp_0 is discrete (see [13, Proposition 4.3, Corollary 4.4]).

Let $w_0 \in \mathrm{Lie}(W)$ such that $\mathbb{R}w_0 = \mathrm{Lie}(W)$. Let

$$\mathcal{A} := \{v \in V : v \wedge w_0 = 0\}. \qquad \text{Then } N(H, W) = \{g \in G : gp_0 \in \mathcal{A}\} \tag{70}$$

(see [13, Remark 4.1]).

By (69), there exists a compact set $C \subset N(H, W)$ such that $\pi(C) \subset \pi(N(H, W)) \smallsetminus \pi(S(H, W))$ and such that $\lambda(\pi(C)) > 0$. Put

$$\epsilon = \frac{\lambda(\pi(C))}{4} > 0. \tag{71}$$

Let $\mathcal{C} = Cp_0 \cup -C \cdot p_0 \subset \mathcal{A}$. Let $\mathcal{D} \subset \mathcal{A}$ be a compact set as in the conclusion of Proposition 2.3. We replace $\mathcal{D}$ by $\mathcal{D} \cup -\mathcal{D}$, and we define

$$S(\mathcal{D}) = \big\{g \in N(H, W) : g\gamma p_0 \in \mathcal{D} \text{ for some } \gamma \in \Gamma \setminus N_G(H^{\mathrm{nc}})\big\}. \tag{72}$$

By [13, Proposition 4.5], $S(\mathcal{D}) \subset S(H, W)$ and $\pi(S(\mathcal{D}))$ is closed in G/Γ. We choose a compact neighborhood $\mathcal{K}$ of $\pi(C)$ contained in $G/\Gamma \smallsetminus \pi(S(\mathcal{D}))$. By [13, Proposition 4.5], there exists bounded a symmetric open neighborhood Φ of $\mathcal{D}$ in V such that for any $g \in G$ and for $\gamma_1, \gamma_2 \in \Gamma$, we have the following:

$$\text{if } g\gamma_1 p_0, g\gamma_2 p_0 \in \Phi \text{ and if } \pi(g) \in \mathcal{K}, \text{ then } \gamma_1 p_0 = \pm\gamma_2 p_0. \tag{73}$$

Let R_0 and R_1 be as in Proposition 2.3, and let

$$\Sigma = \big\{v \in \Gamma p_0 : \|v\| \ge R_0 R_1 \sup\{\|w\| : w \in \Phi\}\big\}. \tag{74}$$

Then there exists k_2 and a symmetric open neighborhood Ψ of $\mathcal{C}$ in V such that the conclusion of Proposition 2.3 holds. Let $k \ge k_2$, and let $\psi(s) = z(s)a_{t_k}u(\varphi_k(s))g_k$ for all $s \in I$. Then in view of (73), by linearization technique [13, Proposition 4.7], and

by Proposition 2.3 (a crucial step), we have

$$\left|\left\{s \in I : \psi(s)\Sigma \cap \Psi \neq \emptyset,\ \pi(\psi(s)) \in \mathcal{K}\right\}\right| \leq 2\epsilon|I|. \tag{75}$$

Now $\mathcal{O} = \{\pi(g) : gp_0 \in \Psi, \pi(g) \in \mathcal{K}\}$ is a neighborhood of $\pi(C)$. Let

$$E(k) = \left\{s \in I : \pi(z(s)a_{t_k}u(\varphi_k(s))g_k) \in \mathcal{O}\right\}, \quad \forall k \in \mathbb{N}. \tag{76}$$

Since $\lambda_k \to \lambda$, by (71) we have

$$|E(k)| > 3\epsilon|I|, \quad \forall \text{ large } k \in \mathbb{N}. \tag{77}$$

Let $\Sigma_1 = \Gamma p_0 \setminus \Sigma$. By (75) and (77), we have

$$\left|\left\{s \in E(k) : \psi(s)\Sigma_1 \cap \Psi \neq \emptyset\right\}\right| \geq \epsilon|I| \quad \text{for all large } k \in \mathbb{N}. \tag{78}$$

PROPOSITION 4.2
There exists $v \in \Sigma_1 = \Gamma p_0 \setminus \Sigma$ such that

$$\left|\left\{s \in I : u(\varphi(s))g_0 v \in V^- + V^0\right\}\right| \geq \epsilon|I|/\#(\Sigma_1) > 0. \tag{79}$$

Proof
Since Γp_0 is discrete, Σ_1 is finite. By (78), after passing to a subsequence, there exists $v \in \Sigma_1$ such that

$$|E_v(k)| \geq \epsilon|I|/\#(\Sigma_1), \quad \forall \text{ large } k \in \mathbb{N}, \tag{80}$$

where $E_v(k) = \{s \in E(k) : z(s)a_{t_k}u(\varphi_k(s))g_k v \in \Psi\}$. In view of (16), let $V^- = \sum_{\mu<0} V_\mu$, let $V^0 = V_0$, and let $V^+ = \sum_{\mu>0} V_\mu$. Then $V = V^- \oplus V^0 \oplus V^+$. Let $q_+ : V \to V^+$ denote the corresponding projection. Let

$$E^\delta = \left\{s \in I : q_+(u(\varphi(s))g_0 v) \geq \delta\right\}. \tag{81}$$

Since $\varphi_k \to \varphi$ uniformly on I, there exists $k_3 \in \mathbb{N}$ such that if $k \geq k_3$ and if $s \in E^\delta$, then $q_+\big(z(s)u(\varphi_k(s))g_k v\big) \geq \delta/2$, and hence

$$z(s)a_{t_k}u\big(\varphi_k(s)\big)g_k v = a_{t_k}\big(z(s)u(\varphi(s))g_k v\big) \notin \Psi. \tag{82}$$

Therefore, $E_v(k) \cap E^\delta = \emptyset$. By (80) and (81), we have

$$\left|\left\{s \in I : u(\varphi(s))g_0 v \in V^- + V^0\right\}\right| = \lim_{\delta\to 0}|I \setminus E^\delta| \geq \epsilon|I|/\#(\Sigma_1). \qquad \square$$

Proof of Theorem 4.1
Since $\dim H < \dim G$, by [13, Proposition 4.9] the set

$$S_{g_0 v} := \left\{x \in \mathbb{R}^{n-1} : u(x) g_0 v \in V^0 + V^-\right\}$$

is contained in a proper affine subspace of $\mathbb{R}^{n-1}$ or in a sphere in $\mathbb{R}^{n-1}$. By Proposition 4.2,

$$\left|\left\{s \in I : \varphi(s) \in S_{g_0 v}\right\}\right| \geq \epsilon |I| / \#(\Sigma_1). \tag{83}$$

The two preceding statements contradict our assumption on φ as stated in (66). Therefore, (69) fails to hold. Hence λ is G-invariant. This completes the proof of Theorem 4.1. □

Remark 4.1
If $g_0 = e$, then in (83) we have $S_{g_0 v} = S_{\gamma p_0}$ for some $\gamma \in \Gamma$. By [13, Proposition 4.9], there exist $h \in G$ and a noncompact simple subgroup F containing A such that

$$S_{\gamma p_0} = \left\{x \in \mathbb{R}^{n-1} : u(x) \in P^- F h\right\}$$

and $h^{-1} F h(\gamma p_0) = \gamma p_0$. Therefore, $N_G(F) h \Gamma = h \gamma N_G(H^{\mathrm{nc}}) \Gamma$ is closed. Now $P^- N_G(F) h = P^- F h$. Hence by (11), $\mathcal{I}(Fh) \in \mathscr{S}$. Hence $S_{\gamma p_0} \in \mathcal{S}^{-1}(\mathscr{S})$. In other words, if $g_k \to e$, then in the statement of Theorem 4.1 it is enough to assume that (66) holds for all $S_1 \in \mathcal{S}^{-1}(\mathscr{S})$.

5. Deduction of the main results

The following observation allows us to deduce the results stated in the introduction from Theorem 4.1.

Let $U^- = \{h \in G : a_{t_k} h a_{t_k}^{-1} \overset{k\to\infty}{\longrightarrow} \to e\}$. Then $P^- = U^- Z_G(A)$.

PROPOSITION 5.1
Let $\{\theta_k\}$ and $\{\psi_k\}$ be uniformly convergent sequences of continuous maps from $I \to G$ such that $P^- \theta_k(s) = P^- \psi_k(s)$ for all $s \in I$. Let $\{x_k\}$ be a sequence in G/Γ, and let $t_k \to \infty$ be a sequence in $\mathbb{R}$. Suppose that there exists a probability measure μ on G/Γ which is $Z_G(A)$-invariant, and suppose that for any subinterval $J \subset I$ with nonempty interior and for any $f \in \mathrm{C_c}(G/\Gamma)$, the following holds:

$$\lim_{k\to\infty} \frac{1}{|J|} \int_J f\left(a_{t_k} \theta_k(s) x_k\right) \mathrm{d}s = \int_{G/\Gamma} f \, \mathrm{d}\mu. \tag{84}$$

Then for any $f \in \mathrm{C_c}(G/\Gamma)$, we have

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f\big(a_{t_k}\psi_k(s)x_k\big)\,\mathrm{d}s = \int_{G/\Gamma} f\,\mathrm{d}\mu. \tag{85}$$

Proof
The statement can be proved using uniform continuity of f as in [13, proof of Proposition 4.11]. □

Proof of Theorem 1.4
Let $\theta_k(s) = \zeta(s)u(\varphi_k(s))$, and let $\psi_k(s) = u(\varphi_k(s))$ for all $s \in I$. Then by Theorem 4.1, (84) holds for $\mu = \mu_G$. Therefore by Proposition 5.1, we have (85), which is the same as (9). □

Proof of Theorem 1.5
Due to the regularity of Lebesgue measure, it is enough to prove the theorem under the assumption that

$$(\mathcal{I} \circ \psi)^{(1)} \neq 0, \quad \forall s \in I. \tag{86}$$

The map $\mathcal{S} : \mathbb{R}^{n-1} \to \mathbb{S}^{n-1}$ defined by $\mathcal{S}(x) = \mathcal{I}(u(x))$ is the inverse stereographic projection. Without loss of generality, we may therefore assume that there exists a sequence $\varphi_k \to \varphi$ in $C^n(I, \mathbb{R}^{n-1})$ such that $\mathcal{I}(\psi_k(s)) = \mathcal{I}(u(\varphi_k(s))$ and $\mathcal{I}(\psi(s)) = \mathcal{I}(u(\varphi(s))$ for all $s \in I$. Then by (86) and (12), we have

$$\varphi^{(1)}(s) \neq 0 \quad (\forall\, s \in I) \qquad \text{and} \qquad \big|\big\{s \in I : \varphi(s) \notin \mathcal{S}^{-1}(S)\big\}\big| = 0 \quad (\forall\, S \in \mathscr{S}).$$

Therefore, by Remark 4.1, the conclusion of Theorem 4.1 holds in the case of $x_k \to x_0 = e\Gamma$. Therefore, since $P^-\psi_k(s) = P^-u(\varphi_k(s))$ for all $s \in I$, (14) follows from Proposition 5.1. □

Proof of Theorem 1.6
If the result fails to hold, then there exist $f \in \mathrm{C_c}(G/\Gamma)$, $\epsilon > 0$, a sequence $x_k \to x$ in G/Γ, a sequence $\{\psi_k\}$ of functions from $I \to G$ such that $\mathcal{I} \circ \psi_k \mapsto \psi$ in $C^n(I, \mathbb{S}^{n-1})$, and an unbounded sequence $g_k \to \infty$ such that

$$\Big|\frac{1}{|I|} \int_I f\big(g_k\psi_k(s)x_k\big)\,\mathrm{d}s - \int_I f\,\mathrm{d}\mu_G\Big| \geq \epsilon. \tag{87}$$

Since $G = KA^+K$, by passing to a subsequence, for each $k \in \mathbb{N}$ we have $g_k = h'_k a_{t_k} h_k$, where $h_k \to h$ and $h'_k \to h'$ in K as $k \to \infty$, and where $t_k \to \infty$ in $\mathbb{R}$. Let $\tilde{x} \in G$ and $\tilde{x}_k \in G$ be such that $x_k = \tilde{x}_k\Gamma$, $x = \tilde{x}\Gamma$, and $\tilde{x}_k \to \tilde{x}$ as $k \to \infty$.

Let $\bar{\psi}(s) = h\psi(s)\tilde{x}$, and let $\bar{\psi}_k(s) = h_k\psi_k(s)\tilde{x}_k$ for all $s \in I$ and $k \in \mathbb{N}$. Then the condition of Theorem 1.6 is satisfied for $\bar{\psi}$ in place of ψ and for $\bar{\psi}_k$ in place of ψ_k; note that we have used a stronger condition on ψ so that (15) holds for all proper subspheres S of $\mathbb{S}^{n-1}$ and for $h \in G$. Therefore,

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f\big(h' a_{t_k} \bar{\psi}_k(s)\Gamma\big)\,\mathrm{d}s = \int_{G/\Gamma} f(h'y)\,\mathrm{d}\mu_G(y) = \int_{G/\Gamma} f\,\mathrm{d}\mu_G. \tag{88}$$

Since $h'_k \to h'$ and since f is uniformly continuous, this equality contradicts (87). □

Proof of Theorem 1.2

As in the proof of Theorem 1.6, we need to show that, given sequences $\psi_k \xrightarrow{k\to\infty} \psi$ in $C^n(I, T^1(M))$ and $t_k \xrightarrow{k\to\infty} \infty$ in $\mathbb{R}$,

$$\lim_{k\to\infty} \frac{1}{|I|} \int_I f\big(g_{t_k}\psi_k(s)\big)\,\mathrm{d}s = \int_{T^1(M)} f\,\mathrm{d}\mu, \quad \forall f \in \mathrm{C_c}\big(T^1(M)\big). \tag{89}$$

We deduce this statement from Theorem 1.5.

There exists a lattice Γ in $G = \mathrm{SO}(n, 1)$ such that $T^1(M) \cong \mathrm{SO}(n-1)\backslash G/\Gamma$ and $T^1(\mathbb{H}^n) \cong \mathrm{SO}(n-1)\backslash G$. Moreover, the geodesic flow $\{g_t\}$ on $T^1(M)$ corresponds to the translation action of $\{a_t\}$ on $\mathrm{SO}(n-1)\backslash G/\Gamma$ from the left; the action is well defined because $\mathrm{SO}(n-1) \subset Z_G(\{a_t\})$. Now the maps $\mathrm{Vis} : T^1(\mathbb{H}^n) \to \mathbb{S}^{n-1}$ and $\bar{\mathcal{I}} : \mathrm{SO}(n-1)\backslash G \to \mathbb{S}^{n-1}$ are the same under the above identifications. Also, the sets $\mathscr{S}$ defined in (1) and (10) as subsets of $\partial\mathbb{H}^n$ and $P^-\backslash G$, respectively, are the same under the above identification.

The convergent sequence $\psi_k \to \psi$ in $C(I, T^1(M))$ can be lifted to a convergent sequence $\tilde{\psi}_k \to \tilde{\psi}$ in $C^n(I, T^1(\mathbb{H}^n))$. Via the above correspondence, we obtain a convergent sequence $\tilde{\tilde{\psi}}_k \to \tilde{\tilde{\psi}}$ in $C^n(I, G)$ such that $\mathrm{Vis}(\tilde{\psi}_k(s)) = \mathcal{I}(\tilde{\tilde{\psi}}_k(s))$ and $\mathrm{Vis}(\tilde{\psi}(s)) = \mathcal{I}(\tilde{\tilde{\psi}}(s))$. Therefore, the conditions (a) and (b) on ψ imply condition (12) of Theorem 1.5 for the map $\tilde{\tilde{\psi}}$. Also, the required convergence property (13) is satisfied for $\{\tilde{\tilde{\psi}}_k\}$. Now any $f \in \mathrm{C_c}(T^1(M))$ can be treated as a $\mathrm{SO}(n-1)$-invariant function on G/Γ. In this case, the conclusion (14) of Theorem 1.5 holds. Therefore, (89) follows. □

Theorem 1.1 is a special case of Theorem 1.2.

6. Action of $\{a_t\}$ on shrinking curves

Let $S \in \mathscr{S}$, or let $S = \mathbb{S}^{n-1}$. Define

$$S^* = S \setminus \bigcup_{\substack{S' \subset S,\ \dim S' < \dim S \\ S' \in \mathscr{S}}} S'. \tag{90}$$

Let $\varphi \in C^n(I, \mathbb{R}^{d-1})$, and let $g_0 \in G$. We define

$$I(S) = \{s \in I : \varphi(s) \in \mathcal{S}^{-1}(S^* g_0^{-1})\}. \tag{91}$$

By the Lebesgue density theorem, almost every $x \in I(S)$ is a *density point* of $I(S)$; that is, if I_k is any sequence of intervals in I containing x such that $|I_k| \to 0$, then $|I(S) \cap I_k|/|I_k| \to 1$ as $k \to \infty$.

THEOREM 6.1
Let $x \in I(\mathbb{S}^{n-1})$ such that $\varphi^{(1)}(x) \neq 0$ and such that x is a density point for $I(\mathbb{S}^{n-1})$. Then for any sequences $\varphi_k \to \varphi$ in $C^n(I, \mathbb{R}^{n-1})$, $g_k \to g_0$ in G, $t_k \to \infty$ in $\mathbb{R}$, and any sequence of intervals $I_k \subset [a, b]$ such that $x \in I_k$, $|I_k| \to 0$, and $|I_k|^n e^{t_k} \to \infty$, the following holds: for any $f \in \mathrm{C_c}(G/\Gamma)$, we have

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k} u(\varphi_k(s)) g_k \Gamma\big)\, \mathrm{d}s = \int_{G/\Gamma} f \, \mathrm{d}\mu_G. \tag{92}$$

Let $\pi : G \to G/\Gamma$ denote the natural quotient map. Let $S \in \mathscr{S}$, and let $m = 1 + \dim S$. Let $g \in G$ be such that $S = \mathcal{I}(\mathrm{SO}(m, 1)g)$ and such that $N_G(\mathrm{SO}(m, 1))\pi(g)$ is closed. Due to the following claim, the coset $N_G(\mathrm{SO}(m, 1))g$ is uniquely defined.

We claim that if $F = \{h \in G : \mathcal{I}(\mathrm{SO}(m, 1)h) = \mathcal{I}(\mathrm{SO}(m, 1))\}$, then $F = N_G(\mathrm{SO}(m, 1))$. To prove the claim, we note that $N_G(\mathrm{SO}(m, 1)) \subset F$. In particular, F is a reductive group. Since $N_G(\mathrm{SO}(m, 1))$ is a symmetric subgroup of G, by [5, Corollary 4.7] $N_G(\mathrm{SO}(m, 1))$ is a maximal reductive subgroup of G. Therefore, $F = N_G(\mathrm{SO}(m, 1))$.

Let L be the subgroup of $N_G(\mathrm{SO}(m, 1))$ such that $L\pi(g) = \overline{\mathrm{SO}(m, 1)\pi(g)}$. Let μ_L denote the unique L-invariant probability measure on $L\pi(g)$.

THEOREM 6.2
Let $x \in I(S)$ be such that $\varphi^{(1)}(x) \neq 0$ and x is a density point for the set $I(S)$. Then for any sequence $t_k \to \infty$ in $\mathbb{R}$ and for any sequence of intervals $I_k \subset [a, b]$ such that $x \in I_k$, $|I_k| \to 0$, and $|I_k|^n e^{t_k} \to \infty$, the following holds: for any $f \in \mathrm{C_c}(G/\Gamma)$, we have

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k} u(\varphi(s)) g_0\big)\, \mathrm{d}s = \int_{L\pi(g)} f(zy)\, \mathrm{d}\mu_L(y), \tag{93}$$

where $z \in Z_G(A) \cap \mathrm{SO}(n)$ is such that $u(\varphi(x))g_0 \in U^- zLg$ and $u(\varphi^{(1)}(x)) \in zLz^{-1}$; z depends only on φ, x, g_0, and Lg.

Proof of Theorems 6.1 and 6.2
Let $y_0 = \pi(g_0)$. For $k \in \mathbb{N}$, let $y_k = \pi(g_k)$ and let λ_k be a probability measure on G/Γ such that for any $f \in \mathcal{C}(G/\Gamma)$, we have

$$\int_{G/\Gamma} f \, \mathrm{d}\lambda_k = \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k} u(\varphi_k(s) y_k)\big) \, \mathrm{d}s.$$

Proposition 2.3 is valid for $J \subset I_k$. Therefore, the proof of Theorem 3.2 is valid in this case, and we obtain that after passing to a subsequence $\lambda_k \to \lambda$ in the space of probability measure on G/Γ.

Let $W = \{u(r\varphi^{(1)}(x)) : r \in \mathbb{R}\}$. We claim that λ is W-invariant. To prove this, let $r \in \mathbb{R}$, $\epsilon > 0$, and $f \in \mathrm{C_c}(G/\Gamma)$. Now f is uniformly continuous, the family $\{\varphi_k^{(1)}(s)\}$ is equicontinuous, and $|I_k| \to 0$. Therefore, for sufficiently large $k \in \mathbb{N}$ and any $s \in I_k$, the following holds; we write $\eta_1 \overset{\epsilon}{\approx} \eta_2$ to indicate that $|\eta_1 - \eta_2| \le \epsilon$, and we have

$$\begin{aligned} f\big(u(r\varphi^{(1)}(x)) a_{t_k} u(\varphi_k(s)) y_k\big) &\overset{\epsilon}{\approx} f\big(u(r\varphi_k^{(1)}(s)) a_{t_k} u(\varphi_k(s)) y_k\big) \\ &= f\big(a_{t_k} u(\varphi_k(s) + e^{-t_k} r\varphi^{(1)}(s)) y_k\big) \\ &= f\big(a_{t_k} u(\varphi_k(s + re^{-t_k}) + O(e^{-2t_k})) y_k\big) \\ &= f\big(u(O(e^{-t_k})) a_{t_k} u(\varphi_k(s + re^{-t_k})) y_k\big) \\ &\overset{\epsilon}{\approx} f\big(a_{t_k} u(\varphi_k(s + re^{-t_k})) y_k\big). \end{aligned} \tag{94}$$

Therefore, for sufficiently large $k \in \mathbb{N}$, we have

$$\begin{aligned} \int_{G/\Gamma} f\big(u(r\varphi^{(1)}(x)) y\big) \, \mathrm{d}\lambda_k(y) &\overset{\epsilon}{\approx} \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k} u(\varphi_k(s + re^{-t_k})) y_k\big) \, \mathrm{d}s \\ &\overset{\epsilon}{\approx} \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k} u(\varphi_k(s)) y_k\big) \, \mathrm{d}s = \int f \, \mathrm{d}\lambda_k; \end{aligned}$$

the last approximation holds because

$$\frac{2re^{-t_k} \sup|f|}{|I_k|} = \frac{2r \sup|f|}{e^{t_k}|I_k|} \to 0.$$

This implies that λ is W-invariant.

The proof of Proposition 3.4 goes through in this case. If λ is not G-invariant, we apply Ratner's classification of ergodic invariant measures exactly as in the earlier case and follow the proof of Theorem 4.1 for I_k in place of I. All the arguments are valid up to (83) to conclude that

$$\big|\big\{s \in I_k : \mathcal{S}(\varphi(s)) \in S' g_0^{-1}\big\}\big| \ge \epsilon |I_k| / \#(\Sigma_1) \tag{95}$$

for some $S' \in \mathscr{S}$. By our hypothesis, x is a density point of $I(S)$ (see (91)). Therefore, in view of the definition of S^*, we deduce that $S \subset S'$.

For Theorem 6.1, $S = \mathbb{S}^{n-1}$, which contradicts the above conclusion, and hence λ is G-invariant.

Now for Theorem 6.2, we have $\varphi_k = \varphi$ and $g_k = g_0$ for all $k \in \mathbb{N}$. For any $s \in I(S)$, there exists $b(s) \in M$ such that $b(s) \to z$ as $s \to x$ and

$$u\big(\varphi(s)\big)g_0 \in U^- b(s) L g.$$

Therefore, since x is a density point of $I(S)$ and $\lambda_k \to \lambda$, from the definition of λ_k we conclude that $\operatorname{supp}\lambda \subset zL\pi(g)$. Since $\lambda\big(\pi(N(H, W))\big) > 0$, we conclude that $S' g_0^{-1} \subset S g_0^{-1}$. Thus, $S = S'$ and $H^{\mathrm{nc}} \cong \mathrm{SO}(m, 1)$, and $\operatorname{supp}(\lambda) \subset g' N_G(H^{\mathrm{nc}})\pi(e)$ for some $g' \in G$ such that $AW \subset g' N_G(H^{\mathrm{nc}})(g')^{-1}$. Since $\lambda\big(\pi(S(H, W))\big) = 0$, we deduce that each W-ergodic component of λ is invariant under $g' H^{\mathrm{nc}}(g')^{-1}$. Therefore since

$$\operatorname{supp}(\lambda) \subset zL\pi(g) \qquad \text{and} \qquad L\pi(g) = \overline{\mathrm{SO}(m, 1)\pi(g)},$$

by dimension consideration, we conclude that $\operatorname{supp}(\lambda) = zL\pi(g)$ and that $\lambda = z\mu_L$. This completes the proof of Theorem 6.2. □

7. Evolution of shrinking curves under geodesic flow

Let the notation be as in Section 1.1. As a consequence of Theorem 6.2, we deduce the following.

THEOREM 7.1

Let $\psi \in C^n(I, T^1(M))$. Let $M_1 \in \bar{\mathscr{S}}$, and let $S \in \mathscr{S}(M_1)$. Let $x \in I$ be such that $(\mathrm{Vis} \circ \tilde{\psi})^{(1)}(x) \neq 0$ and such that x is a density point of $I(S)$. Then, given any sequence $t_k \to \infty$ in $\mathbb{R}$ and a sequence of subintervals I_k of I containing x such that $|I_k| \to 0$ and $|I_k|^n e^{t_k} \to \infty$, the following holds:

$$\lim_{k\to\infty} \frac{1}{|I_k|} \int_{I_k} f\big(a_{t_k}\psi(s)\big)\,\mathrm{d}s = \int_{T^1(M_1)} f\,\mathrm{d}\mu_{M_1}, \quad \forall f \in \mathrm{C_c}\big(T^1(M)\big). \tag{96}$$

The conclusion of Theorem 1.3 can be deduced from Theorem 7.1 using the regularity of Lebesgue measure and standard arguments of measure theory. As a consequence of Theorem 6.1, we can also obtain a version of Theorem 7.1 where we assume that x is a density point of $I(M)$ and where we allow $\psi_k \to \psi$ in $C^n(I, T^1(M))$.

7.1. Geodesic evolution of faster shrinking curve

We can also obtain the following variation of Theorem 7.1.

THEOREM 7.2
In Theorem 7.1, suppose that $\psi \in C^{2n-2}(I, T^1(M))$. Then, given a sequence $t_k \to \infty$ and a sequence of subintervals I_k of I containing x such that $|I_k| \to 0$ and $|I_k|^2 e^{t_k} \to \infty$, equation (96) holds.

In the case when x is assumed to be a generic point of $I(M)$, we can consider $\psi_k \to \psi$ in C^{2n-2} and obtain the analogous conclusion. It is interesting to compare these statements with the results in [14].

To prove Theorem 7.2 using the method of this article, the only property required to be verified is the following variation of Proposition 2.1.

PROPOSITION 7.3 (Basic lemma, II)
Let $\varphi_k \to \varphi$ in $C^{2n-2}(I, \mathbb{R}^{d-1})$. Given $C > 0$, there exists $R_0 > 0$ such that for any sequence $t_k \to \infty$ in $\mathbb{R}$ there exists $k_0 \in \mathbb{N}$ such that for any $x \in I = [a, b]$ and for $v \in V$, there exists an interval $[s_k, s_k'] \subset I$ containing x such that for any $k \geq k_0$, the following conditions are satisfied:

$$e^{t_k}(s_k' - s_k)^2 < C, \tag{97}$$

$$\left\| a_{t_k} u(\varphi_k(s_k))v \right\| \geq \|v\|/R_0 \quad \text{if } s_k > a, \tag{98}$$

$$\left\| a_{t_k} u(\varphi_k(s_k'))v \right\| \geq \|v\|/R_0 \quad \text{if } s_k' < b. \tag{99}$$

Proof
We follow the strategy of the proof of Proposition 2.1. We now highlight some crucial modification required in the proof.

First, (23) is replaced by $e^{t_k}\delta_k^2 \geq C$. We put $\mu = n - 1$ in (25) to get

$$\sup_{s \in [0, \delta_k]} \left\| q_{n-1}\big(u(\varphi_{k,r_k}(s))w_k\big) \right\| \leq R_k^{-1} C^{-1} \delta_k^{2(n-1)}.$$

Therefore in place of (32), we have

$$\lim_{s \to 0} \left\| q_{n-1}\big(u(\varphi_{0,r_0}(s))w_0\big) \right\| / s^{2n-2} = 0. \tag{100}$$

Now following the further arguments using the $\mathrm{SL}(2, \mathbb{R})$-representation theory, we obtain an analogue of (40) for q_{n-1} involving $h^{(n-1)-\mu_0}$ in the highest-order term. Therefore, (41) becomes

$$\lim_{s \to 0} \left\| q_{n-1}\big(u(\varphi_{0,r_0}(s))w_0\big) \right\| / s^{(n-1)-\mu_0} \geq \eta_0 \rho_0^{n-1-\mu_0} > 0.$$

Since $n - 1 - \mu_0 \leq 2n - 2$, this contradicts (100). □

Remark 7.1
Using Proposition 7.3, we can obtain an analogue of Corollary 2.2 for $P_{k,x}(s) = \varphi_k(x) + s\varphi_k^{(1)}(s)$. Thus for linearization technique, we can approximate a $C^{2(n-1)}$-curve φ_k at any $s \in I$ by its tangent line, rather than a polynomial curve.

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