LIMITS OF TRANSLATES OF DIVERGENT GEODESICS AND INTEGRAL POINTS ON ONE-SHEETED HYPERBOLOIDS

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Abstract. For any non-uniform lattice Γ in $\mathrm{SL}_2(\mathbb{R})$, we describe the limit distribution of orthogonal translates of a *divergent* geodesic in $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$. As an application, for a quadratic form Q of signature $(2,1)$, a lattice Γ in its isometry group, and $v_0 \in \mathbb{R}^3$ with $Q(v_0) > 0$, we compute the asymptotic (with a logarithmic error term) of the number of points in a discrete orbit $v_0\Gamma$ of norm at most T, when the stabilizer of v_0 in Γ is finite. Our result in particular implies that for any non-zero integer d, the smoothed count for number of integral binary quadratic forms with discriminant d^2 and with coefficients bounded by T is asymptotic to $c \cdot T \log T + O(T)$.

1. Introduction

1.1. **Motivation.** Let $Q \in \mathbb{Z}[x_1, \cdots, x_n]$ be a homogeneous polynomial and set $V_m := \{x \in \mathbb{R}^n : Q(x) = m\}$ for an integer m. It is a fundamental problem to understand the set $V_m(\mathbb{Z}) = \{x \in \mathbb{Z}^n : Q(x) = m\}$ of integral solutions.

In particular, we are interested in the asymptotic of the number $N(T) := \#\{x \in V_m(\mathbb{Z}) : \|x\| < T\}$ as $T \to \infty$, where $\|\cdot\|$ is a fixed norm on $\mathbb{R}^n$.

The answer to this question depends quite heavily on the geometry of the ambient space V_m. We suppose that the variety V_m is homogeneous, i.e., there exist a connected semisimple real algebraic group G defined over $\mathbb{Q}$ and a $\mathbb{Q}$-rational representation $\iota : G \to \mathrm{SL}_n$ such that $V_m = v_0.\iota(G)$ for some non-zero $v_0 \in \mathbb{Q}^n$.

Let $\Gamma < G(\mathbb{Q})$ be a subgroup commensurable with $G(\mathbb{Z})$ preserving $V_m(\mathbb{Z})$. Since V_m is Zariski closed, by Borel and Harish-Chandra [3, Theorems 6.9 and 7.8], the co-volume of Γ in G is finite and there are only finitely many Γ-orbits in $V_m(\mathbb{Z})$. Hence understanding the asymptotic of $N(T)$ is reduced to the orbital counting problem of estimating $\#(v_0\Gamma \cap B_T)$ for $B_T = \{x \in V_m : \|x\| < T\}$ and $v_0 \in V_m(\mathbb{Z})$.

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Theorem 1.1 (Duke-Rudnick-Sarnak [10])**.** *Set H to be the stabilizer subgroup of v_0 in G. Suppose that H is a symmetric subgroup of G. If the volume of $(H \cap \Gamma)\backslash H$ is finite, i.e., if $H \cap \Gamma$ is a lattice in H, we have, as $T \to \infty$,*

$$\#(v_0\Gamma \cap B_T) \sim \frac{\mathrm{vol}_H(H \cap \Gamma\backslash H)}{\mathrm{vol}_G(\Gamma\backslash G)} \mathrm{vol}_{H\backslash G}(B_T),$$

that is, the ratio of both the sides converges to 1 as $T \to \infty$, where the volumes on H, G and $v_0G \simeq H\backslash G$ are computed with respect to invariant measures chosen compatibly; that is, $d\,\mathrm{vol}_G = d\,\mathrm{vol}_H \times d\,\mathrm{vol}_{H\backslash G}$ locally.

A simpler proof of this result using mixing of geodesic flow was given by Eskin and McMullen [11]. When H is any maximal reductive $\mathbb{Q}$-subgroup with $\mathrm{vol}((H \cap \Gamma)\backslash H) < \infty$, the same conclusion was obtained by Eskin, Mozes and Shah in [12] using Ratner's description [17] of measures invariant under unipotent flows. An effective version of Theorem 1.1 has also been obtained in [10] for $(H \cap \Gamma)\backslash H$ compact and in [2] in general.

In view of the the main term of the asymptotic, it is crucial to assume that $\mathrm{vol}(H \cap \Gamma\backslash H) < \infty$ in Theorem 1.1. The main aim of this paper is to break this barrier and investigate the counting problem in the case when $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$.

If H is a semisimple Lie group with no compact factors in a semisimple Lie group G, then any closed $\Gamma\backslash\Gamma H$ in $\Gamma\backslash G$ must be of finite volume by Dani [6] and Margulis [16] (see also [20]).

If Q is a quadratic form of signature (p, q) with $p + q \geq 3$, $p \geq q$ and G is the special orthogonal group of Q, the case of $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$ for $H = \mathrm{Stab}_G(v_0)$ arises only when $(p, q) = (2, 1)$ and $Q(v_0) = m > 0$, that is, when the variety $V_m = \{x \in \mathbb{R}^3 : Q(x) = m\}$ is a one-sheeted hyperboloid. To prove this claim, note that any non-compact stabilizer H of $v_0 \in \mathbb{R}^n$ in G is either locally isomorphic to $\mathrm{SO}(p - 1, q)$, $\mathrm{SO}(p, q - 1)$ or a compact extension of a horospherical subgroup. Except for the case of $SO(1, 1)$ the groups $\mathrm{SO}(p - 1, q)$ and $\mathrm{SO}(p, q - 1)$ are compact or semi-simple. Also any closed orbit of a compact extension of a horospherical subgroup in $\Gamma\backslash G$ is compact (cf. [8]). Therefore in view the above remarks it follows that the case of $\mathrm{vol}(H \cap \Gamma\backslash H) = \infty$ arises only when $H \simeq \mathrm{SO}(1, 1)$; hence $n = 3$ and $Q(v_0) > 0$.

In the next subsection, we state our main theorem in a greater generality, not necessarily in the arithmetic situation.

1.2. **Counting integral points on a one-sheeted hyperboloid.** Let $Q(x_1, x_2, x_3)$ be an real quadratic form of signature $(2, 1)$. Denote by G the identity component of the special orthogonal group $\mathrm{SO}_Q(\mathbb{R})$. Let $\Gamma < G$ be a lattice and $v_0 \in \mathbb{R}^3$ be such that $Q(v_0) > 0$ and the orbit $v_0\Gamma$ is discrete. As before, we fix a norm $\|\cdot\|$ on $\mathbb{R}^3$ and set $B_T := \{x \in v_0G : \|w\| < T\}$.

To present our theorem with a best possible error term, we consider the following smoothed counting function: fixing a non-negative function $\psi \in$

$C_c^\infty(G)$ with integral one, let

$$\tilde{N}_T := \sum_{v \in v_0\Gamma} (\chi_{B_T} * \psi)(v)$$

where $\chi_{B_T} * \psi(x) = \int_G \chi_{B_T}(xg)\psi(g)\, dg$, $x \in v_0 G$, is the convolution of the characteristic function of B_T and ψ. Note that $\tilde{N}_T \asymp \#(v_0\Gamma \cap B_T)$ in the sense that their ratio is in between two uniform constants for all sufficiently large $T \gg 1$.

Denoting by $H \simeq \mathrm{SO}(1,1)^\circ$ the identity component of the one-dimensional stabilizer subgroup of v_0 in G, note that $\mathrm{vol}(H \cap \Gamma \backslash H) < \infty$ if and only if $H \cap \Gamma$ is infinite. In order to state our theorem, we write H as a one-parameter subgroup $\{h(s) : s \in \mathbb{R}\}$ so that the Lebesgue measure ds defines the Haar measure on H: $\int_{-\log T}^{\log T} ds = \mathrm{vol}_H(\{h(s) : |s| < \log T\})$.

Theorem 1.2. *If the volume of $(H\cap\Gamma)\backslash H$ is infinite, we have the following:*

(1) *For $T \gg 1$,*

$$N_T = \frac{2\log T \cdot \mathrm{vol}_{H\backslash G}(B_T)}{\mathrm{vol}_G(\Gamma\backslash G)}(1 + O((\log T)^{-0.25})),$$

where $d\,\mathrm{vol}_G = ds \times d\,\mathrm{vol}_{H\backslash G}$ locally.

(2) *For $T \gg 1$,*

$$\tilde{N}_T = c \cdot T \log T + O(T),$$

where $c = \lim_{T\to\infty} \frac{2\,\mathrm{vol}_{H\backslash G}(B_T)}{T\,\mathrm{vol}_G(\Gamma\backslash G)}$.

We note that when $\mathrm{vol}(H\cap\Gamma\backslash H) < \infty$, $\tilde{N}_T = c\cdot T + O(T^\alpha)$ for $0 < \alpha < 1$ is obtained in [10]. We believe, as suggested by Z. Rudnick to us, that $\tilde{N}_T = c \cdot T \log T + c' \cdot T + O(T^\alpha)$ for some $c' > 0$ and $0 < \alpha < 1$ and hence the order of the second term for $\tilde{N}_T$ cannot be improved.

Theorem 1.2 can be generalized to the orbital counting for more general representations of $\mathrm{SL}_2(\mathbb{R})$ (see section 6).

Remark 1.3. In the case when $Q = x_1^2 + x_2^2 - d^2x_3^2$ for $d \in \mathbb{Z}$, $v_0 = (1,0,0)$, and $\Gamma = \mathrm{SO}_Q(\mathbb{Z})$, it was pointed out in [10] that an elementary number theoretic computation of [19] leads to the asymptotic

$$\#\{(x_1,x_2,x_3) \in v_0\Gamma : \sqrt{x_1^2 + x_2^2 + d^2x_3^2} < T\} = c\cdot T\log T + O(T\log(\log T)).$$

However this deduction seems to work only for this very special case; for instance, we are not aware of any other approach than ours which can deal with non-arithmetic situations.

1.3. **Arithmetic case and Integral binary quadratic forms.** In the arithmetic case, Theorem 1.2 implies the following:

Corollary 1.4. *Let $Q(x_1,x_2,x_3)$ be an integral quadratic form with signature $(2,1)$. Suppose that for some $v_0 \in \mathbb{Z}^3$ with $Q(v_0) > 0$, the stabilizer subgroup of v_0 is isotropic over $\mathbb{Q}$.*

Then there exists $c = c(\|\cdot\|) > 0$ such that as $T \to \infty$,

$$\#\{x \in \mathbb{Z}^3 : Q(x) = Q(v_0), \|x\| < T\} = c \cdot T \log T + O(T(\log T)^{3/4}).$$

For a binary quadratic form $q(x, y) = ax^2 + bxy + cy^2$, its discriminant $\operatorname{disc}(q)$ is defined to be b^2-4ac. The group $\mathrm{SL}_2(\mathbb{R})$ acts on the space of binary quadratic forms by $(g.q)(x, y) = q((x, y)g)$ and preserves the discriminant. For $d \in \mathbb{Z}$, denote by $\mathcal{B}_d(\mathbb{Z})$ the space of integral binary quadratic forms with discriminant d. Note that $\mathcal{B}_d(\mathbb{Z}) \neq \emptyset$ if and only if d congruent to 0 or 1 mod 4. Now d is a square *if and only if* the stabilizer of every $q \in \mathcal{B}_d(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{Z})$ is infinite *if and only if* every $q \in \mathcal{B}_d(\mathbb{Z})$ is decomposable over $\mathbb{Z}$. (cf. [4]).

Therefore Theorem 1.2 implies the following:

Theorem 1.5. . *For any non-zero square $d \in \mathbb{Z}$, there exists $c_0 > 0$ such that*

$$\#\{q \in \mathcal{B}_d(\mathbb{Z}) : \operatorname{disc}(q) = d, \|q\| < T\} = c_0 \cdot T \log T + O(T(\log T)^{3/4})$$

where $\|ax^2 + bxy + cy^2\| = \|(a, b, c)\|$.

1.4. **Orthogonal translates of a divergent geodesic.** Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a non-uniform lattice in G. For $s \in \mathbb{R}$, define

$$h(s) = \begin{bmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{bmatrix}, \; a(s) = \begin{bmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{bmatrix} \tag{1.1}$$

and set $H = \{h(s) : s \in \mathbb{R}\}$.

In the case when the orbit $\Gamma\backslash\Gamma H$ is closed and of finite length, the limiting distribution of the translates $\Gamma\backslash\Gamma H a(T)$ as $T \to \infty$ is described by the unique G-invariant probability measure $d\mu(g) = dg$ on $\Gamma\backslash G$ [10], that is, if s_0 is the period of $\Gamma \cap H\backslash H$, then for any $\psi \in C_c(\Gamma\backslash G)$,

$$\lim_{T\to\pm\infty} \frac{1}{s_0} \int_{s=0}^{s_0} \psi(h(s)a(T))ds = \int_{\Gamma\backslash G} \psi \, dg.$$

Similarly, understanding the limit of the translates $\Gamma\backslash\Gamma H a(T)$, when $\Gamma\backslash\Gamma H$ is of infinite length, is the main new ingredient in our proofs of Theorem 1.2.

Theorem 1.6. *Let $x_0 \in \Gamma\backslash G$ and suppose that $x_0 h(s)$ diverges as $s \to +\infty$, that is, $x_0 h(s)$ leaves every compact subset for all sufficiently large $s \gg 1$. For a given compact subset $\Omega \subset \Gamma\backslash G$, there exists $M = M(\Omega, x_0) > 0$ such that for any $\psi \in C^\infty(\Gamma\backslash G)$ with support in Ω, we have, as $|T| \to \infty$,*

$$\int_0^\infty \psi(x_0 h(s)a(T))ds = \int_0^{|T|+M} \psi(x_0 h(s)a(T))ds = |T| \int \psi \, d\mu + O(1),$$

where the implied constant depends on x_0, Ω and $\mathcal{S}^\dagger(\psi) = \max\{\|\psi\|_{C^1}, \mathcal{S}_1(\psi)\}$; here $\|\psi\|_{C^1}$ denotes the C^1-norm (see (3.3)) and $\mathcal{S}_1(\psi)$ denotes the L^2-Sobolev norm of ψ of degree one (see (3.1)).

Remark 1.7. Consider the hyperbolic plane $\mathbb{H}^2$. A parabolic fixed point for Γ is a point in the geometric boundary $\partial_\infty(\mathbb{H}^2)$ fixed by a parabolic element of Γ. If $\mathcal{F} \subset \mathbb{H}^2$ is a finite sided Dirichlet region for Γ, then the parabolic fixed points of Γ are precisely the Γ-orbits of vertices of $\overline{\mathcal{F}}$ lying in $\partial_\infty(\mathbb{H}^2)$. Let $\pi : G \to \mathbb{H}^2$ denote the orbit map $g \mapsto g(i)$. For $x_0 = \Gamma g_0 \in \Gamma\backslash G$, the image $\pi(g_0 H)$ is a geodesic in $\mathbb{H}^2$ with two endpoints $g_0 H(+\infty) := \lim_{s\to\infty} \pi(g_0 h(s))$ and $g_0 H(-\infty) := \lim_{s\to-\infty} \pi(g_0 h(s))$ in $\partial_\infty(\mathbb{H}^2)$. We remark that $x_0 h(s)$ diverges as $s \to +\infty$ (resp. $s \to -\infty$) if and only if $g_0 H(+\infty)$ (resp. $g_0 H(-\infty)$) is a parabolic fixed point for Γ (cf. Theorem 2.1).

Corollary 1.8. *Suppose that $x_0 H$ is closed and non-compact. For any $\psi \in C_c(\Gamma\backslash G)$,*

$$\lim_{T\to\pm\infty} \frac{1}{2|T|} \int_{-\infty}^{\infty} \psi(x_0 h(s) a(T))\, ds = \int_{\Gamma\backslash G} \psi\, d\mu.$$

It is well-known due to the work of Duke, Rudnick and Sarnak [10] that Theorem 1.2 follows once we establish the equidistribution of the translates $\Gamma\backslash\Gamma H a(T)$ of a divergent H-orbit as in Theorem 1.6. Our proof of Theorem 1.6 is based on the description of divergent orbits due to Dani [7] and the effective equidistribution of the translates of a small piece of $\Gamma\backslash\Gamma H$ due to Kleinbock and Margulis [15]. The new ingredient of our work is mainly in the section 4.

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2. Structure of cusps in $\Gamma\backslash G$ and divergent trajectory

Let $G = \mathrm{SL}_2(\mathbb{R})$ and Γ be a non-uniform lattice in G. We will keep the notation for $h(s)$ and $a(s)$ from (1.1) in the introduction. Let

$$N = \{\left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right) : s \in \mathbb{R}\} \quad \text{and} \quad U = wNw^{-1}$$

where $w = \left[\begin{smallmatrix} \cos(\pi/4) & \sin(\pi/4) \\ -\sin(\pi/4) & \cos(\pi/4) \end{smallmatrix}\right]$. Note that $h(s) = wa(s)w^{-1}$ for all $s \in \mathbb{R}$. For $\eta > 0$, let

$$H_\eta = \{h(s) : s/2 > -\log \eta\}.$$

Let $K = \mathrm{SO}(2) = \{g \in G : gg^t = I\}$. Then the multiplication map $U \times H \times K \to G$: $(u, h, k) \mapsto uhk$ is a diffeomorphism.

The following classical result may be found at [13, Thm. 0.6] or [9]:

Theorem 2.1. *There exists a finite set $\Sigma \subset G$ such that the following holds:*

1. *$\Gamma\backslash\Gamma\sigma U$ is compact for every $\sigma \in \Sigma$.*
2. *For any $\eta > 0$, the set*

$$\Omega_\eta := \Gamma\backslash G \setminus \bigcup_{\sigma\in\Sigma} \Gamma\backslash\Gamma\sigma U H_\eta K$$

is compact; and any compact subset of $\Gamma\backslash G$ is contained in Ω_η for some $\eta > 0$.

(3) *There exists* $\eta_0 > 0$ *such that for* $i = 1, 2$, *if* $\sigma_i \in \Sigma$, $u_i \in U$, $h_i \in H_{\eta_0}$, *and* $\Gamma\sigma_1 u_1 h_1 k_1 = \Gamma\sigma_2 u_2 h_2 k_2$, *then* $\sigma_1 = \sigma_2$, $k_1 = \pm k_2$ *and* $h_1 = h_2$.

Consider the standard representation of $G = \mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$: $((v_1, v_2), g) \mapsto (v_1, v_2)g$. Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. Let

$$p = (0, 1)w^{-1} = (-\sin(\pi/4), \cos(\pi/4)).$$

Then $pU = p$, and $ph(s) = (0, 1)a(s)w^{-1} = e^{-s/2}p$ for all $s \in \mathbb{R}$. Also

$$g \in UH_\eta K \Leftrightarrow \|pg\| < \eta. \tag{2.1}$$

Proposition 2.2 (Dani [7])**.** *Let* $x_0 \in \Gamma\backslash G$ *be such that the trajectory* $\{x_0 h(s) : s \geq 0\}$ *is divergent. Then there exist* $\sigma_0 \in \pm I\Sigma$, $s_0 \in \mathbb{R}$ *and* $u \in U$ *such that* $x_0 = \Gamma\sigma_0 u h(s_0)$.

Proof. By Theorem 2.1, there exists $s_1 > 0$ and $\sigma \in \Sigma$ such that $x_0 h(s) = \Gamma\sigma U H_{\eta_0/2} K$ for all $s \geq s_1$. Let $g_1 \in UH_{\eta_0/2}K$ be such that $x_0 h(s_1) = \Gamma\sigma g_1$. We claim that $pg_1 \in \mathbb{R}p$. If not, then $\|pg_1 h(s)\| \to \infty$ as $s \to \infty$, and hence there exists $s > 0$ such that $\eta_0/2 < \|pg_1 h(s)\| < \eta_0$. By (2.1), $g_1 h(s) = uhk$ for some $u \in U$, $h \in H_{\eta_0}$ and $k \in K$. Therefore

$$\Gamma\sigma uhk = \Gamma\sigma g_1 h(s) = x_0 h(s_1 + s) \in \Gamma\sigma U H_{\eta_0/2} K.$$

By Theorem 2.1(iii), we have that $h \in H_{\eta_0/2}$. But then $\|pg_1 h(s)\| = \|puhk\| < \eta_0/2$, a contradiction. Therefore our claim that $pg_1 \in \mathbb{R}p$ is valid. Hence $g_1 = u_1 h(s)\{\pm I\}$ for some $u_1 \in U$ and $s/2 \geq -\log(\eta_0/2)$. Thus $x_0 h(s_1) = \Gamma\sigma u_1 h(s)\{\pm I\}$, and hence $x_0 = \Gamma\sigma_0 u_1 h(s - s_1)$, where $\sigma_0 = \pm I\sigma$. $\square$

Proposition 2.3. *Let* $x_0 \in \Gamma\backslash G$ *be such that the trajectory* $\{x_0 h(s) : s \geq 0\}$ *is divergent. Let* $\Omega \subset \Gamma\backslash G$ *be a compact subset. There exists* $M_1 = M_1(\Omega, x_0) > 0$ *such that*

$$x_0 h(s) a(T) \notin \Omega$$

for any $T \in \mathbb{R}$ *and* $s > 0$ *satisfying* $s > |T| + M_1$. *In particular, for any* $f \in C(\Gamma\backslash G)$ *with support inside* Ω,

$$\int_0^\infty f(x_0 h(s) a(T))\, ds = \int_0^{|T|+M_1} f(x_0 h(s) a(T))\, ds.$$

Proof. By Proposition 2.2, $x_0 = \Gamma\sigma_0 u h(s_0)$ for some $\sigma_0 \in \pm\Sigma, u \in U, s_0 \in \mathbb{R}$. By Theorem 2.1(ii), let $\eta > 0$ be such that $\Omega \subset \Omega_\eta$. Let $M_1 = -s_0 - 2\log(\eta)$. Since $s - |T| > -s_0 - 2\log\eta$, we have

$$\begin{aligned} \|puh(s_0)h(s)a(T)\| &= \|ph(s + s_0)a(T)\| \\ &= e^{-(s+s_0)/2}\|pa(T)\| \\ &< e^{-(s+s_0)/2} e^{|T|/2} \\ &= e^{-(s+s_0-|T|)/2} < \eta. \end{aligned} \tag{2.2}$$

Therefore by (2.1), $uh(s_0)h(s)a(T) \in UH_\eta K$, and hence

$$x_0 h(s) a(T) \in \Gamma\sigma_0 U H_\eta K \subset \Gamma\backslash G \smallsetminus \Omega_\eta.$$

□

3. Uniform mixing on compact sets

Let $G = \mathrm{SL}_2(\mathbb{R})$ and $\Gamma < G$ be a lattice. Let μ denote the G-invariant probability measure on $\Gamma\backslash G$. For an orthonormal basis X_1, X_2, X_3 of $\mathfrak{sl}(2,\mathbb{R})$ with respect to an Ad-invariant scalar product, and $\psi \in C^\infty(\Gamma\backslash G)$, we consider the Sobolev norm

$$\mathcal{S}_m(\psi) = \max\{\|X_{i_1}\cdots X_{i_j}(\psi)\|_2 : 1 \le i_j \le 3, 0 \le j \le m\}. \tag{3.1}$$

The well-known spectral gap property for $L^2(\Gamma\backslash G)$ says that the trivial representation is isolated (see [1, Lemma 3]) in the Fell topology of the unitary dual of G. It follows that there exist $\theta > 0$ and $c > 0$ such that for any $\psi_1, \psi_2 \in C^\infty(\Gamma\backslash G)$ with $\int \psi_i d\mu = 0$, $\mathcal{S}_1(\psi_i) < \infty$ and for any $T > 0$,

$$|\langle a(T)\psi_1, \psi_2\rangle| \le ce^{-\theta|T|}\mathcal{S}_1(\psi_1)\mathcal{S}_1(\psi_2) \tag{3.2}$$

(cf. [5], [21])

Write $\mathcal{O}_\epsilon = \{g \in G : \|g - I\|_\infty \le \epsilon\}$. For a compact subset $\Omega \subset \Gamma\backslash G$, let $0 < \epsilon_0(\Omega) \le 1$ be the injectivity radius of Ω, that is, $\epsilon_0(\Omega)$ is the supremum of $0 < \epsilon \le 1$ such that for any $x \in \Omega$, the map $\mathcal{O}_\epsilon \ni g \mapsto xg \in \Gamma\backslash G$ is injective.

For $s \in \mathbb{R}$, let

$$n_+(s) = \left(\begin{smallmatrix} 1 & 0 \\ s & 1 \end{smallmatrix}\right) \quad \text{and} \quad n_-(s) = \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right).$$

Let $\|f\|_{C^1}$ denotes the C^1-norm of f, that is,

$$\|f\|_{C^1} = \|f\|_\infty + \sum_{i=1}^{3} \|X_i(f)\|_\infty. \tag{3.3}$$

Set $\mathcal{S}^\dagger(f) := \max\{\mathcal{S}_1(f), \|f\|_{C^1}\}$; in fact, $\mathcal{S}_1(f) \le \|f\|_{C^1}$.

The following is a special case of [15, Prop. 2.4.8]:

Theorem 3.1. *Let $\Omega \subset \Gamma\backslash G$ be a compact subset and $\eta > 0$. There exists $c = c(\Omega) > 0$ such that for any $\psi \in C^\infty(\Gamma\backslash G)$ with support in Ω, for any $|T| \ge 1$, $x \in \Omega$, and $0 < r_0 < \epsilon_0(\Omega)$, we have*

$$\left|\int_0^{r_0} \psi(xn_\nu(s)a(T))\, ds - r_0 \int \psi\, d\mu\right| \le c \cdot (\mathcal{S}^\dagger(\psi) + 1)e^{-\theta_0|T|} \tag{3.4}$$

for some $\theta_0 > 0$ depending only on the spectral gap for $L^2(\Gamma\backslash G)$. Here and in what follows, the sign $\nu = +$ if $T > 0$ and $\nu = -$ if $T < 0$.

4. Translates of divergent orbits

Let $x_0 \in \Gamma\backslash G$ be such that $x_0h(s)$ diverge as $s \to \infty$.

Theorem 4.1. *For any $|T| > 1$ and any $\psi \in C_c^\infty(\Gamma\backslash G)$*

$$\int_0^{|T|} \psi(x_0h(s)a(T))\, ds = |T| \int \psi\, d\mu + O(1)\mathcal{S}^\dagger(\psi)$$

where the implied constant depends on x_0, $\mathcal{S}^\dagger(\psi)$ and the support of ψ.

Proof. Let $R_0 = -\log \eta_0$. Due to Proposition 2.2, replacing x_0 by another point in x_0H, we may assume that $x_0 = \Gamma\sigma_0 h(R_0)$. For any $S > 0$, $\|ph(R_0)h(S)a(S)\| \in [\eta_0/\sqrt{2}, \eta_0]$. Hence $x_0h(R_0)h(S)a(S) \in \Omega_{\eta_0/\sqrt{2}}$.

Let r_0 be the injectivity radius of $\Omega_{\eta_0/\sqrt{2}}$, that is, $r_0 = \epsilon_0(\Omega_{\eta_0/\sqrt{2}})$. Let $S_0 = 0$, and choose S_i such that $r_0e^{-S_i} \le \delta_i := S_{i+1} - S_i \le 2r_0e^{-S_i}$ for each i. We will choose $S_i = \log(2r_0i+1)$ for each i. Then $x_0h(S_i)a(S_i) \in \Omega_{\eta_0/\sqrt{2}}$. We put $R_i = T - S_i$.

We will express $x_0h([S_i, S_{i+1}])a(T) = x_ih^{a(S_i)}([0,\delta_i])a(R_i)$, where $x_i = x_0h(S_i)a(S_i)$, $h^{a(S_i)}(s) = a(-S_i)h(s)a(S_i) = n(e^{S_i}s/2)w_i(s)$, and $|w_i(s)| = O(e^{-2S_i})$. Note that $r_0/2 \le e^{S_i}\delta_i/2 \le r_0$.

By Theorem 3.1, we have

$$\int_0^{r_0} \psi(x_in(s)a(R_i))ds - r_0\int \psi d\mu = \mathcal{S}^\dagger(\psi)\cdot O(e^{-\theta_0 R_i})$$

and hence

$$\int_{S_i}^{S_{i+1}} \psi(x_0h(s)a(T))ds = \frac{\delta_i}{r_0}\int_0^{r_0} \psi(x_in(s)a(R_i))ds + \mathcal{S}^\dagger(\psi)\cdot O(e^{-2S_i}\delta_i).$$

Let $k = k(T)$ be such that $S_k \le T < S_k + r_0e^{-S_k}$. Therefore, since $\delta_ir_0^{-1} \le 2e^{-S_i}$,

$$\begin{aligned}
&\int_0^T \psi(xh(s)a(T))ds = \sum_{i=0}^{k-1}\int_{S_i}^{S_{i+1}} \psi(xh(s)a(T))ds + O(e^{-S_k})\\
&= \sum_{i=0}^{k-1}\delta_i\frac{1}{r_0}\int_0^{r_0}\psi(x_in(s)a(T))ds + \mathcal{S}^\dagger(\psi)\cdot O(e^{-2S_i}\delta_i) + O(1)\\
&= \sum_{i=0}^{k-1}\delta_i\mu(\psi) + \sum_{i=0}^{k-1}\delta_ir_0^{-1}\mathcal{S}^\dagger(\psi)\cdot O(e^{-\theta_0R_i}) + \mathcal{S}^\dagger(\psi)\cdot O(e^{-2S_i}\delta_i) + O(1)\\
&= T\mu(\psi) + O(\sum_{i=1}^{k-1} e^{-S_i}e^{-\theta_0R_i} + \sum_{i=1}^{k} e^{-3S_i})\mathcal{S}^\dagger(\psi) + O(1)\\
&= T\mu(\psi) + O(e^{-\theta_0T}\sum_{i=0}^{k-1} e^{(1-\theta_0)S_i} + \sum_{i=0}^{k-1} e^{-3S_i})\mathcal{S}^\dagger(\psi) + O(1).
\end{aligned}$$

Since $S_i = \log(2r_0i+1)$, $0 < T - S_k < 2e^{-T}$ implies that $k < \frac{e^T-1}{2r_0} < k+1$, and hence

$$\sum_{i=0}^{k-1} e^{-3S_i} \ll \sum_{i=1}^{k-1}\frac{1}{(2r_0i+1)^3} = O(k^{-2}+1) = O(e^{-2T}+1) < \infty$$

and

$$\sum_{i=0}^{k-1} e^{(1-\theta_0)S_i} \ll \int_0^{e^T} \frac{1}{(2r_0x+1)^{1-\theta_0}} dx = O(e^{\theta_0 T}).$$

Hence

$$e^{-\theta_0 T} \sum_{i=0}^{k-1} e^{(1-\theta_0)S_i} + \sum_{i=0}^{k-1} e^{-3S_i} = O(1).$$

Therefore

$$\int_0^T \psi(xh(s)a(T))ds = T\mu(\psi) + O(1)\mathcal{S}^\dagger(\psi).$$

□

Theorem 1.6 follows from the following:

Theorem 4.2. *Let $x_0h(s)$ diverge as $s \to \infty$. For a given compact subset $\Omega \subset \Gamma\backslash G$, and $\psi \in C^\infty(\Gamma\backslash G)$ with support in Ω, we have*

$$\int_0^\infty \psi(x_0h(s)a(T))ds = |T| \cdot \int \psi \, d\mu + O(1)\mathcal{S}^\dagger(\psi)$$

with the implied constant depending only on x_0, $\mathcal{S}^\dagger(\psi)$, and the support of ψ.

Proof. Since $x_0h(s)$ diverges as $s \to \infty$, by Proposition 2.3, there exists $M_1 = M_1(\Omega) > 0$ such that

$$\begin{aligned}
&\int_0^\infty \psi(x_0h(s)a(T))ds = \int_0^{|T|+M_1} \psi(x_0h(s)a(T))ds \\
&= (|T| + M_1) \int \psi \, d\mu + O(1)\mathcal{S}^\dagger(\psi) \\
&= |T| \int \psi \, d\mu + O(1)\mathcal{S}^\dagger(\psi).
\end{aligned}$$

□

By a similar argument, we also deduce the following:

Corollary 4.3. *If $x_0h(s)$ diverges as $s \to -\infty$, then*

$$\int_{-\infty}^0 \psi(x_0h(s)a(T))ds = |T| \int \psi d\mu + O(1)\mathcal{S}^\dagger(\psi)$$

with the implied constant depending only on x_0, $\mathcal{S}^\dagger(\psi)$, and Ω.

Lemma 4.4. *If $x_0h(\mathbb{R})$ is closed and non-compact, then $x_0h(s)$ diverges as $s \to \pm\infty$.*

Proof. We use a well-known fact that for a closed subgroup H of a locally compact second countable group G and a discrete subgroup Γ of G, if ΓH is closed in G, then the canonical projection map $H \cap \Gamma\backslash H \to \Gamma\backslash G$ is a proper map (cf. [18, Remark 7.9(2)]). Since $x_0h(\mathbb{R})$ is non-compact and $h(\mathbb{R})$ is one-dimensional with no non-trivial finite subgroups, the stabilizer of x_0 in $h(\mathbb{R})$ is trivial. Therefore the map $h(\mathbb{R}) \to \Gamma\backslash G$ given by $h \to x_0 h$ is a proper injective map. This implies that $x_0 h(s)$ diverges as $s \to \pm\infty$. $\square$

Proof of Corollary 1.8. As the set $C_c^\infty(\Gamma\backslash G)$ is dense in $C_c(\Gamma\backslash G)$, the claim follows from Lemma 4.4, Theorem 4.1, and Corollary 4.3. $\square$

5. Counting: Proof of Theorem 1.2

Let Q be a real quadratic form in 3 variables of signature $(2,1)$ and Γ_0 a lattice in the identity component G_0 of $\mathrm{SO}_Q(\mathbb{R})$. We assume that $v_0\Gamma_0$ is discrete for some vector $v_0 \in \mathbb{R}^3$ with $Q(v_0) = d > 0$ and that the stabilizer H_0 of v_0 in G_0 is finite.

It suffices to prove Theorem 1.2 in the case when $Q = x^2 + y^2 - z^2$ and $v_0 = (\sqrt{d}, 0, 0)$ by the virtue of Witt's theorem.

Consider the spin double cover map $\iota : G := \mathrm{SL}_2(\mathbb{R}) \to G_0$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a^2-b^2-c^2+d^2}{2} & ac-bd & \frac{a^2-b^2+c^2-d^2}{2} \\ ab-cd & bc+ad & ab+cd \\ \frac{a^2+b^2-c^2-d^2}{2} & ac+bd & \frac{a^2+b^2+c^2+d^2}{2} \end{pmatrix}.$$

For $s \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, we set

$$h(s) = \begin{pmatrix} \cosh(s/2) & \sinh(s/2) \\ \sinh(s/2) & \cosh(s/2) \end{pmatrix}; \ a(s) = \begin{pmatrix} e^{s/2} & 0 \\ 0 & e^{-s/2} \end{pmatrix} \text{ and } k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Recall that $H := \{h(s) : s \in \mathbb{R}\}$, $A := \{a(t) : t \in \mathbb{R}\}$ and $K_1 := \{k(\theta) : \theta \in [0, 2\pi)\}$, here K_1 is half of the circle group. Observing that

$$\iota(h(s)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} \quad \text{and} \quad \iota(a(t)) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & 1 & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix},$$

the subgroup $\tilde{H} := \pm H$ is the stabilizer of v_0 in G. We have a generalized Cartan decomposition $G = \tilde{H}AK_1$ in the sense that every g is of the form hak for unique $h \in \tilde{H}, a \in A, k \in K_1$. And for $g = h(s)a(t)k$, $d\mu(g) = \sinh(t)dsdtdk$ defines a Haar measure on G, where $dk = (1/2\pi)dk(\theta)$, and ds, dt and $d\theta$ are Lebesgue measures. As $v_0 G = \pm H\backslash G \simeq A \times K_1$, $\sinh(t)dtdk$ defines an invariant measure on $v_0 G$. We consider the volume forms on G and $v_0 G$ with respect to these measures. Via the map ι, these define invariant measures on G_0 and $v_0 G_0$ as well.

Denote by Γ the pre-image of Γ_0 under ι. Then $\mathrm{Stab}_\Gamma(v_0) = \tilde{H} \cap \Gamma = \{\pm I\}$.

For each $T > 1$, define a function on $\Gamma\backslash G$:

$$F_T(g) := \sum_{\gamma \in \pm I\backslash\Gamma} \chi_{B_T}(v_0\gamma g).$$

Proposition 5.1. *For any $\Psi \in C_c^\infty(\Gamma\backslash G)$,*

$$\langle F_T, \Psi \rangle = \frac{T \log T \mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G)} \cdot 2 \int_{K_1} \frac{1}{\|v^+ k\|} dk + O(\mathcal{S}^\dagger(\Psi)T)$$

where $v^{\pm} = \frac{\sqrt{d}}{2}(e_1 \pm e_3)$. *Here the implied constant depends only on the support of* Ψ *and* $\mathcal{S}^{\dagger}(\Psi)$.

Proof. Note that $v_0 = v^+ + v^-$ and $v_0 a(t) = e^t v^+ + e^{-t} v^-$. Since $B_T = \{v_0 a(t) k : \|v_0 a(t) k\| < T,\, t \in \mathbb{R},\, k \in K_1\}$, we have

$$\begin{aligned}
\langle F_T, \Psi \rangle &= \int_{\Gamma\backslash G} \sum_{\gamma \in \pm I \backslash \Gamma} \chi_{B_T}(v_0 \gamma g) \Psi(g) d\mu(g) \\
&= \int_{k \in K_1} \int_{\|v_0 a(t) k\| < T} \left(\int_{h(s) \in \pm I \backslash \tilde{H}} \Psi(h(s) a(t) k) ds \right) \sinh(t) dt dk \\
&= \int_{k \in K_1} \int_{\|v_0 a(t) k\| < T} \left(\int_{s \in \mathbb{R}} \Psi(h(s) a(t) k) ds \right) \sinh(t) dt dk.
\end{aligned}$$

Since $v_0\Gamma$ is discrete and $H \cap \Gamma$ is trivial, it follows that $\Gamma\backslash\Gamma H$ is closed and non-compact in $\Gamma\backslash G$. Now fix any $k \in K_1$. Hence by Theorem 4.2 and Lemma 4.4,

$$\begin{aligned}
&\int_{t \gg 1, \|v_0 a(t) k\| < T} \left(\int_{s \in \mathbb{R}} \Psi(h(s) a(t) k) ds \right) \sinh(t) dt \\
&= \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{t \gg 1, e^t \|v^+ k\| < T + O(1)} (2t\mu(\Psi) + O(1)\mathcal{S}^{\dagger}(\Psi))(e^t/2 + O(1)) dt \\
&= \frac{T \log T \mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G) \cdot \|v^+ k\|} + O(T)\mathcal{S}^{\dagger}(\Psi).
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_{t \ll -1, \|v_0 a(t) k\| < T} \left(\int_{s \in \mathbb{R}} \Psi(h(s) a(t) k) ds \right) \sinh(t) dt \\
&= \int_{t \gg 1, \|v_0 a(-t) k\| < T} \left(\int_{s \in \mathbb{R}} \Psi(h(s) a(-t) k) ds \right) \sinh(t) dt \\
&= \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{t \gg 1, e^t \|v^- k\| < T + O(1)} (2t\mu(\Psi) + O(1)\mathcal{S}^{\dagger}(\Psi))(e^t/2 + O(1)) dt \\
&= \frac{T \log T \mu(\Psi)}{\mathrm{vol}(\Gamma\backslash G) \|v^- k\|} + O(T)\mathcal{S}^{\dagger}(\Psi).
\end{aligned}$$

Since $v^- k(\pi) = -v^+$,

$$\int_{k \in K_1} \|v^- k\|^{-1} dk = \int_{k \in K_1} \|v^+ k(\pi) k\|^{-1} dk = \int_{K_1} \|v^+ k\|^{-1} dk.$$

The required formula can be deduced in a straightforward manner from this. $\square$

Fix a non-negative function $\psi \in C_c^{\infty}(G)$ whose support injects to $\Gamma\backslash G$ and with integral $\int \psi(g)\, d\mu(g) = 1$. Consider a function ξ_T on $\mathbb{R}^3$ defined by

$$\xi_T(x) = \int_{g\in G} \chi_{B_T}(xg)\psi(g)d\mu(g).$$

Then the sum $\sum_{\gamma\in\pm I\backslash\Gamma} \xi_T(v_0\gamma)$ is a smoothed over counting satisfying

$$\sum_{\gamma\in\pm I\backslash\Gamma} \xi_T(v_0\gamma) \asymp \# v_0\Gamma \cap B_T.$$

Theorem 5.2. *As $T\to\infty$,*

$$\sum_{\gamma\in\pm I\backslash\Gamma} \xi_T(v_0\gamma) = \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \cdot \int_{k\in K_1} \frac{1}{\|v^+k\|} dk + O(T).$$

Proof. It is not hard to verify that

$$\sum_{\gamma\in\pm I\backslash\Gamma} \xi_T(v_0\gamma) = \langle F_T, \Psi\rangle$$

where $\Psi(\Gamma g) = \sum_{\gamma\in\Gamma}\psi(\gamma g)$. Therefore the claim follows from Proposition 5.1. □

Theorem 5.3. *For $T\gg 1$, we have*

$$\#\{w\in v_0\Gamma : \|w\| < T\} = \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|v^+k\|} dk \cdot (1 + (\log T)^{-\alpha})$$

where $\alpha = 0.25$.

Proof. Note that $F_T(I) = \#\{w\in v_0\Gamma : \|w\| < T\}$. For each $\epsilon>0$, let $\mathcal{O}_\epsilon = \{g\in G : \|g-I\|_\infty \le \epsilon\}$. There exists $0<\ell\le 1$ such that for all small $\epsilon>0$,

$$\mathcal{O}_{\ell\epsilon}B_T \subset B_{(1+\epsilon)T}, \quad B_{(1-\epsilon)T} \subset \cap_{u\in\mathcal{O}_{\ell\epsilon}} uB_T. \tag{5.1}$$

Let ψ^ϵ be a non-negative smooth function on G supported in $\mathcal{O}_{\ell\epsilon}$ and with integral $\int\psi^\epsilon d\mu = 1$ and define $\Psi^\epsilon\in C_c^\infty(\Gamma\backslash G)$ by $\Psi^\epsilon(\Gamma g) := \sum_{\gamma\in\Gamma}\psi^\epsilon(\gamma g)$.

Using (5.1), we have

$$\langle F_{(1-\epsilon)T}, \Psi^\epsilon\rangle \le F_T(I) \le \langle F_{(1+\epsilon)T}, \Psi^\epsilon\rangle.$$

Note that $\mathcal{S}_1(\Psi^\epsilon) = O(\epsilon^{-3/2})$ and $\|(\Psi^\epsilon)\|_{C^1} = O(\epsilon^{-3})$ so that $\mathcal{S}^\dagger = O(\epsilon^{-3})$. Therefore by Proposition 5.1

$$\begin{aligned}\langle F_{(1\pm\epsilon)T}, \Psi^\epsilon\rangle &= \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|v^+k\|} dk + O(\epsilon T\log T) + O(\mathcal{S}^\dagger(\Psi^\epsilon)T) \\ &= \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|v^+k\|} dk + O(\epsilon T\log T) + O(\epsilon^{-3}T) \\ &= \frac{2T\log T}{\mathrm{vol}(\Gamma\backslash G)} \int_{K_1} \frac{1}{\|v^+k\|} dk(1 + (\log T)^{-1/4}),\end{aligned}$$

where the last equality follows by we putting $\epsilon = (\log T)^{-1/4}$. □

Proof of Theorem 1.2. The computation in the proof of Proposition 5.1 also shows that

$$\mathrm{vol}(B_T) = \int_{k\in K_1}\int_{\|v_0 a(t)k\|<T} \sinh(t)dtdk = T\int_{k\in K}\frac{1}{\|v^+k\|}dk + O(\log T). \tag{5.2}$$

From Theorem 5.3, it follows that

$$F_T(I) = \frac{2\log T\,\mathrm{vol}(B_T)}{\mathrm{vol}(\Gamma\backslash G)}(1+O(\log T)^{-\alpha}). \tag{5.3}$$

Since $F_T(I) = \#(v_0\Gamma\cap B_T)$, this completes the proof of the first claim (1). The second claim (2) follows from Proposition 5.1. □

6. Orbital counting for general representations of $\mathrm{SL}_2(\mathbb{R})$

Let $G = \mathrm{SL}_2(\mathbb{R})$. For $s\in\mathbb{R}$, define

$$h(s) = \begin{bmatrix}\cosh(s/2) & \sinh(s/2)\\ \sinh(s/2) & \cosh(s/2)\end{bmatrix},\ a(s) = \begin{bmatrix}e^{s/2} & 0\\ 0 & e^{-s/2}\end{bmatrix},\ k(\theta) = \begin{bmatrix}\cos(\theta/2) & -\sin(\theta/2)\\ \sin(\theta/2) & \cos(\theta/2)\end{bmatrix}$$

Put $H = \{h(s) : s\in\mathbb{R}\}$, $A^+ = \{a(t) : t>0\}$, and $K_1 = \{k(\theta) : \theta\in[0,2\pi]\}$, here K_1 is half of the circle group. Put $w_0 = k(\pi)$. Then $\{\pm I\}\backslash G = HA^+K_1\cup Hw_0A^+K_1$, $w_0^{-1}h(s)w_0 = h(-s)$ and $w_0^{-1}a(t)w_0 = a(-t)$.

Let V be any finite dimensional representation of G and $v_0\in G$ be such that H is the stabilizer subgroup of v_0 in G, i.e., $H = G_{v_0}$ where $G_{v_0} = \{g\in G : v_0g = v_0\}$. Assume that V is linearly spanned by v_0G. Then if e^{mt} is the highest eigenvalue for $a(t)$-action on V, then $m\in\mathbb{N}$, and the G action factors through $\{\pm I\}\backslash G = \mathrm{PSL}_2(\mathbb{R})\cong\mathrm{SO}(2,1)^0$.

For example, let V_m denote the $(2m+1)$-dimensional space of real homogeneous polynomials of degree $2m$ in two variables, and consider the standard right action of $g\in\mathrm{SL}(2,\mathbb{R})$ on $P(x,y)\in V_m$ by $(Pg)(x,y) = P((x,y)g)$, where $(x,y)\left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] = (ax+cy, bx+dy)$. Let $v_0(x,y) = (x^2-y^2)^m$. Then $G_{v_0} = H\mathcal{W}$, where $\mathcal{W} = \{\pm I\}$ if m is odd and $\mathcal{W} = \{\pm I, \pm w_0\}$ if m is even. Moreover, $\{P\in V_m : Ph = P,\ \text{for all } h\in H\} = \mathbb{R}v_0$. A general finite dimensional representation of G with a nonzero H-fixed vector is a direct sum of such irreducible representations, and v_0 is a sum of one nonzero H-fixed vector from each of the irreducible representations; we assume that V is a span of v_0G.

Theorem 6.1. *Let V, v_0 and m be as above. Suppose that Γ is a lattice in G, $v_0\Gamma$ is discrete, and $\Gamma_{v_0} := \Gamma\cap G_{v_0}$ is finite. Let $\|\cdot\|$ be any norm on V, and $v_0^+ = \lim_{t\to\infty} v_0a_t/\|v_0a_t\|$. Let C be an open subset of $\{v\in V : \|v\| = 1\}$ such that $\Theta = \{\theta\in[0,2\pi] : v_0^+k(\theta)\in\mathbb{R}C\}$ has positive Lebesgue measure, and $\{\theta\in[0,2\pi] : v_0^+k(\theta)\in\mathbb{R}(\overline{C}\smallsetminus C)\}$ has zero Lebesgue measure. Then for $T\gg 1$,*

$$\#(v_0\Gamma\cap[0,T]C) \tag{6.1}$$
$$= \frac{4(2\pi)^{-1}\int_\Theta \|v_0^+k(\theta)\|^{-1/m}\,d\theta}{|\Gamma_{v_0}|\cdot\mathrm{vol}_G(\Gamma\backslash G)}\times\frac{\log T}{m}T^{1/m}(1+O((\log T)^{-\alpha}))$$

where vol_G *is given by the Haar integral* $dg = \sinh(t)dtdsd\theta$ *on* G*, where* $g = h(s)a(t)k(\theta)$*, and* $\alpha = \frac{1}{4}$.

Moreover, if $C \subset V$ *satisfies* $\mathbb{R}\overline{C} \cap v_0^+ K_1 = \emptyset$*, then* $\#(v_0\Gamma \cap \mathbb{R}C) < \infty$.

Proof. The result can be deduced by the arguments as in the proof of Theorem 5.3; one may also use the basic ideas from [14] about using the highest weight. □

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