

UNIPOTENT FLOWS ON PRODUCTS OF $\mathrm{SL}(2, K)/\Gamma$ 'S

by

Nimish A. Shah

Abstract. — We will give a simplified and a direct proof of a special case of Ratner's theorem on closures of individual orbits of unipotent flows; namely, the case of orbits of the diagonally embedded unipotent subgroup acting on $\mathrm{SL}(2, K)/\Gamma_1 \times \cdots \times \mathrm{SL}(2, K)/\Gamma_n$, where K is a locally compact field of characteristic 0 and each Γ_i is a cocompact discrete subgroup of $\mathrm{SL}(2, K)$. This special case of Ratner's theorem plays a crucial role in the proofs of uniform distribution of Heegner points by Vatsal, and Mazur conjecture on Heegner points by C. Cornut; and their generalizations in their joint work on CM-points and quaternion algebras. We will also deduce the corresponding uniform distribution theorem assuming Ratner's classification of ergodic invariant measures as a consequence of our proof. A purpose of this article is to make the ergodic theoretic results accessible to a wide audience.

Résumé (Flots unipotents sur des produits de $\mathrm{SL}(2, K)/\Gamma$). — Nous présentons une preuve directe et simplifiée d'un cas particulier du théorème de Ratner sur les ensembles d'adhérence de certaines orbites du flot unipotent. Plus précisément nous étudions les orbites de sous-groupes unipotents agissant sur $\mathrm{SL}(2, K)/\Gamma_1 \times \cdots \times \mathrm{SL}(2, K)/\Gamma_n$, où K est un corps localement compact de caractéristique 0 et où chaque Γ_i est un réseau cocompact de $\mathrm{SL}(2, K)$. Ce cas particulier du théorème de Ratner joue un rôle important dans les preuves de Vatsal et Cornut de résultats sur les points de Heegner. À partir de notre démonstration du théorème de Ratner sur la classification des mesures ergodiques invariantes, nous obtenons également un théorème de distribution uniforme. Un des buts de cet article est de rendre accessibles à un large public des théorèmes ergodiques difficiles.

1. Introduction

In the mid seventies M.S. Raghunathan conjectured that dynamical properties of individual orbits of unipotent flows on finite volume homogeneous spaces of semisimple Lie groups show a remarkable algebraic behaviour; namely, the closure of any

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non-periodic orbit is a finite volume homogeneous space of a larger subgroup. This conjecture was motivated by an approach to resolve Oppenheim conjecture on values of quadratic forms at integral points. A precise form of Raghunathan's conjecture, and its important measure theoretic analogues were formulated by S.G. Dani, who also verified those conjectures for horospherical flows in the early eighties. This work attracted greater attention to the Raghunathan conjecture and its extensions. It generated a lot of excitement when in the late eighties G.A. Margulis fully settled the Oppenheim conjecture in affirmation by verifying Raghunathan's conjecture for certain very specific cases. This seems to be the first major triumph of the power of ergodic theoretic methods in solving long standing number theoretic problems. Soon after, by the beginning of the nineties M. Ratner obtained complete affirmative resolution of the above mentioned conjectures on unipotent flows, and also proved the uniform distribution for individual orbits, through a series of papers [15, 16, 17, 18] involving many original deep ideas from dynamics and Lie theory. Ratner's theorems were very powerful tools ready to be used. Since then several types of new Diophantine approximation results have been proved using the algebraic properties of unipotent dynamics. The dynamical results were also generalized for p -adic Lie groups by Ratner [19], as well as Margulis and Tomanov [12] whose work also provided more transparent and conceptual proofs of these results making effective use of techniques of from algebraic groups and entropy theory.

What really surprises me about the p -adic case of Ratner theorem is the way it gets utilized in the work of V. Vatsal [24] on uniform distribution of Heegner points. Using a combination of remarkable number theoretic results and his observations, Vatsal reduced the study of distribution of Heegner points to the following combinatorial problem:

Let \mathcal{T} be a $p+1$ -regular tree for a prime p , and $\mathcal{G} = \mathcal{T}/\Gamma$ be a finite quotient graph, where Γ is group of automorphisms of \mathcal{T} with finite stabilizers of vertices. Let Γ' be a conjugate of Γ in $\text{Aut}(\mathcal{T})$ such that Γ and Γ' do not have a common subgroup of finite index; that is, they are not commensurable. Fix a base point v_0 in \mathcal{T} , and let $\mathcal{T}(n)$ denote the vertices of \mathcal{T} at the distance n from v_0 . Consider the finite graph $\mathcal{G}' = \mathcal{T}/\Gamma'$, and let $q : \mathcal{T} \rightarrow \mathcal{G}$ and $q' : \mathcal{T} \rightarrow \mathcal{G}'$ denote the natural quotient maps. We embed \mathcal{T} diagonally in $\mathcal{T} \times \mathcal{T}$, and project it onto $\mathcal{G} \times \mathcal{G}'$; more precisely we consider the map $\bar{\Delta} : \mathcal{T} \rightarrow \mathcal{G} \times \mathcal{G}'$ given by $\bar{\Delta}(v) = (q(v), q'(v))$. The question is whether $\bar{\Delta}(\mathcal{T}(n))$ surjects onto $\mathcal{G} \times \mathcal{G}'$ for large n , and does it visit all points of the product graph with the correct limiting frequency as $n \rightarrow \infty$?

His question was motivated by the fact that on a finite non-bipartite regular graph, a random walk of step n is uniformly distributed as $n \rightarrow \infty$. On the other hand in this case it is already a question whether the image of the diagonally embedded \mathcal{T} is surjective on $\mathcal{G} \times \mathcal{G}'$. In the actual situation of interest, $\mathcal{T} \cong \text{SL}_2(\mathbb{Z}_p) \backslash \text{SL}_2(\mathbb{Q}_p)$, realized as the Bruhat-Tits tree, and Γ is a cocompact discrete subgroup of $\text{SL}_2(\mathbb{Q}_p)$ so that \mathcal{G} is associated to the quotient by the right action of Γ , and Γ' is a conjugate of Γ in $\text{SL}_2(\mathbb{Q}_p)$. Therefore the surjectivity of the diagonal embedding follows if we

can show that the set $\Gamma\Gamma'$ is dense in $\mathrm{SL}_2(\mathbb{Q}_p)$; or more generally, if the element-wise product of any two non-commensurable lattices in $\mathrm{SL}_2(\mathbb{Q}_p)$ is a dense subset of $\mathrm{SL}_2(\mathbb{Q}_p)$.

Vatsal asked this question to Raghunathan, who realizing this as a question about orbit closures for Γ -action on $\mathrm{SL}_2(\mathbb{Q}_p)/\Gamma'$ consulted Dani. The same question was earlier posed and answered in author's Masters thesis [22] for lattices in $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{C})$, and later in [23] for the lattices in arbitrary real semisimple Lie groups using Ratner's theorem. Dani informed Vatsal that his guess was indeed correct, and showed how to deduce the density result using orbit closure results for actions of semisimple subgroup on p -adic homogeneous spaces. Later using Ratner's uniform distribution results for unipotent flows on the homogeneous space $\mathrm{SL}_2(\mathbb{Q}_p)/\Gamma \times \mathrm{SL}_2(\mathbb{Q}_p)/\Gamma'$, Vatsal also deduced the uniform distribution for the set $\bar{\Delta}(\mathcal{T}(n))$ as $n \rightarrow \infty$ in $\mathcal{G} \times \mathcal{G}'$.

It is remarkable that the above seemingly combinatorial question about products of certain finite graphs turns out to be intimately connected to deep algebraic behaviour of ergodic properties of unipotent flows; and these flows are analysed using local arguments involving the adjoint actions on the Lie algebra near the origin.

In what follows, we would like to give a self-contained proof of the above surjectivity of the diagonal embedding of a tree in the product of several regular finite graphs. The published proofs of Ratner's theorem for p -adic Lie groups are quite intricate and they require taking care of many different possibilities associated to the general case. Our purpose here is to follow the original arguments of Margulis [10] used in his proof of Oppenheim conjecture, as well as those used in its extensions by Dani and Margulis [6], along with additional observations to give an elementary proof.

In later works [2, 3, 25], Vatsal and Cornut also require the closure and the uniform distribution results for products of several copies of $\mathrm{SL}_2(K)$ for any finite extension K of \mathbb{Q}_p . To take care of this, we have given our proofs for all local fields K of characteristic 0 in place of \mathbb{Q}_p , without introducing any extra complications.

After the introduction, the article gets divided into two independent parts. In the first half (§§2–4) we give a proof of the orbit closure result. For the case of $G = \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ this proof is self-contained. It may be noted that this case suffices to prove that $\overline{\Gamma\Gamma'} = \mathrm{SL}_2(K)$ as needed by Vatsal [24] and Cornut [2]. For the case of $\mathrm{SL}_2(K)^n$ ($n \geq 3$), near the end of §4, we need to assume a technical result on 'uniform recurrence in linear time' on the 'non-singular' set for the case of $\mathrm{SL}_2(K)^{n-1}$. The results on closures of orbits of diagonally embedded $\mathrm{SL}_2(K)$ and density of products of non-commensurable lattices are proved in §5.

In the second half of this article (§§6–9) we provide a complete proof of the above mentioned result on uniform recurrence in linear time (Theorem 1.9). In fact, the main content of the proof is precisely the same result which allows one to deduce Ratner's equidistribution theorem for unipotent flows from Ratner's description of ergodic invariant measures. Therefore instead of our proof, the reader could refer to particular parts of Ratner's original proof (see [18, pp. 257–264] for the archimedean case and [19, pp. 376–380] for the non-archimedean case). On the other hand, in the product case we are able to reduce the problem to the product of only two copies

of $\mathrm{SL}_2(K)$'s. Further simplification comes from the fact that in the case of compact quotients of $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ the singular sets are non-self intersecting.

As an additional consequence of the proof, in §10 we also deduce the equidistribution result by assuming the measure classification theorem.

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1.1. Notation. — Let K denote a local field of characteristic zero. Let $n \geq 1$ be given. Let $G = \mathrm{SL}_2(K)^n$. For $\emptyset \neq J \subset \{1, \dots, n\}$, let

$$\begin{aligned} G_J &= \{(g_1, \dots, g_n) \in G : g_k = e, \forall k \notin J\} \\ H_J &= \{(g_1, \dots, g_n) \in G_J : g_i = g_j, \forall i, j \in J; g_k = e, \forall k \notin J\}. \end{aligned}$$

Then $G_J \cong \mathrm{SL}_2(K)^{|J|}$ and $H_J \cong \mathrm{SL}_2(K)$, which is diagonally embedded in $\mathrm{SL}_2(K)^{|J|}$, where $|J|$ denotes the cardinality of J . If $|J| = 1$ then $H_J = G_J$.

Let \mathcal{C} denote the collection of sets of the form $\mathcal{J} = \{J_1, \dots, J_m\}$, where $1 \leq m \leq n$, $J_i \subset \{1, \dots, n\}$, $J_i \neq \emptyset$, and $J_i \cap J_j = \emptyset$ for all $i \neq j$. Define

$$H_{\mathcal{J}} = H_{J_1} \cdots H_{J_m}.$$

Let

$$\begin{aligned} w_1(t) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \forall t \in K; & d_1(\alpha) &= \begin{pmatrix} \alpha & \\ & \alpha^{-1} \end{pmatrix}, \forall \alpha \in K^\times; \\ W &= \{w(\mathbf{t}) = (w_1(t_1), \dots, w_1(t_n)) : \mathbf{t} = (t_1, \dots, t_n) \in K^n\} \\ A &= \{(d_1(\alpha_1), \dots, d_1(\alpha_n)) : \alpha_j \in K^\times\}. \end{aligned}$$

We also consider

$$\begin{aligned} H &= \{(g, \dots, g) \in G : g \in \mathrm{SL}_2(K)\} = H_{\{1, \dots, n\}} \\ U &= \{u(t) = w(t, \dots, t) : t \in K\} = W \cap H \\ D &= \{d(\alpha) = (d_1(\alpha), \dots, d_1(\alpha)) : \alpha \in K^\times\} = A \cap H, \\ U^\perp &= \{w(t_1, \dots, t_{n-1}, 0) : t_j \in K\}. \end{aligned}$$

Assumption. — For $j = 1, \dots, n$, let Γ_j be a discrete subgroup of $G_{\{j\}}$ such that $G_{\{j\}}/\Gamma_j$ is compact, and let $\Gamma = \Gamma_1 \cdots \Gamma_n$. Then

$$(1) \quad G/\Gamma \cong G_{\{1\}}/\Gamma_1 \times \cdots \times G_{\{n\}}/\Gamma_n.$$

In this article, we will consider the action of G on G/Γ by left translations; that is, if $g \in G$ and $x \in G/\Gamma$ then $gx := (gg_1)[\Gamma]$, where $g_1 \in G$ is such that $x = g_1[\Gamma]$ is the coset of g_1 in G/Γ . Also for any $A \subset G$ and $X \subset G/\Gamma$, we define $AX = \{ax : a \in A, x \in X\} \subset G/\Gamma$.

We endow G/Γ with the quotient topology; that is, a set $X \subset G/\Gamma$ is closed (or open) if and only if its inverse image in G is closed (resp. open). Thus, given any $A \subset G$, and $x = g[\Gamma] \in G/\Gamma$ for some $g \in G$, the set Ax is closed in G/Γ if and only if $Ag\Gamma$ is a closed subset of G .

Let

$$\mathcal{C}_0 := \{\mathcal{J} \in \mathcal{C} : \cup_{J \in \mathcal{J}} J = \{1, \dots, n\}\} = \{\mathcal{J} \in \mathcal{C} : H_{\mathcal{J}} \supset U\}.$$

1.2. Statements of the main results

Theorem 1.1. — Given $n \geq 1$, let G , Γ , U , and the other notation be as above. For any $x \in G/\Gamma$, there exists $\mathcal{J} \in \mathcal{C}_0$ and $w \in W$ such that

$$\overline{Ux} = (wH_{\mathcal{J}}w^{-1})x.$$

Definition 1.1. — A multi-parameter subgroup of W is a subgroup of W of the form $V = \{w(\mathfrak{t}) : \mathfrak{t} \in \mathfrak{V}\}$, where \mathfrak{V} is a subspace of K^n . We define $\dim V := \dim_K(\mathfrak{V})$.

Corollary 1.2. — Given $n \geq 1$, let G , Γ , and the other notation be as above. Let V be a multi-parameter subgroup of W . Then for any $x \in G/\Gamma$, there exists $\mathcal{J} \in \mathcal{C}$ and $w \in W$ such that $\overline{Vx} = wH_{\mathcal{J}}w^{-1}x$.

Corollary 1.3. — For any $x \in G/\Gamma$, there exists $\mathcal{J} \in \mathcal{C}_0$ such that

$$\overline{Hx} = H_{\mathcal{J}}x.$$

In order to describe the relation between H , Γ_i 's, and \mathcal{J} , we need some definitions.

In a topological group, two infinite discrete subgroups Λ and Λ' are said to be *commensurable*, if $\Lambda \cap \Lambda'$ is a subgroup of finite index in both, Λ and Λ' .

For $i = 1, \dots, n$, let $p_i : G \rightarrow \mathrm{SL}_2(K)$ denote the projection on the i -th factor. Let $x_0 = e\Gamma$ denote the coset of the identity in G/Γ .

Proposition 1.4. — Suppose that $H_{\mathcal{J}}x_0$ is compact for some $\mathcal{J} \in \mathcal{C}_0$. Then for any $J \in \mathcal{J}$ and any $i, j \in J$, the lattices $p_i(\Gamma_i)$ and $p_j(\Gamma_j)$ in $\mathrm{SL}_2(K)$ are commensurable.

Combining this fact with Corollary 1.3 immediately gives the next result. Note that $H_{\mathcal{J}} = G$ if and only if $\mathcal{J} = \{\{1\}, \dots, \{n\}\}$.

Corollary 1.5. — If $p_i(\Gamma_i)$ and $p_i(\Gamma_j)$ are not commensurable for all $i \neq j$ then $\overline{H\Gamma} = G$. □

More generally, we will show the following:

Corollary 1.6. — Let $\mathcal{J} \in \mathcal{C}_0$ be the partition of $\{1, \dots, n\}$ such that for any i, j , we have $i, j \in J$ for some $J \in \mathcal{J}$ if and only if $p_i(\Gamma_i)$ and $p_i(\Gamma_j)$ are commensurable. Then $\overline{Hx_0} = H_{\mathcal{J}}x_0$.

Corollary 1.7. — Suppose that Γ_1 and Γ_2 are cocompact noncommensurable lattices in $\mathrm{SL}_2(K)$. Then $\Gamma_1\Gamma_2$ is dense in $\mathrm{SL}_2(K)$.

1.3. Singular set for the U -action. — In the proof of Theorem 1.1 we will need to understand the set of points for which the closure of the U -orbit is contained in a closed orbit of a strictly lower dimensional subgroup of G .

More precisely, we say that a point $x \in G/\Gamma$ is *singular* (for the U -action on G/Γ) if $Ux \subset (wH_{\mathcal{J}}w^{-1})x$ and $(wH_{\mathcal{J}}w^{-1})x$ is compact for some $\mathcal{J} \in \mathcal{C}_0$ and $w \in U^{\perp}$, such that $H_{\mathcal{J}} \neq G$.

The set of singular points (for the U -action on G/Γ) is denoted by $\mathcal{S}(U, \Gamma)$.

Note that if $n = 1$ then $\mathcal{S}(U, \Gamma) = \emptyset$.

Proposition 1.8. — *There always exists a non-singular point for the U -action on G/Γ ; that is $G/\Gamma \neq \mathcal{S}(U, \Gamma)$.*

This fact can be proved quickly as follows: There exists a unique G -invariant probability measure ν on G/Γ ; that is, $\nu(gE) = \nu(E)$ for any measurable set $E \subset G/\Gamma$ and any $g \in G$. By Moore's ergodicity theorem, U -acts ergodically on G/Γ with respect ν . Since $\nu(E) > 0$ for any nonempty open subset of G/Γ , by Hedlund's lemma, $\overline{Uy} = G/\Gamma$ for ν -almost all $y \in G/\Gamma$. Hence $\nu(\mathcal{S}(U, \Gamma)) = 0$.

In subsection 7 we will also give a simple proof of Proposition 1.8 (without using Moore's ergodicity) by showing that $\mathcal{S}(U, \Gamma)$ is the image of a union of countably many algebraic subvarieties of G of strictly lower dimension.

As mentioned before following property of unipotent flows, called *uniform recurrence in linear time* in [6], at the end of the proof of Theorem 1.1.

Theorem 1.9. — *Let $x_i \rightarrow x$ be a sequence in G/Γ such that $x \notin \mathcal{S}(U, \Gamma)$. Then for any sequence $t_i \rightarrow \infty$ in K and a compact neighbourhood \mathfrak{D} of 0 in K , there exists $t'_i \in (1 + \mathfrak{D})t_i$ for every $i \in \mathbb{N}$, such that, after passing to a subsequence, $u(t'_i)x_i \rightarrow y$ for some $y \in G/\Gamma \setminus \mathcal{S}(U, \Gamma)$.*

Note that if $G = \mathrm{SL}_2(K)$; that is $n = 1$, then Theorem 1.9 is a triviality, because $\mathcal{S}(U, \Gamma) = \emptyset$ in this case.

Moreover for proving Theorem 1.1 for any given n , we will need to use Theorem 1.9 only for $G = \mathrm{SL}_2(K)^m$, where $m < n$.

Therefore the proof of Theorem 1.1 for $n = 2$ uses only the trivial case of Theorem 1.9; that is for $n = 1$. It may be noted that Corollary 1.7, which is of special interest, depends only on this case of Theorem 1.1.

The Theorem 1.9 is actually derived as a consequence of a more general result about limiting distribution of a sequence of U -trajectories on the singular set. Since the techniques of proving this result are very different from the remaining part of the proof of Theorem 1.1 we have included all those results in a second part of this article. In the second part of this article we will also prove the uniform distribution result assuming Ratner's description of ergodic U -invariant measures. In fact, the first part of this article uses some of the ideas which have their analogues in the classification of ergodic invariant measures for the U -action.

2. Preliminaries

2.1. A result in ergodic theory. — We recall a result due to G.D. Mostow, see [8, Prop. 1.5].

Proposition 2.1. — *For any $x \in G/\Gamma$, the orbit DWx is dense in G/Γ .*

Proof. — Take any $\alpha \in K$ such that $|\alpha|_p > 1$ and let $a = d(\alpha)$. By Mautner's Phenomenon (see [1, 13]), a acts ergodically on G/Γ . Therefore by Hedlund's lemma there exists $y \in G/\Gamma$ such that

$$(2) \quad \overline{\{a^i : i > 0\}y} = G/\Gamma.$$

Let a sequence $\{y_k\} \in \{a^i : i > 0\}y$ be such that $y_k \rightarrow x$ as $k \rightarrow \infty$.

Let $z \in G/\Gamma$ be given. Then by (2) there exists a sequence $i_k \rightarrow \infty$ such that $a^{i_k}y_k \rightarrow z$, as $k \rightarrow \infty$.

Let a sequence $g_k \rightarrow e$ in G be such that $y_k = g_kx$ for all k . Since $\mathrm{Lie}(G) = \mathbb{T}[\mathrm{Lie}(W)] \oplus \mathrm{Lie}(A) \oplus \mathrm{Lie}(W)$, there exist sequences $\mathbf{s}_k \rightarrow 0$ and $\mathbf{t}_k \rightarrow 0$ in K^n , $d_k \rightarrow e$ in A , and a $k_0 \in \mathbb{N}$ such that

$$g_k = \mathbb{T}[w(\mathbf{s}_k)]d_kw(\mathbf{t}_k), \quad \forall k \geq k_0.$$

Therefore $a^{i_k}y_k = (a^{i_k}g_ka^{i_k})a^{i_k}y_k$. Now

$$a^{i_k}g_ka^{-i_k} = \mathbb{T}[w(\alpha^{-2i_k}\mathbf{s}_k)]d_kw(\alpha^{2i_k}\mathbf{t}_k) =: \delta_k w_k, \quad \forall k \geq k_0,$$

where $\delta_k := \mathbb{T}[w(\alpha^{-2i_k}\mathbf{s}_k)]d_k$ and $w_k := w(\alpha^{2i_k}\mathbf{t}_k) \in W$. Thus $a^{i_k}y_k = \delta_k w_k a^{i_k}x$ and $\delta_k \rightarrow e$. Therefore

$$w_ka^{i_k}x = \delta_k^{-1}(a^{i_k}y_k) \rightarrow \lim_{k \rightarrow \infty} a^{i_k}y_k = z.$$

Thus $z \in \overline{WDx} = \overline{DWx}$. This shows that $G/\Gamma \subset \overline{DWx}$. \square

The proofs of Mautner's phenomenon and Hedlund's lemma are very nice and short [1]. The above result deviates from the classical ergodic theory results in one essential way; namely it tells something about the dynamical property of each individual orbit, rather than of almost every orbit. It is due to this reason we are able to use the above result for problems in number theory.

2.2. Basic lemmas on minimal sets for group actions. — In this subsection let G be a locally compact second countable topological group acting continuously on a topological space Ω . For a subgroup F of G , a subset X of Ω is called *F-minimal* if X is closed, F -invariant, and does not contain any proper closed F -invariant subset. Thus if X is F -minimal then $\overline{Fx} = X$ for every $x \in X$. By Zorn's lemma, any compact F -invariant subset of Ω contains an F -minimal subset.

Lemma 2.2 (Margulis [11]). — *Let F , P and P' be subgroups of G such that $F \subset P \cap P'$. Let Y, Y' be closed subsets of Ω , and $M \subset G$ be any set. Suppose that*

1. $PY \subset Y, P'Y' \subset Y'$,
2. $mY \cap Y' \neq \emptyset$ for all $m \in M$, and

3. Y is compact and F -minimal.

Then $gY \subset Y'$ for all $g \in N_G(F) \cap \overline{P'MP}$.

In particular, if $Y' = Y$ then Y is invariant under the closed subgroup generated by $N_G(F) \cap \overline{P'MP}$.

Proof. — Let $g \in \overline{P'MP}$. There exist sequences $\{p'_i\} \subset P'$, $\{m_i\} \subset M$, and $\{p_i\} \subset P$ such that $p'_i m_i p_i \rightarrow g$ as $i \rightarrow \infty$.

By 2), for each m_i there exists a $y_i \in Y$ such that $m_i y_i \in Y'$. Since $\{p_i^{-1} y_i\} \subset Y$ and Y is compact, by passing to subsequences, we may assume that $p_i^{-1} y_i \rightarrow y$ for some $y \in Y$. Now $\{p'_i m_i y_i\} \subset Y'$. Therefore as $i \rightarrow \infty$,

$$p'_i m_i y_i = (p'_i m_i p_i)(p_i^{-1} y_i) \rightarrow gy \in Y'.$$

Further if $g \in N_G(F)$, then

$$Y' \supset \overline{Fgy} = \overline{gFy} = g\overline{Fy} = gY,$$

where $\overline{Fy} = Y$ because Y is F -minimal. □

Lemma 2.3 (Margulis [11]). — Assume that G acts transitively on Ω . Let F and P , where $F \subset P$, be closed subgroups of G , and Y be a compact F -minimal subset of Ω . Suppose there exists $y \in Y$ and a neighbourhood Φ of the identity in G such that

$$(3) \quad \{g \in \Phi : gy \in Y\} \subset P.$$

Then $\overline{\eta(F)}$ is compact in P/P_y , where $P_y = \{g \in P : gy = y\}$ and $\eta : P \rightarrow P/P_y$ is the natural quotient map.

Proof. — It is enough to show that given a sequence $\{f_i\} \subset F$, the sequence $\{\eta(f_i)\}$ has a convergent subsequence.

To show this, we note that after passing through a subsequence, $f_i y \rightarrow z$ for some $z \in Y$. Since Ω is a homogeneous space of G , Φy is a neighbourhood of y in Ω . Now since Y is F -minimal, Fy is dense in Y , and hence there exists $f \in F$ such that $fz \in \Phi y$. Therefore by (3), $fz = p'y$ for some $p' \in P$. Hence $z = py$, where $p = f^{-1}p' \in P$. Thus $f_i y \rightarrow py$. Hence $(p^{-1}f_i)y \rightarrow y$.

Again by (3) there exists a sequence $p_i \rightarrow e$ in P such that $(p^{-1}f_i)y = p_i y$ for all large i . Thus $f_i y = pp_i y$; and hence $f_i^{-1}pp_i \in P_y$ for all large i . Therefore $\eta(f_i) = \eta(pp_i) \rightarrow \eta(p)$ as $i \rightarrow \infty$. □

2.3. Limit set of a sequence of unipotent trajectories on a vector space. — Later after applying Lemma 2.2, we will proceed further using the following result.

Proposition 2.4. — *Let $M \subset G \setminus \mathrm{N}_G(U)$ such that $e \in \overline{M}$. Then the closure of the subgroup generated by $\overline{UMU} \cap \mathrm{N}_G(U)$ contains either wDw^{-1} for some $w \in W$, or a nontrivial one-parameter subgroup of U^\perp .*

The proof of this proposition is based on the following general result [6, 10]: Let V be a finite dimensional vector space over K and $U = \{u(t)\}_{t \in K}$ be a nontrivial one-parameter unipotent subgroup of $\mathrm{GL}(V)$ and $\{p_i\}$ be a sequence of points in V such that each of the trajectories $\{u(t)p_i\}_{t \in K}$ is non-constant. Let L denote the space of U -fixed vectors in V . Now if $p_i \rightarrow p$ for some $p \in L$ then, after passing to a subsequence, the following holds: there exist a sequence $t_i \rightarrow \infty$ in K and a non-constant polynomial map $\phi : K \rightarrow V$ such that for any $s \in K$, we have $u(st_i)p_i \rightarrow \phi(s)$ as $i \rightarrow \infty$.

We will prove this only for the cases needed for our purpose.

Let $V = K^2$ and consider the standard linear action of $\{w_1(t)\}$ on K^2 . Let $I_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $L_0 = \{tI_0 : t \in K\}$ is the space of $\{w_1(t)\}$ -fixed vectors.

Lemma 2.5. — *Let $\{p_i\} \subset K^2 \setminus L_0$ be a sequence such that $p_i \rightarrow I_0$ as $i \rightarrow \infty$. Then, after passing to a subsequence, there exists a sequence $t_i \rightarrow \infty$ such that the following holds: Then for any $s \in K$,*

$$\lim_{i \rightarrow \infty} w_1(st_i) \cdot p_i = (1 + s)I_0.$$

Proof. — Write $p_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}$, $\forall i$. Since $p_i \notin L_0$, $b_i \neq 0$, $\forall i$. Put $t_i = b_i^{-1}$. Then for any $s \in K$, as $i \rightarrow \infty$,

$$w_1(st_i) \begin{pmatrix} a_i \\ b_i \end{pmatrix} = \begin{pmatrix} a_i + s \\ b_i \end{pmatrix} \rightarrow \begin{pmatrix} 1 + s \\ 0 \end{pmatrix}.$$

□

Let $I_1 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$. For $1 \leq m \leq n$, put

$$\begin{aligned} E_m &= M_2(K)^m \\ I_m &= (I_1, \dots, I_1) \in E_m \\ w_m(t) &= (w_1(t), \dots, w_1(t)) \in \mathrm{SL}_2(K)^m \\ L_m &= \{X \in E_m : w_m(t)Xw_m(-t) = X, \forall t \in K\} = (L_1)^m, \end{aligned} \tag{4}$$

Lemma 2.6 (Margulis). — *Let $\{X_i\} \subset E_m \setminus L_m$ be a sequence such that $X_i \rightarrow I_m$ as $i \rightarrow \infty$. Then after passing to a subsequence, there exist a sequence $t_i \rightarrow \infty$, and a nonconstant polynomial map $\psi : K \rightarrow K^m$ of degree at most 2 such that given any $s \in K$ and a sequence $s_i \rightarrow s$ in K ,*

$$(5) \quad \lim_{i \rightarrow \infty} w_m(s_i t_i) X_i w_m(-s_i t_i) = w_m(\psi(s)).$$

In particular, $\psi(0) = 0$.

Proof. — If we write $X_i = (X_i(1), \dots, X_i(m))$, where $X_i(j) \in M_2(K)$ for $1 \leq j \leq m$, and $X_i(j, t) = w_1(t)X_i(j)w_1(-t)$ then

$$w_m(t)X_iw_m(-t) = (X_i(1, t), \dots, X_i(m, t)).$$

Fix any $1 \leq j \leq m$. If $X_i(j) = \begin{pmatrix} a_i(j) & b_i(j) \\ c_i(j) & d_i(j) \end{pmatrix}$, then

$$(6) \quad X_i(j, t) = X_i(j) + \begin{pmatrix} c_i(j) & d_i(j) - a_i(j) \\ 0 & -c_i(j) \end{pmatrix} t + \begin{pmatrix} 0 & -c_i(j) \\ 0 & 0 \end{pmatrix} t^2,$$

If $X_i(j, t) = X_i(j)$ for all t , then $c_i(j) = 0 = d_i(j) - a_i(j)$, and we put $t_i(j) = \infty$. If $c_i(j) \neq 0$ or $d_i(j) - a_i(j) \neq 0$, then there exists $t_i(j) \in K$ such that

$$(7) \quad \max\{|(d_i(j) - a_i(j))t_i(j)|, |c_i(j)t_i(j)^2|\} = 1.$$

As $i \rightarrow \infty$, since $X_i(j) \rightarrow 0$, we have $a_i(j) - d_i(j) \rightarrow 1 - 1 = 0$ and $c_i(j) \rightarrow 0$. Therefore $t_i(j) \rightarrow \infty$, and hence $|c_i(j)t_i(j)| \leq |t_i(j)|^{-1} \rightarrow 0$ as $i \rightarrow \infty$.

Put

$$(8) \quad t_i = \min\{t_i(1), \dots, t_i(m)\}.$$

Since $X_i \notin (L_1)^m$, we have that $t_i < \infty$. Since $X_i \rightarrow I_m$, we have $t_i \rightarrow \infty$. By (7) and (8), after passing to a subsequence, for each $1 \leq j \leq m$, there exist $\alpha_j, \beta_j \in K$ such that

$$(9) \quad \lim_{i \rightarrow \infty} (d_i(j) - a_i(j))t_i = \alpha_j \quad \text{and} \quad \lim_{i \rightarrow \infty} -c_i(j)t_i^2 = \beta_j.$$

In particular, $c_i(j)t_i \rightarrow 0$ for all j . Now (5) follows from (6) and (9), where

$$\psi(s) = (\alpha_1 s + \beta_1 s^2, \dots, \alpha_m s + \beta_m s^2).$$

Due to (7), $|\alpha_{j_0}| = 1$ or $|\beta_{j_0}| = 1$ for some j_0 . Therefore ψ is nonconstant. \square

Proof of Proposition 2.4. — Let

$$E = E_{n-1} \times K^2 \quad \text{and} \quad \mathbf{p} = (I_{n-1}; I_0).$$

Define the linear action of G on E as follows: For any

$g = (g(1), \dots, g(n)) \in G$, and $X = (X(1), \dots, X(n-1); Y) \in E$,

$$(10) \quad g \cdot X = (g(1)X(1)g(2)^{-1}, \dots, g(n-1)X(n-1)g(n)^{-1}; g(n)Y).$$

Then

$$(11) \quad U = \{g \in G : g \cdot \mathbf{p} = \mathbf{p}\}.$$

Let $L = \{X \in E : U \cdot X = X\} = L_{n-1} \times L_0$. Then

$$(12) \quad N_G(U) = \{g \in G : g \cdot \mathbf{p} \in L\}.$$

We note that $N_G(U) = Z(G)DW$, where $Z(G) = \{(\pm I_1, \dots, \pm I_1) \in G\}$ is the center of G . Also

$$(13) \quad G \cdot \mathbf{p} = \mathrm{SL}_2(K)^{n-1} \times K^* I_0.$$

For $g \in G$ if $g \cdot \mathbf{p} \in \overline{UM \cdot \mathbf{p}}$ then there exist $u_i \in U$ and $m_i \in M$ such that $u_i m_i \cdot \mathbf{p} \rightarrow g \cdot \mathbf{p}$ as $i \rightarrow \infty$. Then $(g^{-1} u_i m_i) \cdot \mathbf{p} \rightarrow \mathbf{p}$. Therefore, by (13) there exists a sequence $\delta_i \rightarrow e$ in G such that $(g^{-1} u_i m_i) \cdot \mathbf{p} = \delta_i \cdot \mathbf{p}$ for all i . By (11) there exist $u'_i \in U$ such that $g^{-1} u_i m_i u'_i = \delta_i$ for each i . Therefore $u_i m_i u'_i \rightarrow g$.

Thus for any $g \in G$,

$$(14) \quad g \cdot \mathbf{p} \in \overline{UM \cdot \mathbf{p}} \cap L \Leftrightarrow g \in \overline{UMU} \cap N_G(U).$$

By (12), $M \cdot \mathbf{p} \cap L = \emptyset$ and $e \in \overline{M}$. Therefore there exists a sequence

$$(15) \quad \{X_i\} \subset M \cdot \mathbf{p} \subset E \setminus L,$$

such that $X_i \rightarrow \mathbf{p}$ as $i \rightarrow \infty$. By combining Lemma 2.5 and Lemma 2.6, after passing to subsequences, there exists a sequence $t_i \rightarrow \infty$ in K such that for any $s \in K$,

$$(16) \quad \lim_{i \rightarrow \infty} u(st_i) \cdot X_i = (w_{n-1}(\psi(s)); \phi(s)I_0) \in L,$$

where $\phi(s)$ is a polynomial of degree at most 1, $\phi(0) = 1$ and

$$\psi(s) = (\psi_1(s), \dots, \psi_{n-1}(s)) \in K^{n-1}$$

is a polynomial map of degree at most 2, $\psi(0) = \mathbf{0}$, and ψ or ϕ is non-constant. We define $\psi'_k = \sum_{j=k}^{n-1} \psi_j$ for $1 \leq k \leq n-1$, and

$$\psi'(s) = (\psi'_1(s), \dots, \psi'_{n-1}(s), 0) \in K^n.$$

Then $\psi' : K \rightarrow K^n$ is a polynomial of degree at most 2, and ψ' is constant if and only if ψ is constant.

For any $s \in K$ such that $\phi(s) \neq 0$, we put

$$(17) \quad \Phi(s) = w(\psi'(s))d(\phi(s)).$$

Therefore due to (10),

$$(18) \quad \Phi(s) \cdot \mathbf{p} = (w_1(\psi_1(s)), \dots, w_1(\psi_{n-1}(s)); \phi(s)I_0) \in L.$$

Therefore by (15)–(18),

$$\Phi(s) \cdot \mathbf{p} \in \overline{U \cdot (M \cdot \mathbf{p})} \cap L.$$

Hence by (14) and (17), for all $s \in K$ with $\phi(s) \neq 0$,

$$\Phi(s) \in DU^\perp \cap \overline{UMU}.$$

Now the conclusion of the proposition follows from Lemma 2.9 proved below. \square

2.4. Some more elementary lemmas. — It is straightforward to verify the following.

Lemma 2.7. — *Let $m \in \mathbb{N}$ and $\psi : K \rightarrow K^m$ be a polynomial map such that $\deg(\psi) \geq 1$. Then there exists a nonzero vector $v \in K^m$ such for any $s \in K$,*

$$(19) \quad \psi(t + st^{-q}) - \psi(t) \rightarrow sv \quad \text{as } t \rightarrow \infty,$$

where $q = \deg(\psi) - 1$. In particular, any closed additive subgroup generated by $\psi(K)$ contains a nonzero subspace of K^m . \square

Lemma 2.8. — *Let F be an abelian subgroup of DW . Then either $F \subset \{d(\pm 1)\}W$ or there exists $v \in W$ such that $F \subset vDv^{-1}$.*

Proof. — Suppose $d(\alpha)w(t) \in F$ for some $\alpha \in K^*$ such that $\alpha \neq \pm 1$. Let $v = w((1 - \alpha^2)^{-1}t)$. Then $v^{-1}d(\alpha)w(t)v = d(\alpha)$. Therefore $v^{-1}Fv$ is contained in the centralizer of $d(\alpha)$.

Now for any $\beta \in K^\times$ and $s \in K^n$, we have

$$d(\alpha)[d(\beta)w(s)]d(\alpha)^{-1} = d(\beta)w(\alpha^2 s).$$

Therefore, since $\alpha^2 \neq 1$, we have $vFv^{-1} \subset D$. \square

Lemma 2.9. — *Let $\phi : K \rightarrow K$ be a affine map, and $\psi : K \rightarrow K^{n-1} \times \{0\}$ be a polynomial map such that at least one of them is non-constant, $\phi(0) = 1$ and $\psi(0) = 0$. Let F be the closed subgroup of DU^\perp generated by*

$$\{\Phi(t) := w(\psi(t))d(\phi(t)) : t \in K, \phi(t) \neq 0\}.$$

Then either F contains a nontrivial one-parameter subgroup of U^\perp or $F = vDv^{-1}$ for some $v \in U^\perp$.

Proof. — If $F \subset U^\perp$ then the result follows from Lemma 2.7. Otherwise ϕ is a non-constant affine map. Therefore $\phi(K) = K$. In particular, $F \not\subset Z(G)W$.

If F is abelian, then by Lemma 2.8 there exists $v \in W$ such that $F \subset vDv^{-1}$. Since ϕ is linear and nonconstant, $F = vDv^{-1}$.

Now we can further assume that F is not abelian. Since the commutator

$$[F, F] \subset [DU^\perp, DU^\perp] \subset U^\perp,$$

there exists $s \in K^{n-1} \times \{0\}$, $s \neq 0$ such that $w(s) \in F$. Therefore

$$\Phi(t)w(s)\Phi(t)^{-1} = w(\phi(t)^2 s) \in F, \quad \forall t \in K.$$

Put $\tilde{\psi}(t) := \phi^2(t)s$. Then $\tilde{\psi} : K \rightarrow K^{n-1} \times \{0\}$ is a non-constant polynomial map. Therefore by Lemma 2.7 applied to $\tilde{\psi}$ we conclude that F contains a nontrivial one-parameter subgroup of U^\perp . This completes the proof. \square

The following is a special case of the general fact that cocompact discrete subgroups in semisimple Lie groups do not contain unipotent elements having nontrivial Adjoint action on the Lie algebra.

Proposition 2.10. — *$W \cap G_x = \{e\}$ for all $x \in G/\Gamma$.*

Proof. — Let C be a compact subset of G such that $C\Gamma = G$. Since Γ is discrete, there exists a neighbourhood Ω of e in G such that $cZ(G)\Gamma^{-1} \cap \Omega = \{e\}$ for all $c \in C$. Therefore

$$G_y \cap \Omega = \{e\}, \quad \forall y \in C\Gamma/\Gamma = G/\Gamma.$$

Suppose that $w(\mathbf{t}) \in G_x$ for some $\mathbf{t} \in K$. Let $\alpha \in K^\times$ such that $|\alpha| < 1$. Then

$$G_{d(\alpha^i)_x} = d(\alpha^i)G_x d(\alpha^{-i}) \ni d(\alpha^i)w(\mathbf{t})d(\alpha^{-i}) = w(\alpha^{2i}\mathbf{t}) \rightarrow e$$

as $i \rightarrow \infty$. Therefore $w(\alpha^{2i}\mathbf{t}) \in G_{d(\alpha^i)_x} \cap \Omega = \{e\}$ for some i . Hence $w(\mathbf{t}) = e$. \square

Proposition 2.11. — *Let Δ be a discrete subgroup of DW such that $\Delta \cap W = \{e\}$. Then W acts properly on DW/Δ .*

Proof. — We have

$$[\Delta, \Delta] \subset [DW, DW] \cap \Delta \subset W \cap \Delta = \{e\}.$$

Hence Δ is an abelian subgroup of DW . If $g = d(-1)w(\mathbf{t}) \in \Delta$ for some $\mathbf{t} \in K^n$, then $g^2 = w(2\mathbf{t}) \in \Delta \cap W = \{e\}$; and hence $\mathbf{t} = 0$. Therefore by Lemma 2.8 there exists $v \in W$ such that $\Delta \subset vDv^{-1}$.

Since $DW = (vDv^{-1})W = W(vDv^{-1})$, we have that

$$DW/\Delta = W(vDv^{-1})/\Delta \cong W \times (vDv^{-1}/\Delta)$$

is a W -equivariant isomorphism, where W acts on the space $W \times (vDv^{-1}/\Delta)$ by translation on the first factor and trivially on the second factor; and this action is proper. \square

3. U -minimal sets

In order to understand closed U -invariant sets, especially the closures of U -orbits, we begin with the study of U -minimal sets.

Theorem 3.1. — *Let X be a U -minimal subset of G/Γ . Then X is invariant under either vDv^{-1} for some $v \in U^\perp$ or a nontrivial one-parameter subgroup of U^\perp .*

Proof. — Let $M = \{g \in G : gX \cap X \neq \emptyset\}$. For any $g \in M \cap N_G(U)$, $gX \cap X$ is a nonempty closed U -invariant set. Hence by minimality $gX = X$. Thus $M \cap N_G(U)$ is a closed subgroup of G . We note that DW is a subgroup of finite index in $N_G(U) = Z(G)DW$. Therefore $M_1 := M \cap DW$ is a closed subgroup of DW and an open subgroup of $M \cap N_G(U)$.

First suppose that $e \notin \overline{M \setminus N_G(U)}$. Then every orbit of $M \cap N_G(U)$ in X is open. Therefore every orbit of M_1 on X is open, and hence it is compact. Let $x \in X$. Since $U \subset M_1$, and X is U -minimal, $X = M_1x$. Hence $M_1/(M_1)_x \cong M_1x = X$ is compact. By Proposition 2.10 and Proposition 2.11, U acts properly on $M_1/(M_1)_x$, which is a contradiction.

Therefore $e \in \overline{M \setminus N_G(U)}$. By Lemma 2.2, X is invariant under the subgroup generated by $N_G(U) \cap \overline{UMU}$. Now the conclusion of the theorem follows from Proposition 2.4. \square

Corollary 3.2. — *Let $n = 1$; that is, $G \cong \mathrm{SL}_2(K)$ and Γ is a cocompact discrete subgroup of G . Then Ux is dense in G/Γ for every $x \in G/\Gamma$.*

In other words, Theorem 1.1 is valid for $n = 1$.

Proof. — Since \overline{Ux} is a closed U -invariant subset of G/Γ , there exists a compact U -minimal subset $X \subset \overline{Ux}$. By Theorem 3.1, X is invariant under D , because for the case of $n = 1$, we have $W = U$ and $U^\perp = \{e\}$. Thus X is a closed DW -invariant subset of G/Γ . Therefore by Proposition 2.1, $X = G/\Gamma$. Thus $\overline{Ux} = G/\Gamma$. \square

3.1. D -invariant U -minimal sets. — In view of Theorem 3.1, we first suppose that the U -minimal set is invariant under wDw^{-1} for some $w \in U^\perp$. Now $Y := w^{-1}X$ is U -minimal and D -invariant. Therefore for simplicity of notation we will further investigate Y , rather than X .

We need the following group theoretic result.

Proposition 3.3. — *Let sequences $\{h_i\}$ in $\mathrm{SL}_2(K)$ and $\{t_i\}$ in K , $|t_i| \rightarrow \infty$ be given. Then, after passing to a subsequence, there exists at most one $s^* \in K$ such that for any $s \in K$, $s \neq s^*$, the following holds:*

$$(20) \quad w_1(st_i)h_iB \rightarrow eB \quad \text{as } i \rightarrow \infty,$$

where B is the group of all upper triangular matrices in $\mathrm{SL}_2(K)$, and the limit is considered in the quotient space $\mathrm{SL}_2(K)/B$.

In fact, if $\{h_i\}$ is a constant sequence, then (20) holds for all $s \in K$.

Proof. — Consider the projective linear action of $\mathrm{SL}_2(K)$ on the projective space $\mathcal{P} = (K^2 \setminus \{0\})/K^\times$. Let $\langle v \rangle$ denote the image of $v \in K^2$ on \mathcal{P} . The stabilizer of $\langle e_1 \rangle$ is B , where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We can express $h_i \langle e_1 \rangle = \langle \begin{pmatrix} a_i \\ b_i \end{pmatrix} \rangle$, where $|a_i|^2 + |b_i|^2 = 1$. Then for any $s \in K$,

$$(21) \quad w_1(st_i)h_i \langle e_1 \rangle = \langle \begin{pmatrix} a_i + st_i b_i \\ b_i \end{pmatrix} \rangle = \langle \begin{pmatrix} 1 \\ b_i/a_i + st_i b_i \end{pmatrix} \rangle, \quad \text{if } s \neq -a_i/(t_i b_i).$$

After passing to a subsequence, either $-a_i/(t_i b_i) \rightarrow s^*$ for some $s^* \in K$, or $|-a_i/(t_i b_i)| \rightarrow \infty$. By (21), if $s \neq s^*$, then since $|t_i| \rightarrow \infty$,

$$w_1(st_i)h_i \langle e_1 \rangle \rightarrow \langle e_1 \rangle.$$

From this (20) follows, because the action of $\mathrm{SL}_2(K)$ on \mathcal{P} is transitive, and the stabilizer of $\langle e_1 \rangle$ is B . \square

The next proposition is very similar to Proposition 2.4, and it will allow us to investigate further after an application of Lemma 2.2.

Proposition 3.4. — *Let $M \subset G$ such that $e \in \overline{M \setminus H}$. Then the closed subgroup generated by $\overline{DUMDU} \cap W$ contains a nontrivial one-parameter subgroup of U^\perp .*

Proof. — First we suppose that $e \notin \overline{M \setminus U^\perp H}$. Since $e \in \overline{M \setminus H}$, there exist $v \in U^\perp \setminus \{e\}$ and $h \in H$ such that $vh \in M$. By Proposition 3.3, applied to $H \cong \mathrm{SL}_2(K)$ and $DU \cong B$, there exists a sequence $\{u_i\} \subset U$ such that $u_i h DU \rightarrow e DU$ in H/DU . Hence

$$u_i v h DU = v u_i h DU \rightarrow v DU, \quad \text{as } i \rightarrow \infty.$$

Therefore $v \in \overline{UMDU}$. We can write $v = w(\mathbf{t})$, $\mathbf{t} \in K^n \setminus \{0\}$. Then $d(a)vd(-a) = w(a^2\mathbf{t})$ for all $a \in K^\times$. By Lemma 2.7, the closure of the additive subgroup generated by $\{a^2\mathbf{t} : a \in K\}$ in K^n contains $K\mathbf{t}$. Hence the subgroup generated by $\overline{DUMDU} \cap W$ contains a nontrivial one-parameter subgroup of U^\perp .

Now we may assume that $e \in \overline{M \setminus U^\perp H}$. Let a sequence $\{g_i\} \subset M \setminus U^\perp H$ be such that $g_i \rightarrow e$. Since $G = G_{\{1, \dots, n-1\}}H$, we can write $g_i = X_i h_i$, where $X_i \in G_{\{1, \dots, n-1\}} \setminus U^\perp$, $X_i \rightarrow 0$, and $h_i \rightarrow e$ in H . By Lemma 2.6, after passing to a subsequence, there exist a sequence $t_i \rightarrow \infty$ in K and a non-constant polynomial map $\psi : K \rightarrow K^n$ of degree at most 2 such that for any $s \in K$,

$$(22) \quad \lim_{i \rightarrow \infty} u(st_i)X_i u(-st_i) = w(\psi(s)) \in U^\perp.$$

By Proposition 3.3, there exists at most one $s^* \in K$ such that for all $s \in K$ with $s \neq s^*$, the following holds:

$$(23) \quad u(st_i)h_i(DU) \rightarrow DU, \quad \text{as } i \rightarrow \infty.$$

By (22) and (23), $\forall s \in K$ with $s \neq s^*$, as $i \rightarrow \infty$,

$$u(st_i)g_i DU = (u(st_i)X_i u(-st_i))(u(st_i)h_i DU) \rightarrow w(\psi(s))DU,$$

in G/DU . Thus $w(\psi(s)) \in \overline{UMDU}$, $\forall s \in K$. Since $W \cong K^n$, and $\psi(s)$ is a non-constant polynomial map, the conclusion of this proposition follows from Lemma 2.7. \square

Theorem 3.5. — *Let X be a U -minimal subset of G/Γ . Then either X is a closed orbit of wHw^{-1} for some $w \in U^\perp$, or X is invariant under a nontrivial one-parameter subgroup of U^\perp .*

Proof. — By Theorem 3.1, we are reduced to considering the case that X is wDw^{-1} -invariant for some $w \in W$.

We put $Y = w^{-1}X$. Then Y is DU -invariant and U -minimal. Let

$$M = \{g \in G : gY \cap Y \neq \emptyset\}.$$

By Lemma 2.2, applied to $Y' = Y$, $P = P' = DU$ and $F = U$, we have that Y is invariant under the subgroup generated by $\overline{DUMDU} \cap N_G(U)$.

Now if $e \in \overline{M \setminus H}$ then by Proposition 3.4, there exists a nontrivial one-parameter subgroup, say V , of U^\perp such that $VY = Y$. Therefore

$$VX = V(w^{-1}Y) = w^{-1}(VY) = w^{-1}Y = X,$$

and the conclusion of the theorem holds.

Next suppose that $e \notin \overline{M \setminus H}$. Fix $y \in Y$ and let $\Delta = H_y$. Then by Lemma 2.3, $\overline{DU\Delta}/\Delta$ is compact in H/Δ . Since H/DU is compact, we have that H/Δ is compact.

Therefore by Proposition 2.1 applied to the case of $G := H \cong \mathrm{SL}_2(K)$, $\Gamma := \Delta$, $W := U$, and $D := D$, we conclude that $\overline{DU\Delta} = H$. Since $Hy \cong H/\Delta$, we have that Hy is compact and $Hy = \overline{DUy} = Y$. Hence $X = (wHw^{-1})(wy)$, which is a closed orbit of wHw^{-1} . \square

3.2. Minimal sets for actions of at least 2 dimensional subgroups of W

Remark 3.1. — For any $x \in G/\Gamma$, there exists $g_j \in G_{\{\{j\}\}}$ for $1 \leq j \leq n$ such that $x = (g_1 \dots g_n)\Gamma$, and

$$G_x = (g_1\Gamma_1g_1^{-1}) \cdots (g_n\Gamma_ng_n^{-1}).$$

In particular, for any $J \subset \{1, \dots, n\}$, we have

$$(24) \quad G_Jx \cong \prod_{j \in J} G_{\{j\}}/g_j\Gamma_jg_j^{-1}.$$

For $\mathcal{J} \in \mathcal{C}$, define $\cup\mathcal{J} = \cup_{J \in \mathcal{J}} J$, $|\mathcal{J}| = |\cup\mathcal{J}|$, $G_{\mathcal{J}} = G_{\cup\mathcal{J}}$, $W_{\mathcal{J}} = W \cap G_{\mathcal{J}}$, $U_{\mathcal{J}} = W \cap H_{\mathcal{J}} = \prod_{J \in \mathcal{J}} U_J$, where $U_J = W \cap H_J$, and $D_{\mathcal{J}} = A \cap H_{\mathcal{J}} = \prod_{J \in \mathcal{J}} D_J$, where $D_J = D \cap H_J$.

Theorem 3.6. — Assume that for any $k < n$ the Theorem 1.1 is true for k in place of n . Let $\mathcal{J} \in \mathcal{C}$ such that $\mathcal{J} \neq \{\{1, \dots, n\}\}$. Then for any $x = G/\Gamma$, we have $\overline{U_{\mathcal{J}}x} = wH_{\mathcal{J}'}w^{-1}x$ for some $\mathcal{J}' \in \mathcal{C}$ and $w \in W$.

Proof. — We intend to prove this result by induction on n .

By our choice of \mathcal{J} there exists $\mathcal{J}_1 \subset \mathcal{J}$, where $\mathcal{J}_1 \in \mathcal{C}$ and $1 \leq n_1 := |\cup\mathcal{J}_1| < n$. Put $G_1 = G_{\mathcal{J}_1}$ and $U_1 = U_{\mathcal{J}_1}$.

By Remark 3.1, for any $y \in G/\Gamma$,

$$G_1y \cong \prod_{j \in \cup\mathcal{J}_1} G_{\{j\}}/(G_{\{j\}})_y.$$

We claim that there exists $\mathcal{J}'_1 \in \mathcal{C}$ and $w_1 \in W_{\mathcal{J}_1}$ such that, if we put $H_1 = w_1H_{\mathcal{J}'_1}w_1^{-1}$ then

$$(25) \quad \overline{U_1x} = H_1x.$$

Here $U_1 \subset H_1 \subset G_1$.

If $\mathcal{J}_1 = \{\{\cup\mathcal{J}_1\}\}$, then the claim follows by applying the assumption that Theorem 1.1 is valid for $n_1 < n$, G_1 in place of G , and U_1 in place of U .

If $\mathcal{J}_1 \neq \{\{\cup\mathcal{J}_1\}\}$ then the claim follows by applying the induction hypothesis of this theorem to $n_1 < n$ in place of n , G_1 in place of G , and \mathcal{J}_1 in place of \mathcal{J} . Thus the claim is proved in all the cases.

If $\mathcal{J} \notin \mathcal{C}_0$, then $|\mathcal{J}| < n$, and hence if we choose $\mathcal{J}_1 = \mathcal{J}$ then the conclusion of the theorem follows from (25).

Therefore we can assume that $\mathcal{J} \in \mathcal{C}_0$. Let $\mathcal{J}_2 = \mathcal{J} \setminus \mathcal{J}_1 \neq \emptyset$. Then $n_2 = |\mathcal{J}_2| = |\mathcal{J}| - |\mathcal{J}_1| = n - n_1 < n$. By the same argument as above for \mathcal{J}_2 in place of \mathcal{J}_1 the

following holds: there exists $\mathcal{J}'_2 \in \mathcal{C}$ and $w_2 \in W_{\mathcal{J}'_2}$ such that, if we put $G_2 = G_{\mathcal{J}_2}$, $U_2 = U_{\mathcal{J}_1}$ and $H_2 = w_2 H_{\mathcal{J}'_2} w_2^{-1}$, then $U_2 \subset H_2 \subset G_2$ and

$$(26) \quad \overline{U_{\mathcal{J}_2} x} = H_2 x.$$

Since $(\cup \mathcal{J}_1) \cap (\cup \mathcal{J}_2) = \emptyset$, we have that $G_{\mathcal{J}_1} \subset Z_G(G_{\mathcal{J}_2})$. Therefore for any $g_2 \in G_{\mathcal{J}_2}$, we have

$$(27) \quad \overline{U_{\mathcal{J}_1} g_2 x} = \overline{g_2 U_{\mathcal{J}_1} x} = g_2 (w_1 H_{\mathcal{J}'_1} w_1^{-1} x) = w_1 H_{\mathcal{J}'_1} w_1^{-1} (g_2 x).$$

Moreover

$$(28) \quad H_1 H_2 x \cong H_1 / (H_1)_x \times H_2 / (H_2)_x$$

which is compact. Hence by (25)–(28),

$$\overline{U_{\mathcal{J}} x} = \overline{U_1 U_2 x} = \overline{U_1 H_2 x} = H_1 H_2 x = w H_{\mathcal{J}'} w^{-1} x,$$

where $w = w_1 w_2$ and $\mathcal{J}' = \mathcal{J}'_1 \cup \mathcal{J}'_2$. This completes the proof of the theorem. \square

Remark 3.2. — By the condition of Theorem 3.6, $n \geq 2$. Therefore to begin the induction, we have $n = 2$ and for this case $\mathcal{J} = \{\{1\}, \{2\}\}$, $\mathcal{J}_1 = \{\{1\}\}$ and $\mathcal{J}_2 = \{\{2\}\}$, and the result follows from the assumption that Theorem 1.1 is valid for $n = 1$; in fact, this assumption was verified in Corollary 3.2.

Theorem 3.7. — Assume that for all $k < n$, the Theorem 1.1 is true for k in place of n . Let V be a multi-parameter subgroup of W of dimension at least 2 and containing U . Let X be a compact V -minimal subset of G/Γ . Then there exists $\mathcal{J} \in \mathcal{C}_0$ and $w \in W$ such that $X = (w H_{\mathcal{J}} w^{-1})x$.

Proof. — Without loss of generality we may assume that V is the largest multi-parameter subgroup of W whose action preserves X .

If $n = 2$ then $V = W$ and the theorem follows from Theorem 3.6. We intend to prove this theorem by induction on n .

Let $V_1 = V \cap G_{\{1, \dots, n-1\}}$. Then $V = V_1 U$, and $\dim V_1 \geq 1$ (see Definition 1.1). Let J be the smallest subset of $\{1, \dots, n-1\}$ such that $V_1 \subset G_J$. Then there exists $a \in A \cap G_J$ such that $U_J \subset a V_1 a^{-1}$.

Let Y be a compact V_1 -minimal subset of X . Take $y \in Y$. Then by Remark 3.1, $G_J y$ is compact, and

$$G_J y \cong G_J / \prod_{j \in J} (G_{\{j\}})_y.$$

In particular, $Y \subset G_J y$.

Let $y_1 = ay$. We claim that there exists $\mathcal{J} \in \mathcal{C}$ and $w_1 \in W \cap G_J$ such that $H_{\mathcal{J}} \subset G_J$ and

$$(29) \quad \overline{(a V_1 a^{-1}) y_1} = w_1 H_{\mathcal{J}} w_1^{-1} y_1.$$

If $\dim V_1 = 1$, then $U_J = a V_1 a^{-1}$. Since $|J| < n$, the claim follows from our first hypothesis that Theorem 1.1 is valid for $|J|$ in place of n , G_J in place of G , and U_J in place of U .

If $\dim V_1 \geq 2$, then the claim follows from the induction hypothesis of this theorem applied to G_J in place of G and aV_1a^{-1} in place of V in the statement. This completes the proof of the claim in both the cases.

From (29) we have that

$$Y \supset \overline{V_1 y} = \overline{V_1 a^{-1} y_1} = a^{-1} \overline{(aV_1 a^{-1}) y_1} = a^{-1} w_1 H_{\mathcal{J}} w_1^{-1} a y.$$

Let $w = a^{-1} w_1 a \in W \cap G_J$. Then $w D_{\mathcal{J}} w^{-1} y \subset Y \subset X$, where $D_{\mathcal{J}} = A \cap H_{\mathcal{J}} = \prod_{I \in \mathcal{J}} D_I$.

Therefore by Lemma 2.2,

$$(30) \quad gX = X, \quad \forall g \in N_G(V) \cap \overline{V(w D_{\mathcal{J}} w^{-1})V}.$$

We have

$$(31) \quad N_G(V) \cap \overline{V(w D_{\mathcal{J}} w^{-1})V} \supset W \cap \overline{U D_{\mathcal{J}} U},$$

and

$$(32) \quad \overline{U D_{\mathcal{J}} U} \supset \overline{\{u D_{\mathcal{J}} u^{-1} : u \in U\}} = \prod_{I \in \mathcal{J}} \overline{\{u D_I u^{-1} : u \in U_I\}}.$$

Take any $I \in \mathcal{J}$. Let $\{X_i\}$ be a sequence in $D_I \setminus \{e\}$ such that $X_i \neq e$ as $i \rightarrow \infty$. In view of the identification $D_I \subset H_I \cong \mathrm{SL}_2(K) \subset \mathrm{M}_2(K) = E_1$, we have that

$$\{X_i\} \subset E_1 \setminus L_1$$

(recall (4)). We apply Lemma 2.6 to conclude the following: The subgroup generated by $\{u D_I u^{-1} : u \in U_I\}$ contains U_I (see Lemma 2.7). Therefore by (32), the subgroup generated by $\overline{U D_{\mathcal{J}} U}$ contains $U_{\mathcal{J}}$. Therefore by (30) and (31), $U_{\mathcal{J}} X = X$. By the maximality of V , assumed in the beginning of the proof, $U_{\mathcal{J}} \subset V$. Thus $U_{\mathcal{J}} \subset G_J \cap V = V_1$. Therefore

$$U_{\mathcal{J}} \subset V_1 \subset a^{-1} H_{\mathcal{J}} a \cap W = a^{-1} (H_{\mathcal{J}} \cap W) a = a U_{\mathcal{J}} a^{-1}.$$

Therefore $U_{\mathcal{J}} = a^{-1} U_{\mathcal{J}} a$, and hence $V_1 = U_{\mathcal{J}}$. Thus $V = U_{\mathcal{J}} U = U_{\mathcal{J}'}$, where $\mathcal{J}' = \mathcal{J} \cup \{1, \dots, n\}$. Now the theorem follows from Theorem 3.6. \square

4. Proof of Theorem 1.1

We intend to prove Theorem 1.1 by induction on n .

The case of $n = 1$ is proved in Corollary 3.2.

As an induction hypothesis, we assume that Theorem 1.1 is valid for all k in place of n in its statement, where $k \leq n - 1$. In particular, the hypothesis of Theorem 3.7 is satisfied.

Let $X = \overline{Ux}$. Let V denote a maximal multi-parameter subgroup of W such that $Vx' \subset X$ for some $x' \in X$. Let Z be a compact V -minimal subset contained in $\overline{Vx'}$. Therefore by Theorem 3.5 and by Theorem 3.7, there exists $\mathcal{J} \in \mathcal{C}_0$ and $w \in U^{\perp}$ such that $Z = w H_{\mathcal{J}} w^{-1} z'$, where $z' \in Z$ and $V \subset w H_{\mathcal{J}} w^{-1}$.

Note that $w^{-1}X = \overline{Uw^{-1}x}$. Now if we can show that $w^{-1}X = H_{\mathcal{J}'}(w^{-1}x)$, then $X = wH_{\mathcal{J}'}w^{-1}x$ and the conclusion of the theorem follows. Therefore without loss of generality, we replace X by $w^{-1}X$, Z by $w^{-1}Z$, and z' by $w^{-1}z'$, and assume that $Z = H_{\mathcal{J}}z'$.

If $H_{\mathcal{J}} = G$, then $X = G/\Gamma$ and the theorem is proved.

Therefore we can assume that $H_{\mathcal{J}} \cong \mathrm{SL}_2(K)^m$ for some $m \leq n - 1$. In view of (24), we have

$$(33) \quad \Lambda := (H_{\mathcal{J}})_{z'} = \prod_{J \in \mathcal{J}} H_J / (H_J)_{z'}.$$

We note that

$$(34) \quad H_{\mathcal{J}}/\Lambda \cong H_{\mathcal{J}}z' = Z$$

is compact. Therefore $H_J/(H_J)_{z'}$ is compact for all $J \in \mathcal{J}$. Therefore by Proposition 1.8 applied to $H_{\mathcal{J}}$ in place of G and Λ in place of Γ ,

$$(35) \quad \exists z \in H_{\mathcal{J}}/\Lambda \setminus \mathcal{S}(U, \Lambda).$$

In view of (34) we treat z as an element of Z , and hence

$$(36) \quad H_{\mathcal{J}}z' = Z = H_{\mathcal{J}}z \subset X.$$

We have made such a choice of $z \notin \mathcal{S}(U, \Lambda)$ because later in the proof we intend to apply Theorem 1.9 for the U -action on $H_{\mathcal{J}}/\Lambda$.

We define

$$\mathcal{J}^* = \{J \setminus \max\{J\} : J \in \mathcal{J}, |J| > 1\}.$$

Now $\mathcal{J} \in \mathcal{C}_0$. Therefore $G = G_{\mathcal{J}^*} \cdot H_{\mathcal{J}}$.

Since $z \in Z \subset \overline{Ux}$, there exists a sequence $g_i \rightarrow e$ in G such that $g_i z \in Ux$ for all i . We can express $g_i = X_i h_i$ such that $X_i \in G_{\mathcal{J}^*}$, $h_i \in H_{\mathcal{J}}$, and $X_i \rightarrow e$ and $h_i \rightarrow e$.

If $X_{i_0} \in W$ for some i_0 , then

$$(37) \quad H_{\mathcal{J}}z \subset X = \overline{Ug_i z} = \overline{X_{i_0} U h_i z} \subset X_{i_0} H_{\mathcal{J}} z = X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1} (X_{i_0} z).$$

In particular, z belongs to the closed orbit $(X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1})(X_{i_0} z)$. Therefore

$$H_{\mathcal{J}}z \subset X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1} (X_{i_0} z) = (X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1})z.$$

Hence $H_{\mathcal{J}}$ is an open subgroup of $X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1}$. Since $H_{\mathcal{J}}$ is Zariski closed, we have that $H_{\mathcal{J}} = X_{i_0} H_{\mathcal{J}} X_{i_0}^{-1}$. Therefore the inclusions in (37) are equalities. Hence $X = H_{\mathcal{J}}z$, and the conclusion of the theorem holds.

Now we may assume that $\{X_i\} \subset G_{\mathcal{J}^*} \setminus W$. Put $m = |\cup \mathcal{J}^*|$. In view of the identification, $G_{\mathcal{J}^*} \cong \mathrm{SL}_2(K)^m$, we have that

$$\{X_i\} \subset M_2(K)^m \setminus L_m$$

(recall (4)). Also the conjugation action of $u(t)$ on $G_{\mathcal{J}^*}$ corresponds to the conjugation action of $w_m(t)$ on $M_2(K)^m$. Therefore by Lemma 2.6, there exists a sequence $t_i \rightarrow \infty$

and a non-constant polynomial map $\psi : K \rightarrow K^n$ such that for any sequence $s_i \rightarrow s$ in K ,

$$(38) \quad \lim_{i \rightarrow \infty} u(s_i t_i) X_i u(-s_i t_i) = w(\psi(s)) \in W_{\mathcal{J}^*}.$$

If Z were U -minimal, which would be the case if $H_{\mathcal{J}} \cong \mathrm{SL}_2(K)$, or if $n = 2$ and $m \leq n - 1 = 1$. We would then apply Lemma 2.2 for $Y' = X$, $Y = Z$, $P' = U$, $P = H_{\mathcal{J}}$ and $F = U$; and conclude that $\Psi(s)X \subset X$.

In general, we will have to go deeper into the proof of Lemma 2.2 to see what is exactly required; and that turns out to be Theorem 1.9 as shown below.

In view of (33) and (36), we apply Theorem 1.9 to $H_{\mathcal{J}}$ and Λ in places of G and Γ , respectively, and to the sequence $\{x_i := h_i z\}_{i \in \mathbb{N}} \subset Z$. Since $x_i \rightarrow z$ and $z \notin \mathcal{S}(U, \Lambda)$ (see (35)), we conclude the following: given any compact neighbourhood \mathfrak{D} of 0 in K and $s \in K$, there exists a sequence $t'_i \in s t_i(1 + \mathfrak{D})$ such that, after passing to a subsequence, $u(t'_i)x_i \rightarrow y$ as $i \rightarrow \infty$, where $y \in Z \cong H_{\mathcal{J}}/\Lambda$ and $y \notin \mathcal{S}(U, \Lambda)$.

Since $H_{\mathcal{J}} \cong \mathrm{SL}_2(K)^m$ for some $m \leq n - 1$, by our induction hypothesis, Theorem 1.1 is valid for $H_{\mathcal{J}}$ in place of G . Therefore, since y is nonsingular for the U action on Z , we conclude that

$$(39) \quad \overline{Uy} = Z.$$

Note that this is the second instance of the use of the induction hypothesis in this proof.

We put $s_i = t'_i/t_i \in s(1 + \mathfrak{D})$ for all i . Then $t'_i = s_i t_i$, and after passing to a subsequence, we may assume that $s_i \rightarrow s'$ and $s' \in s(1 + \mathfrak{D})$. Now by (38),

$$\begin{aligned} u(s_i t_i) g_i z &= u(s_i t_i) X_i x_i \\ &= [u(s_i t_i) X_i u(-s_i t_i)] u(s_i t_i) x_i \\ &\rightarrow w(\psi(s')) y. \end{aligned}$$

Thus $w(\psi(s'))y \in X$, and hence by (39)

$$X \supset \overline{Uw(\psi(s'))y} = w(\psi(s'))\overline{Uy} = \Psi(s')Z.$$

Since \mathfrak{D} was an arbitrarily chosen neighbourhood of 0, and $s' \in s(1 + \mathfrak{D})$, we conclude that

$$(40) \quad X \supset w(\psi(s))Z, \quad \forall s \in K.$$

This finishes a major step in the proof, as we have obtained a nontrivial trajectory of a polynomial set in $W_{\mathcal{J}^*}$. Now we will use an idea from [6] to show that X contains a trajectory of a nontrivial one-parameter subgroup of $W_{\mathcal{J}^*}$.

Since X is compact, there exists a sequence $T_i \rightarrow \infty$ in K and $x' \in X$ such that

$$(41) \quad w(\psi(T_i))z \rightarrow x'.$$

Then by Lemma 2.7,

$$\psi(T_i + sT_i^{-q}) - \psi(T_i) \rightarrow s\mathbf{v}, \quad \forall s \in K,$$

where $q = \deg(\psi) - 1 \in \{0, 1\}$ and $\mathbf{v} \in K^n \setminus \{0\}$. Therefore by (41)

$$w(\psi(T_i + sT_i^{-q}))z = w(\psi(T_i + sT_i^{-q} - \psi(T_i)))w(\psi(T_i))z \rightarrow w(s\mathbf{v})x'.$$

Therefore, since $VZ = Z$, for any $u \in V$, by (40),

$$X \ni w(\psi(T_i + sT_i^{-q}))uz = uw(\psi(T_i + sT_i^{-q}))z \rightarrow uw(s\mathbf{v})x'.$$

Thus $VV_1x' \subset X$, where $V_1 = \{w(s\mathbf{v}) : s \in K\}$. We note that $V \subset H_{\mathcal{J}}$ and $\psi(s) \in W_{\mathcal{J}^*}$. Therefore V_1 is a nontrivial one-parameter subgroup of $W_{\mathcal{J}^*}$, which is not contained in V . Thus VV_1 is a multi-parameter subgroup of W which is strictly larger than V , and $VV_1x' \subset X$. This contradicts the maximality property of V assumed at the beginning of the proof. This completes the proof of the theorem. \square

5. Closures of H -orbits

Lemma 5.1. — *If $D \subset wH_{\mathcal{J}}w^{-1}$ for some $w \in W$ and $\mathcal{J} \in \mathcal{C}_0$ then $w \in H_{\mathcal{J}}$.*

Proof. — It easily follows from the facts that $N_G(H_{\mathcal{J}}) = Z(G)H_{\mathcal{J}}$, and that $d(a)w(\mathbf{t})d(a)^{-1} = w(a^2\mathbf{t})$ for any $t \in K^n$ and $a \in K^*$. \square

Define \mathfrak{F} to be the collection of closed subgroups F of G with the following properties: $F/F \cap \Gamma$ is compact, and $F = gH_{\mathcal{J}}g^{-1}$ for some $g \in G$ and $\mathcal{J} \in \mathcal{C}$.

Lemma 5.2. — *\mathfrak{F} is countable.*

Proof. — Let $F \in \mathfrak{F}$. In view of Remark 3.1,

$$F/F \cap \Gamma \cong \prod_{i=1}^r \mathrm{SL}_2(K)/\Lambda_i,$$

where Λ_i is a cocompact discrete subgroup of $\mathrm{SL}_2(K)$ and $1 \leq r \leq n$. It is straightforward to verify that each Λ_i is Zariski dense in $\mathrm{SL}_2(K)$ (this is a very special easy case of the Borel's density theorem (see [4, 5] or [14]). Therefore $\mathrm{Zcl}(F \cap \Gamma) = F$, where $\mathrm{Zcl}(X)$ denotes the Zariski closure of a set X in $\mathrm{M}_2(K)^n$. Now there exists a finite set $S \subset F \cap \Gamma$ such that if $\langle S \rangle$ denotes the subgroup generated by S then

$$\mathrm{Zcl}(\langle S \rangle) = \mathrm{Zcl}(F \cap \Gamma) = F.$$

Thus

$$\mathfrak{F} \subset \{\mathrm{Zcl}(\langle S \rangle) : S \text{ is a finite subset of } \Gamma\}.$$

Since Γ is countable, \mathfrak{F} is countable. \square

Proof of Corollary 1.3. — For any $h \in H$, by Theorem 1.1, there exist $w \in W$ and $\mathcal{J} \in \mathcal{C}_0$ such that

$$\overline{hUh^{-1}x} = \overline{hU(h^{-1}x)} = h(wH_{\mathcal{J}}w^{-1})(h^{-1}x) = F_hx,$$

where $F_h := hwH_{\mathcal{J}}w^{-1}h^{-1}$.

Suppose if $H \subset F_h$ then $H \subset wH_{\mathcal{J}}w^{-1}$, and by Lemma 5.1, we have $w \in H_{\mathcal{J}}$ and $F_h = H_{\mathcal{J}}$. Hence $H_{\mathcal{J}}x$ is compact, and

$$H_{\mathcal{J}}x \supset \overline{Hx} \supset (hUh^{-1})x = H_{\mathcal{J}}x.$$

Thus $\overline{Hx} = H_{\mathcal{J}}x$, and we are through.

Suppose that $H \not\subset F_h$, then $hUh^{-1} \subset F_h \cap H$, which is a proper algebraic subgroup of $H \cong \mathrm{SL}_2(K)$. Therefore $F_h \cap H$ at most 2 dimensional, and any nontrivial algebraic unipotent subgroup of $F_h \cap H$ equals hUh^{-1} . Hence for any $h_1 \in H$, if $F_{h_1} = F_h$ then $h_1Uh_1^{-1} = hUh^{-1}$. Thus,

$$(42) \quad \text{for any } h, h_1 \in H: \text{ if } H \not\subset F_h \text{ and } F_h = F_{h_1}, \text{ then } h_1 \in hN_H(U).$$

Now fix $g \in G$ such that $x = g[\Gamma] \in G/\Gamma$. Since F_hx is compact, we have $gF_hx = gF_hg^{-1}\Gamma/\Gamma$ is compact. Therefore $gF_hg^{-1} \in \mathfrak{F}$. Since \mathfrak{F} is countable, the collection $\{F_h : h \in H\}$ is countable. Hence due to (42), since $H/N_H(U)$ is uncountable, there exists $h \in H$ such that $F_h \supset H$, and we are back to the case considered earlier. \square

Proof of Proposition 1.4. — Since $H_J = G_J \cap H_{\mathcal{J}}$, $Y := G_Jx_0 \cap H_{\mathcal{J}}x_0$ is compact, and the stabilizer of x_0 , which is Γ , is discrete, we conclude that every orbit of H_J in Y is open. Therefore every orbit of H_J in Y is closed. In particular, H_Jx_0 is compact.

Therefore replacing G by G_J , $H_{\mathcal{J}}$ by H_J , and Γ by $G_J \cap \Gamma$, without loss of generality we may assume that Hx_0 is compact.

In view of Remark 3.1, we define the natural projection maps $q_j : G \rightarrow G_{\{j\}}$ and $\bar{q}_j : G/\Gamma \rightarrow G_{\{j\}}/\Gamma_j$. Now $\bar{q}_j^{-1}(e\Gamma_j) \cap Hx_0$ is a compact subset of G/Γ . Since it is countable, it is finite. Therefore

$$\bar{q}_j^{-1}(e\Gamma_j) \cap Hx_0 \cong q_j^{-1}(\Gamma_j) \cap H/q_j^{-1}(\Gamma_j) \cap H \cap \Gamma$$

is finite. Now

$$q_j^{-1}(\Gamma_j) \cap H = \{(\gamma, \dots, \gamma) \in G : \gamma \in \Gamma_j\}$$

and

$$q_j^{-1}(\Gamma_j) \cap H \cap \Gamma = \{(\gamma, \dots, \gamma) \in G : \gamma \in \cap_{i=1}^n \Gamma_i\}.$$

Therefore $\cap_{i=1}^n \Gamma_i$ is a subgroup of finite index in Γ_j . Therefore Γ_i and Γ_j are commensurable for all i and j . \square

Proof of Corollary 1.6. — Let $J \in \mathcal{J}$, and $\Lambda_J = \cap_{j \in J} p_j(\Gamma_j)$. Then by definition Λ_J is a subgroup of finite index in $p_j(\Gamma_j)$ for each $j \in J$, and hence Λ_J is a cocompact lattice in $\mathrm{SL}_2(K)$. Clearly, $H_J/(H_J \cap \Gamma) \cong \mathrm{SL}_2(K)/\Lambda_J$ is compact. Therefore H_Jx_0 is compact.

From this we obtain that $H_{\mathcal{J}}x_0$ is compact. Now for any $J_1, J_2 \in \mathcal{J}$ with $J_1 \neq J_2$, we have that the lattices Λ_{J_1} and Λ_{J_2} are noncommensurable. Therefore applying Corollary 1.5 to $H_{\mathcal{J}}$ in place of G , we conclude that Hx_0 is dense in $H_{\mathcal{J}}x_0$. \square

Proof of Corollary 1.7. — We are in the case of $n = 2$. Let the notation be as above. Let $x_i = e\Gamma_i \in G_i/\Gamma_i$ for $i = 1, 2$ and $x_0 = (x_1, x_2)$. To prove that $\Gamma_1\Gamma_2$ is dense in G , it is enough to show that $\overline{\Gamma_1x_2} = G_2/\Gamma_2$. Let $y \in G_2/\Gamma_2$. By Corollary 1.5, $\overline{Hx_0} = G/\Gamma$. Therefore there exists a sequence $\{h_i\} \subset \mathrm{SL}_2(K)$ such that

$$(h_i, h_i)x_0 = (h_ix_1, h_ix_2) \xrightarrow{i \rightarrow \infty} (x_1, y).$$

Since $h_ix_1 \rightarrow x_1$ as $i \rightarrow \infty$, there exist sequences $\delta_i \rightarrow e$ in $\mathrm{SL}_2(K)$ and $\{\gamma_i\} \subset \Gamma_1$ such that $h_i = \delta_i\gamma_i$ for all i . Since $h_ix_2 \rightarrow y$ as $i \rightarrow \infty$, we have

$$\gamma_ix_2 = \delta_i^{-1}h_ix_2 \xrightarrow{i \rightarrow \infty} y.$$

Thus $y \in \overline{\Gamma_1x_2}$. \square

6. Limiting distributions of sequences of unipotent orbits

As noted in the introduction, we start the second half of the article. First we give the statement of the main result, which says that a unipotent trajectory starting from a non-singular point attaches zero measure on its singular set $\mathcal{S}(U, \Gamma)$ in the limiting distribution.

Notation. — Let $\mathcal{M} = \mathcal{M}(G/\Gamma)$ denote the space of probability measures on G/Γ , which is compact. Then \mathcal{M} is compact with respect to the topology of weak-* convergence; here by definition, a sequence $\mu_i \rightarrow \mu$ in \mathcal{M} if $\int f d\mu_i \rightarrow \int f d\mu$ as $i \rightarrow \infty$, for all $f \in C(G/\Gamma)$.

Let θ denote a Haar measure on K .

Theorem 6.1. — *Let $x_i \rightarrow x$ be a sequence in G/Γ and $t_i \rightarrow \infty$ be a sequence in K . Fix any measurable set $\mathfrak{D} \subset K$ with $0 < \theta(\mathfrak{D}) < \infty$. Let $\mu_i = \mu_i^{\mathfrak{D}} \in \mathcal{M}(G/\Gamma)$ be defined as*

$$(43) \quad \mu_i^{\mathfrak{D}}(E) = \frac{\theta(\{t \in t_i\mathfrak{D} : u(t)x_i \in E\})}{\theta(t_i\mathfrak{D})}, \quad \text{for all Borel sets } E \subset G/\Gamma.$$

Let $\mu \in \mathcal{M}$ be a limit of any subsequence of $\{\mu_i\}_{i=1}^{\infty}$ in \mathcal{M} . Further suppose that $x \notin \mathcal{S}(U, \Gamma)$. Then $\mu(\mathcal{S}(U, \Gamma)) = 0$.

As a first consequence of this result, we deduce the result required in the proof of Theorem 1.1.

Proof of Theorem 1.9. — Given a compact neighbourhood \mathfrak{D} of 0 in K , we apply Theorem 6.1 for $1 + \mathfrak{D}$ in place of \mathfrak{D} in the statement above. Since $\mu(\mathcal{S}(U, \Gamma)) = \emptyset$, we can choose $y \in \text{supp}(\mu) \setminus \mathcal{S}(U, \Gamma)$. Let Ω_i be a sequence of open neighbourhoods of y in G/Γ such that $\cap_i \Omega_i = \{y\}$. Now by the definition of $\mu_i = \mu_i^{1+\mathfrak{D}}$, by passing to a subsequence of i , we may assume that $\text{supp}(\mu_i^{1+\mathfrak{D}}) \cap \Omega_i \neq \emptyset$. Then there exists $t'_i \in (1 + \mathfrak{D})t_i$ such that $u(t'_i)x_i \in \Omega_i$. Therefore $u(t'_i)x_i \rightarrow y$ as $i \rightarrow \infty$. \square

6.1. Uniform distribution of U -orbits. — As another main consequence of Theorem 6.1 we will deduce the uniform distribution of U -orbits using Ratner's measure classification result. We first give an idea of the connection of both the results.

Lemma 6.2. — *Any limit measure μ as obtained in Theorem 6.1 is U -invariant.*

Since invariant measures decompose into its ergodic components, using the description of ergodic U -invariant measures [12, 19] and Theorem 6.1, we will obtain the following uniform distribution result.

Theorem 6.3. — *Let \mathfrak{D} be a measurable subset of K such that $0 < \theta(\mathfrak{D}) < \infty$. Fix any $x \in G/\Gamma$ then there exists $w \in W$ and $\mathcal{J} \in \mathcal{C}$ such that $\overline{Ux} = wH_{\mathcal{J}}w^{-1}x$ and the following holds: For $T \in K \setminus \{0\}$ define $\mu_T \in \mathcal{M}$ as*

$$(44) \quad \mu_T(E) = \frac{\theta(\{t \in T\mathfrak{D} : u(t)x \in E\})}{\theta(T\mathfrak{D})}, \quad \text{for all Borel sets } E \subset G/\Gamma.$$

Then for any continuous function f on G/Γ , we have

$$\int f d\mu_T \rightarrow \int f d\mu \quad \text{as } T \rightarrow \infty \text{ in } K,$$

where μ denotes the unique $wH_{\mathcal{J}}w^{-1}$ -invariant probability measure on the space

$$wH_{\mathcal{J}}w^{-1}x \cong wH_{\mathcal{J}}w^{-1}/(wH_{\mathcal{J}}w^{-1} \cap G_x),$$

where G_x denotes the stabilizer of x in G .

7. A countability theorem and the singular set

Note that for any $g \in G$, and $x = gx_0$, the orbit $G_Jx = gG_{\{j\}}x_0$ is compact for any $j = 1, \dots, n$, where $x_0 \in G/\Gamma$ denotes the coset of the identity. Similarly, G_Jx is compact for any nonempty $J \subset \{1, \dots, n\}$.

Let $\Gamma_J = G_J \cap \Gamma$, and $\bar{\rho}_J : G/\Gamma \rightarrow G_J/\Gamma_J$ denotes the natural projection in view of (1). Note that every fiber of $\bar{\rho}_J$ is a compact orbit of the group G_{J^c} , where $J^c = \{1, \dots, n\} \setminus J$. Therefore $\bar{\rho}_J$ is a proper map; namely, the inverse images of compact sets are compact.

We assume that $n \geq 2$. Let \mathcal{H} denote the collection of all subgroups F of G with the following properties: (i) $F/F \cap \Gamma$ is compact, and (ii) $F = f^{-1}G_{J^c}H_Jf$ for some $f \in G_J$, where $J \subset \{1, \dots, n\}$, $|J| = 2$. Note that $Z(G)F$ is a proper maximal subgroup of G , where $Z(G) = \{(\pm I, \dots, \pm I)\}$ denotes the center of G , and $I = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$.

Note that $F \cap \Gamma = \Gamma_{J^c}(fH_Jf^{-1} \cap \Gamma_J)$. Let $\Lambda = fH_Jf^{-1} \cap \Gamma_J$. Then fH_Jf^{-1}/Λ is compact and admits an fH_Jf^{-1} -invariant probability measure. This measure projects onto an fH_Jf^{-1} -invariant probability measure on fH_Jf^{-1}/L , where L denotes the Zariski closure of Λ in fH_Jf^{-1} . Since $H_J \cong \mathrm{SL}_2(K)$, if L is one dimensional then the quotient cannot be compact, and if L is two dimensional then the quotient is a projective line and does not admit an invariant measure. Therefore $L = fH_Jf^{-1}$; we remark that this conclusion is also a special case of Borel's density theorem [4, 14]. Therefore fH_Jf^{-1} is the Zariski closure of the subgroup generated by a finite subset of Γ_J . Hence $F \in \mathcal{H}$ is determined by J and a finite subset of Γ . Since Γ is countable, we conclude the following:

Lemma 7.1. — *The collection \mathcal{H} is countable.*

For any $F \in \mathcal{H}$, we define (the algebraic variety)

$$X(F) = \{g \in G : U \subset gFg^{-1}\}.$$

Note that for any $F \in \mathcal{H}$ and $g \in G$:

$$(45) \quad g \in X(F) \Leftrightarrow \overline{Ugx_0} \subset gFx_0 = (gFg^{-1})gx_0,$$

where $x_0 = \pi(e)$ and $\pi : G \rightarrow G/\Gamma$ is the natural quotient map.

Lemma 7.2. — $\mathcal{S}(G/\Gamma) = \bigcup_{F \in \mathcal{H}} \pi(X(F))$.

Proof. — By (45), $\pi(X(F)) \subset \mathcal{S}(G/\Gamma)$.

Now let $g \in G$ such that $gx_0 \in \mathcal{S}(G/\Gamma)$. Then there exists $\mathcal{J} \in \mathcal{C}$ and $w \in W$ such that $\bigcup \mathcal{J} = \{1, \dots, n\}$, $H_{\mathcal{J}} \neq G$, $U \subset wH_{\mathcal{J}}w^{-1}$ and $H_{\mathcal{J}}w^{-1}gx_0$ is compact.

Therefore there exists $1 \leq j_1 < j_2 \leq n$ such that, if $g = (g_1, \dots, g_n)$ and $g \in H_{\mathcal{J}}$, then $g_{j_1} = g_{j_2}$. Put $J = \{j_1, j_2\}$. Since $G = G_{J^c}G_J$, there exists $f \in G_J$ such that $g^{-1}w \in G_{J^c}f$.

If we put $F = G_{J^c}fH_Jf^{-1}$, then $F = G_{J^c}(g^{-1}w)H_{\mathcal{J}}(w^{-1}g)$. Since $G_{J^c}z$ is compact for all $z \in G/\Gamma$, Fx_0 is compact. Hence $g^{-1}Ugx_0 \subset Fx_0$. Therefore $g^{-1}Ug \subset F$, and hence $g \in X(F)$. \square

Lemma 7.3. — *Let $F \in \mathcal{H}$, $J = \{j_1, j_2\}$, $1 \leq j_1 < j_2 \leq n$, and $f \in G_J$ such that $F = f^{-1}G_{J^c}H_Jf$. Then $X(F) = WG_{J^c}H_JZ(G)f$. Moreover $H_J(zfx_0)$ is compact for every $z \in Z(G)$.*

Proof. — Take any $g \in X(F)$. Let $U_J = U \cap H_J$. Then $g^{-1}U_Jg \subset f^{-1}H_Jf$. Since $H_J \cong \mathrm{SL}_2(K)$, there exists $h \in H$ such that

$$g^{-1}U_Jg = f^{-1}hU_Jh^{-1}f.$$

Therefore $h^{-1}fg \in D_JW_JZ(G)G_{J^c}$. Multiplying h by an appropriate element of D_J on the right, we may assume that $h^{-1}fg \in W_JG_{J^c}Z(G)$. Hence $g \in hfW_JG_{J^c}Z(G) \subset W_JG_{J^c}H_JZ(G)f$.

Moreover $G_{J^c}H_J(zfx_0) = zfG_{J^c}(f^{-1}H_Jf)x_0 = zfFx_0$ is compact. \square

Proof of Proposition 1.8. — By Lemma 7.3, the set $X(F)\gamma$ cannot contain an open subset of G for any $F \in \mathcal{H}$ and $\gamma \in \Gamma$. Now $X(F)$ can be expressed as a countable union of compact sets, and since \mathcal{H} and Γ are countable sets, by Baire's category theorem we have that $G \neq \bigcup_{F \in \mathcal{H}} X(F)\Gamma$. Therefore $G/\Gamma \neq \mathcal{S}(U, \Gamma)$ by Lemma 7.2. \square

8. Reducing Theorem 6.1 to the case of $n = 2$

By Lemma 7.2 and Lemma 7.3, in order to prove that $\mu(\mathcal{S}(G/\Gamma)) = 0$, it is enough to show that $\mu(WG_{J^c}H_J y) = 0$ for every $J = \{j_1, j_2\}$, $1 \leq j_1 < j_2 \leq n$, and $y \in G/\Gamma$ such that $H_J y$ is compact.

Fix J and y as above. Then $H_J \bar{y}$ is compact in G_J/Γ_J , where $\bar{y} = \bar{\rho}_J(y)$. Also

$$(46) \quad (\bar{\rho}_J)^{-1}(W_J H_J \bar{y}) = W G_{J^c} H_J y.$$

Let $\bar{\mu}$ denote the projection of μ on G_J/Γ_J via $\bar{\rho}_J$; that is, $\bar{\mu}(E) = \mu((\bar{\rho}_J)^{-1}(E))$ for any Borel measurable set $E \subset G_J/\Gamma_J$. Therefore in order to prove that $\mu(WG_{J^c}H_J y) = 0$, it is enough to show that $\bar{\mu}(W_J H_J \bar{y}) = 0$. Further it is enough to show that for any compact set $C \subset W_J$,

$$(47) \quad \bar{\mu}(C H_J \bar{y}) = 0.$$

Note that $G_J \cong \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$, and under this isomorphism H_J corresponds to the diagonally embedded copy of $\mathrm{SL}_2(K)$ in $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$. For the projection homomorphism $\rho_J : G \rightarrow G_J$, let $\bar{u}(t) := \rho_J(u(t)) \in H_J$ for all $t \in K$. Let $\bar{x}_i = \bar{\rho}_J(x_i)$, and let $\bar{\mu}_i \in \mathcal{M}(G_J/\Gamma_J)$ be such that

$$\bar{\mu}_i(E) = \frac{\theta(\{t \in t_i \mathfrak{D} : \bar{u}(t) \bar{x}_i \in E\})}{\theta(t_i \mathfrak{D})}, \quad \text{for all Borel sets } E \subset G_J/\Gamma_J.$$

Then $\bar{\mu}_i$ is the projection of μ_i on G_J/Γ_J . Furthermore whenever $\mu_i \rightarrow \mu$ in $\mathcal{M}(G/\Gamma)$, we have $\bar{\mu}_i \rightarrow \bar{\mu}$. Since $x \notin W G_{J^c} H_J y \subset \mathcal{S}(U, \Gamma)$, by (46), $\bar{x} := \bar{\rho}_J(x) \notin W_J H_J \bar{y}$, and $\bar{x}_i \rightarrow \bar{x}$.

In view of the above explanation, to prove Theorem 6.1 it is enough to prove it for the case of $n = 2$.

For $r > 0$, and $x \in K$, let $B_x(r)$ denote the ball of radius $r > 0$ in K centered at x .

8.1. Reduction to the case of $\mathfrak{D} = B_0(r)$. — Since $0 < \theta(\mathfrak{D}) < \infty$, given any $\beta < 1$ there exists a compact subset $\mathfrak{D}_1 \subset \mathfrak{D}$ such that $\theta(\mathfrak{D}_1)/\theta(\mathfrak{D}) > \beta$. Therefore it will be enough to prove the result under the assumption that $\mathfrak{D} \subset B_0(r)$ for some $r > 0$. Put $B = B_0(r)$.

Let $\lambda_i = \mu^B$ as defined in (43). Then $\mu_i(E) \leq (\theta(B)/\theta(\mathfrak{D}))\lambda_i(E)$ for any Borel set $E \subset G/\Gamma$. By passing to a subsequences we have that $\mu_i \rightarrow \mu$ and $\lambda_i \rightarrow \lambda$ as $i \rightarrow \infty$. Therefore $\mu(E) \leq (\theta(B)/\theta(\mathfrak{D}))\lambda(E)$ for all Borel sets $E \subset G/\Gamma$. Therefore if we prove that $\lambda(\mathcal{S}(U, \Gamma)) = 0$, then $\mu(\mathcal{S}(U, \Gamma)) = 0$. This proves that it is enough to prove Theorem 6.1 for $\mathfrak{D} = B_0(r)$ for all $r > 0$.

9. Theorem 6.1 for $G = \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$

Let $G = \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ and H be the diagonal embedding of $\mathrm{SL}_2(K)$ in G . For $t_1, t_2 \in K$, let $w(t_1, t_2) := \left(\begin{pmatrix} 1 & t_1 \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & t_2 \\ & 1 \end{pmatrix} \right)$. Let $W = \{w(t_1, t_2) : t_i \in K\}$. Define $u(t) = w(t, t) \in G$, $\forall t \in K$, and $U = \{u(t) : t \in K\} = W \cap H$.

Let Γ be a discrete subgroup of G such that G/Γ is compact.

In this section we will prove the following:

Theorem 9.1. — *Let $y \in G/\Gamma$ such that Hy is compact. Let $x_i \rightarrow x$ be a convergent sequence in G/Γ such that $x \notin WHy$. Then given any $\epsilon > 0$ and a compact set $C_1 \subset W$ there exist a neighbourhood Ψ_1 of C_1Hy in G/Γ and a natural number i_0 such that $\forall i \geq i_0$ and $T > 0$,*

$$(48) \quad \theta(\{t \in B_0(T) : u(t)x_i \in \Psi_1\}) \leq \epsilon \theta(B_0(T)).$$

Proof of Theorem 6.1. — Let $\mathfrak{D} = B_0(r)$ for some $r > 0$. In view of (43), $\mu_i(\Psi_1) \leq \epsilon \cdot \theta(B_0(r))$ for all $i \geq i_0$, and hence $\mu(C_1Hy) = 0$. Since C_1 can be chosen to be an arbitrary compact subset of W , we have that $\mu(WHy) = 0$. Thus in view of the discussion in Section 8, the Theorem 9.1 implies Theorem 6.1. \square

9.1. Linearization of the U -action near WHy . — For a group F acting on a set X and an element $x \in X$, let

$$F_x = \{f \in F : fx = x\}, \quad \text{the stabilizer of } x \text{ in } F.$$

Note that $G = G_{\{1\}}H$ and $WH = W_1H$, where $W_1 = G_{\{1\}} \cap W = \{w(t, 0) : t \in K\}$. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 9.2. — *$wHw^{-1} \cap H = U \cup (-I, -I)U$ for all $w \in W_1 \setminus \{e\}$.*

Proof. — Let $h = (x, x) \in H$ and $w = (w_1, I) \in W_1$, $w_1 \neq I$. Then $whw^{-1} \in H \Rightarrow x = w_1xw_1^{-1} \Rightarrow x = \begin{pmatrix} \pm 1 & s \\ 0 & \pm 1 \end{pmatrix}$, $s \in K$. \square

The next observation, which states that the singular set $WHy = W_1Hy$ does not self-intersect along W_1 , makes the study of dynamics near singular sets much simpler in our situation, as compared to the general case [21, Lemma 6.5].

Proposition 9.3. — *For $w_1, w_2 \in W_1$, if $w_1 \neq w_2$ then $w_1Hy \cap w_2Hy = \emptyset$.*

Proof. — Let $Z = w_1Hy \cap w_2Hy$. Suppose that $Z \neq \emptyset$. Put $H_i = w_iHw_i^{-1}$. Then $w_iHy = H_1(w_iy) = H_iz$ is compact for every $z \in Z$. Since G_z is a discrete group, $(H_1 \cap H_2)z$ is open in $Z = H_1z \cap H_2z$. Since $w_1 \neq w_2$, by Lemma 9.2, U is an open subgroup of $H_1 \cap H_2$. Therefore every orbit of U on Z is open in Z . Hence every orbit of U on Z is closed. Since Z is compact, $Uz \cong U/U \cap G_z$ is compact, which contradicts Proposition 2.10. \square

We consider a linear action of G on $E := M_2(K)$ defined as follows: Given $g = (g_1, g_2) \in G$ and $X \in E$,

$$g \cdot X := g_1 X g_2^{-1}.$$

Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in E$. Then

$$(49) \quad H = \{g \in G : g \cdot I = I\}$$

$$(50) \quad G \cdot I = \mathrm{SL}_2(K) \subset E.$$

Let $\mathcal{W} = \{w_1(t) : t \in K\} \subset E$. Then $W_1 \cdot I = \mathcal{W}$, and

$$(51) \quad W_1 H = W H = \{g \in G : g \cdot I \in \mathcal{W}\}.$$

Lemma 9.4. — *The set $G_y \cdot I$ is discrete.*

Proof. — Since $H y$ is compact, $H/H \cap G_y$ is compact, and hence $H G_y$ is closed in G . Therefore $H G_y$ is closed in G . Hence $G_y H = (H G_y)^{-1}$ is closed in G . Due to (50) and (49), the map $G/H \rightarrow \mathrm{SL}_2(K)$ given by $gH \mapsto g \cdot I$ is a homeomorphism. Hence $G_y \cdot I$ is a closed subset of $\mathrm{SL}_2(K)$, and hence of E . Further since G_y is countable, $G_y \cdot I$ is discrete. \square

For any $z \in G/\Gamma$, we define $\mathcal{R}(z) = \{g \cdot I : gz = y, g \in G\}$. Note that if $z = gy$, then $\mathcal{R}(z) = g G_y \cdot I = g \mathcal{R}(y)$. The set $\mathcal{R}(z)$ is called the set of representatives of z in E . By Lemma 9.4, $\mathcal{R}(z)$ is discrete.

Lemma 9.5. — *$\#(\mathcal{R}(z) \cap \mathcal{W}) \leq 1$, for all $z \in G/\Gamma$.*

Proof. — If $g\gamma_1 \cdot I, g\gamma_2 \cdot I \in \mathcal{W}$ for some $\gamma_1, \gamma_2 \in G_y$, then by (49), there exist $w_i \in W_1$ such that $g_i \gamma_i \in w_i H$ for $i = 1, 2$. Then

$$g\gamma_1 y = g\gamma_2 y \in w_1 H y \cap w_2 H y.$$

Therefore by Proposition 9.3, $w_1 = w_2$. Hence $g\gamma_1 H = w_1 H = w_2 H = g\gamma_2 H$. Thus $g\gamma_1 \cdot I = g\gamma_2 \cdot I$. \square

The following observation will allow us to ‘linearize’ the G -action in thin neighbourhoods of compact subsets of $W H y$.

Lemma 9.6. — *Given a compact subset D of \mathcal{W} , there exists a neighbourhood Φ of D in E such that $\#(\mathcal{R}(z) \cap \Phi) \leq 1$ for all $z \in G/\Gamma$.*

Proof. — Let $\{\Phi_i\}$ be a decreasing sequence of relatively compact neighbourhoods of D in E such that $\cap_i \Phi_i = D$. If the lemma is false, then there exists a sequence $\{z_i\} \subset G/\Gamma$ such that $\#(\mathcal{R}(z_i) \cap \Phi_i) \geq 2$ for all i . By passing to a subsequence we may assume that $z_i = g_i y$ for a sequence $g_i \rightarrow g$ in G , and for each i there exist $\gamma_i, \delta_i \in G_y$ such that

$$(52) \quad g_i \gamma_i \cdot I, g_i \cdot \delta_i \cdot I \in \Phi_i \quad \text{and} \quad \gamma_i \cdot I \neq \delta_i \cdot I.$$

Now

$$\{\gamma_i, \delta_i : i \in \mathbb{N}\} \subset \cup_{i=1}^{\infty} \{g_i^{-1} \Phi_i\} \subset (\{g_i : i \in \mathbb{N}\} \cup \{g\}) \overline{\Phi_1},$$

which is compact. Therefore by Lemma 9.4 there exist $\gamma, \delta \in G_y$ such that $\gamma_i \cdot I = \gamma \cdot I$ and $\delta_i \cdot I = \delta \cdot I$ for all large i . Therefore $g_i \cdot \gamma \cdot I \rightarrow g\gamma \cdot I \in D$, and similarly $g\delta \cdot I \in D$. Therefore by Lemma 9.5, $g\gamma \cdot I = g\delta \cdot I$. Hence

$$\gamma_i \cdot I = \gamma \cdot I = \delta \cdot I = \delta_i \cdot I, \quad \text{for all large } i,$$

a contradiction to (52). \square

9.2. Growth properties of polynomial maps. — For any $v \in E$, the coordinate functions of the map $t \mapsto u(t) \cdot v$ are polynomials of degree at most 2. Therefore to study the behaviour of the U -orbits on thin neighbourhoods of compact subsets of \mathcal{W} , we will use the growth properties of the polynomial maps as described in the following basic observations (see [7, 9]).

Let $l \geq 1$ be the dimension of K over the topological closure of \mathbb{Q} in K . For a ball B in K , let $\mathrm{rad}(B)$ denote the radius of B such that $\mathrm{rad}(B) = |\lambda|$ for some $\lambda \in K$. Then for any balls B_1 and B_2 in K ,

$$(53) \quad \theta(B_2) = (r_2/r_1)^l \cdot \theta(B_1), \quad \text{where } r_i = \mathrm{rad}(B_i).$$

Lemma 9.7. — *Let $\epsilon > 0$ and $d \in \mathbb{N}$ be given. Then there exists $c > 0$ such that for any $f \in K[t]$ with $\deg(f) \leq d$, and any ball B in K ,*

$$(54) \quad \theta(\{t \in B : |f(t)| < c \sup_{t \in B} |f(t)|\}) \leq \epsilon \cdot \theta(B).$$

In fact, we can choose $c = C_d^{-1}(\epsilon/d)^{d/l}$, where $C_d = 1$ if K is non-archimedean, and $C_d = (d+1)2^d$ if K is archimedean.

Proof. — Put $M = \sup_{t \in B} |f(t)|$. Fix any $c > 0$. Put $I = \{t \in B : |f(t)| < cM\}$. Suppose that

$$(55) \quad \theta(I) > \epsilon \cdot \theta(B).$$

We claim that there exist points x_0, \dots, x_d in I such that

$$(56) \quad |x_i - x_j| > (\epsilon/d)^{1/l} r, \quad \forall i \neq j,$$

where r denotes the radius of B .

To prove the claim, suppose that x_0, \dots, x_k are chosen so that (56) holds for $0 \leq i, j \leq k$, where $0 \leq k \leq d-1$. Put

$$I' = \bigcup_{j=0}^k B_{x_j}((\epsilon/d)^{1/l} r).$$

Then by (53),

$$\theta(I') \leq (k+1)(\epsilon/d)\theta(B) \leq \epsilon\theta(B).$$

By (55) there exists $x_{i+1} \in I \setminus I'$. Then $|x_{i+1} - x_j| \geq (\epsilon/d)^{1/l}$ for all $j \leq i-1$. This proves the claim.

By Lagrange's interpolation formula,

$$f(x) = \sum_{0 \leq i \leq d} f(x_i) \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

Now $|f(x_i)| < cM$ and $\frac{|x - x_j|}{|x_i - x_j|} \leq 2/(\epsilon/d)^{1/l}$ for all $j \neq i$, and $x \in \overline{B}$. Therefore $M < (d+1)cM2^d/(\epsilon/d)^{d/l}$. This leads to a contradiction if we choose $c = (1/(d+1)2^d)(\epsilon/d)^{d/l}$. Therefore (55) cannot hold. \square

Corollary 9.8. — *For any $f \in K[t]$ with $\deg(f) \leq d$, and balls $B_1 \subset B_2$ in K , let $M_i = \sup_{t \in B_i} |f(t)|$ and $r_i = \text{rad}(B_i)$ for $i = 1, 2$. Then*

$$(57) \quad M_2 \leq C_d(r_2/r_1)^d M_1.$$

Proof. — Let $0 < \epsilon < (r_2/r_1)^{-l}$. Let $F = \{t \in B_2 : |f(t)| < C_d^{-1} \epsilon^{d/l} M_2\}$. Then by Lemma 9.7 and (53),

$$\theta(F) \leq \epsilon \theta(B_2) = \epsilon (r_2/r_1)^l \theta(B_1) < \theta(B_x(r)).$$

Thus $B_x(r) \not\subset F$, and hence $M_1 \geq C_d^{-1} \epsilon^{d/l} M_2$. Hence $M_2 \geq C_d \lambda^d M_1$. \square

Proposition 9.9. — *Given $\epsilon > 0$ and a compact set $C \subset \mathcal{W}$, there exists a compact set $D \subset \mathcal{W}$ containing C such that the following holds: given any neighbourhood Φ of D in \mathcal{W} there exists a neighbourhood Ψ of C in E such that for any $v \in E$ and any ball \mathfrak{B} in K , one of the following holds:*

$$(58) \quad u(\mathfrak{B})v \subset \Phi$$

or

$$(59) \quad \theta(\{t \in \mathfrak{B} : u(t)v \in \Psi\}) \leq \epsilon \cdot \theta(\{t \in \mathfrak{B} : u(t)v \in \Phi\}).$$

Proof. — Let $\{\phi_1, \dots, \phi_4\}$ be linear functionals on E such that

$$\mathbf{y} = \begin{pmatrix} \phi_1(\mathbf{y}) & \phi_2(\mathbf{y}) \\ \phi_3(\mathbf{y}) & \phi_4(\mathbf{y}) \end{pmatrix}, \quad \forall \mathbf{y} \in E.$$

Then

$$\mathcal{W} = \{\mathbf{y} \in E : \phi_i(\mathbf{y} - I) = 0, \forall i \neq 2\}.$$

Note that $\phi_2(\mathbf{y} - I) = \phi_2(\mathbf{y})$ for all $\mathbf{y} \in E$. Define $f_i(t) = \phi_i(u(t)v - I)$ for all i and $t \in K$. Then $f_i \in K[t]$ and $\deg(f_i) \leq 2$.

There exists $\alpha_2 > 0$ such that

$$C \subset \{\mathbf{y} \in \mathcal{W} : |\phi_2(\mathbf{y} - I)| < \alpha_2\}$$

We fix a small $0 < c < 1$, whose value will be specified below. Let $M_2 = c^{-1} \alpha_2$ and put

$$D = \{\mathbf{y} \in \mathcal{W} : |\phi_2(\mathbf{y} - I)| \leq M_2\}.$$

Now given any neighbourhood Φ of D , there exists $M_i > 0$ for each $i \neq 2$, such that

$$\Phi \supset \{\mathbf{y} \in E : |\phi_i(\mathbf{y} - I)| \leq M_i, \forall i\}.$$

We choose $\alpha_i = cM_i$ for each $i \neq 2$, and put

$$\Psi = \{\mathbf{y} \in E : |\phi_i(\mathbf{y} - I)| < \alpha_i, \forall i\}.$$

Then Ψ is a neighbourhood of C .

Define

$$(60) \quad F = \{t \in \mathfrak{B} : |f_i(t)| < M_i, \forall i\}$$

$$(61) \quad \subset \{t \in \mathfrak{B} : u(t)\mathbf{v} \in \Phi\},$$

and

$$(62) \quad F_1 = \{t \in \mathfrak{B} : |f_i(t)| < \alpha_i, \forall i\}$$

$$(63) \quad = \{t \in \mathfrak{B} : u(t)\mathbf{v} \in \Psi\}.$$

Suppose that (58) does not hold. Then

$$\mathfrak{B} \not\subset F.$$

A ball $B \subset F$ is called a *maximal ball* in F , if $B' \not\subset F$ for any ball $B' \subset \mathfrak{B}$ strictly bigger than B .

Let B be a maximal ball in F . We claim that

$$(64) \quad \sup_{t \in B} |f_{i_0}(t)| \geq \tau^{-1}M_{i_0}, \quad \text{for some } i_0,$$

where $\tau = p^2C_2 > 1$ if K is non-archimedean, and $\tau = 1$ if K is archimedean.

Suppose if $\sup_{t \in B} |f_i(t)| < M_i$ for all i , then $B \subset F \subsetneq \mathfrak{B}$. Then there exists a ball $B' \subset \mathfrak{B}$ strictly bigger than B . Hence $B' \not\subset F$. Therefore by (60), for some i_0 ,

$$\sup_{t \in B'} |f_{i_0}(t)| \geq M_{i_0}.$$

If K is a finite extension of \mathbb{Q}_p , we choose B' such that $\text{rad}(B')/\text{rad}(B) = p$; and (57) implies (64). If K is archimedean, (64) is straightforward to conclude.

If K is non-archimedean or $K = \mathbb{R}$, then any two intersecting maximal balls in F are same. Therefore $F = \cup \mathcal{B}$, where \mathcal{B} denotes the collection of disjoint maximal balls of F . If $K = \mathbb{C}$ then there exists a collection \mathcal{B}' of disjoint maximal balls in F such that if we put $\mathcal{B} = \{B_x(3r) : B_x(r) \in \mathcal{B}'\}$ then $F \subset \cup \mathcal{B}$ (cf. [20, Proof of Lemma 8.4]). Therefore

$$(65) \quad \sum_{B \in \mathcal{B}} \theta(B) \leq \kappa \theta(F)$$

where $\kappa = 1$ if $K \neq \mathbb{C}$ and $\kappa = 9$ if $K = \mathbb{C}$.

We specify the value $c = (C_2\epsilon^{2/l})^{-1}\tau^{-1}$. Let $B \in \mathcal{B}$. Therefore by (64), there exists i_0 such that

$$\sup_{t \in B} |f_{i_0}(t)| \geq \tau^{-1}M_{i_0}.$$

Since $\alpha_{i_0} = cM_{i_0}$, by (62) and Lemma 9.7 applied to f_{i_0} :

$$\theta(F_1 \cap B) \leq \theta(\{t \in B : |f_{i_0}(t)| < \alpha_{i_0}\}) \leq \epsilon \cdot \theta(B).$$

Therefore by (65), we get that

$$\begin{aligned}\theta(F_1) &= \theta(F_1 \cap (\cup \mathcal{B})) \\ &\leq \sum_{B \in \cup \mathcal{B}} \theta(F_1 \cap B) \leq \epsilon \sum_{B \in \cup \mathcal{B}} \theta(B) \leq \kappa \epsilon \cdot \theta(F).\end{aligned}$$

Therefore (59) follows from (61) and (63). \square

Proof of Theorem 9.1. — Given $\epsilon > 0$ and a compact set $C_1 \subset W$, put $C = C_1 \cdot I \subset \mathcal{W}$, and obtain $D \subset \mathcal{W}$ as in Proposition 9.9. By Lemma 9.6, there exists a neighbourhood Φ of D in E such that

$$(66) \quad \#(\mathcal{R}(z) \cap \Phi) \leq 1, \quad \forall z \in G/\Gamma.$$

In other words, every element of G/Γ can have at most one representative in Φ .

The set $W_D := \{w \in W_1 : w \cdot I \in D\}$ is compact. Also $\{g : g \cdot I \in D\} = W_D H$. Now $D_1 := W_D H y$ is a compact subset of $W H y \subset G/\Gamma$. Since $x \notin W H y$, and $\mathcal{R}(x) \cap D = \emptyset$. Since $\mathcal{R}(x)$ is discrete and D is compact, there exists a compact neighbourhood V of the identity in G such that $V \mathcal{R}(x) \cap D = \emptyset$. We replace Φ by $\Phi \setminus V \mathcal{R}(x)$, which is an open neighbourhood of D . Since $x_i \rightarrow x$, we have $x_i \in V \mathcal{R}(x)$ for all $i \geq i_0$ for some i_0 . Since $\mathcal{R}(vx) = v \mathcal{R}(x)$ for all $v \in V$, we have that have that

$$(67) \quad \mathcal{R}(x_i) \cap \Phi = \emptyset, \quad \forall i \geq i_0.$$

By Proposition 9.9 and (67), there exists a neighbourhood Ψ of C in E contained in Φ such that for any $T > 0$,

$$(68) \quad \begin{aligned}\theta(\{t \in B_0(T) : u(t)v \in \Psi\}) \\ \leq \epsilon \cdot \theta(\{t \in B_0(T) : u(t)v \in \Phi\}), \quad \forall v \notin \Phi.\end{aligned}$$

Let $\Psi_1 = \{gy : g \cdot I \in \Psi, g \in G\} \subset G/\Gamma$. Since Ψ is a neighbourhood of $C = C_1 H \cdot I$ in E , we conclude that Ψ_1 is a neighbourhood of $C_1 H y$ in G/Γ .

Now fix $T > 0$. For a subset $\Omega \subset E$, define

$$L_\Omega(v) = \{t \in B_0(T) : u(t)v \in \Omega\}, \quad \forall v \in E.$$

We observe that

$$(69) \quad \{t \in B_0(T) : u(t)x_i \in \Psi_1\} = \bigcup_{v \in \mathcal{R}(x_i)} L_\Psi(v).$$

By (67) and (68),

$$(70) \quad \theta(L_\Psi(v)) \leq \epsilon \cdot \theta(L_\Phi(v)), \quad \forall v \in \mathcal{R}(x_i).$$

We claim that

$$(71) \quad L_\Phi(v_1) \cap L_\Phi(v_2) = \emptyset, \quad \forall v_1 \neq v_2, v_i \in \mathcal{R}(x_i).$$

If the claim is false, then there exists $t \in L_\Phi(v_1) \cap L_\Phi(v_2)$. Therefore $\{u(t)v_1, u(t)v_2\} \subset \mathcal{R}(u(t)x_i) \cap \Phi$ and $u(t)v_1 \neq u(t)v_2$. This contradicts (66). This proves the claim.

Now due to (71),

$$(72) \quad \sum_{\mathbf{v} \in \mathcal{R}(x_i)} \theta(L_{\Phi}(\mathbf{v})) \leq \theta(t_i O).$$

Combining (69), (70), and (72), we get

$$\theta(\{t \in B_0(T) : u(t)x \in \Psi_1\}) \leq \epsilon \cdot \Theta(t_i O), \quad \forall i \geq i_0. \quad \square$$

10. Uniform distribution for unipotent orbits

Proof of Lemma 6.2. — Let $\epsilon > 0$ be given. Since $0 < \theta(\mathfrak{D}) < \infty$ by the observation as before, we may assume that \mathfrak{D} is compact. Now since θ is translation invariant and regular, there exists $\delta > 0$, such that for any $s \in K$ with $|s| \leq \delta$, we have

$$\theta((\mathfrak{D} + s) \Delta \mathfrak{D}) / \theta(\mathfrak{D}) \geq \epsilon,$$

where $A \Delta B := (A \setminus B) \cup (B \setminus A)$.

Let $s \in K$ be given. If $t \in K$ such that $|t| \geq \delta^{-1}|s|$, then

$$\begin{aligned} \theta((t\mathfrak{D} + s) \Delta t\mathfrak{D}) / \theta(t\mathfrak{D}) &= \theta(t((\mathfrak{D} + t^{-1}s) \Delta \mathfrak{D})) / \theta(t\mathfrak{D}) \\ &= \theta((\mathfrak{D} + t^{-1}s) \Delta \mathfrak{D}) / \theta(\mathfrak{D}) \leq \epsilon. \end{aligned}$$

Let $i_0 \in \mathbb{N}$ be such that $|t_i| \geq \delta^{-1}|s|$ for all $i \geq i_0$. Then for any Borel set $E \subset G/\Gamma$,

$$|\mu_i(u(-s)E) - \mu_i(E)| \leq \theta((t_i\mathfrak{D} + s) \Delta \mathfrak{D}) / \theta(t_i\mathfrak{D}) \leq \epsilon, \quad \forall i \geq i_0.$$

Therefore $|\mu(u(-s)E) - \mu(E)| \leq \epsilon$. Since ϵ , s , and E are arbitrary, μ is U -invariant. \square

10.1. On the definition of singular set. — We begin with a group theoretic observation.

Proposition 10.1. — *Suppose F is a closed subgroup of G containing U and $x \in G/\Gamma$ such that Fx is compact. Then there exists $\mathcal{J} \in \mathcal{C}$ and $w \in W$ such that $wH_{\mathcal{J}}w^{-1} \subset F \subset Z(G)(wH_{\mathcal{J}}w^{-1})$.*

Proof. — First we consider the case of $n = 1$, that is $G = \mathrm{SL}_2(K)$. Now suppose that $F \subset N_G(U) = DU$. Since $[F, F] \subset U$, by Proposition 2.10, $[F_x, F_x] \subset U \cap G_x = \{e\}$. Therefore F_x is an abelian subgroup of DU . Also since $F_x \cap U = \{e\}$, it is straightforward to verify that $F_x \subset uDu^{-1}$ for some $u \in U$. Since $F = U(uDu^{-1} \cap F)$, it follows that F/F_x cannot be compact, a contradiction.

Therefore there exists $f \in F$ such that $U' := fUf^{-1} \neq U$. Then for the standard $\mathrm{SL}_2(K)$ action on K^2 ,

$$UU' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = K^2 \setminus \begin{pmatrix} K \\ 0 \end{pmatrix}.$$

Hence

$$U'UU' \begin{pmatrix} 1 \\ 0 \end{pmatrix} = K^2 \setminus \{0\}.$$

Since the stabilizer of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is U , we have that $U'UU'U = \mathrm{SL}_2(K)$. Therefore $F = G = H_{\{1\}}$, and the proof is complete.

We intend to prove the general case by induction on n . Therefore we assume that the proposition is valid for k in place of n , where $k = 1, \dots, n-1$.

For $i \in \{1, \dots, n\}$ and $J = \{1, \dots, n\} \setminus \{i\}$, let $p_i : G \rightarrow G_J$ and $\bar{p}_i : G/\Gamma \rightarrow G_J/\Gamma_J$ be the natural quotient maps.

Now if $F \supset G_i$ for some i , then $p_i(F)\bar{p}_i(x) = \bar{p}_i(Fx)$ is compact. Since $p_i(U)$ plays the role of U in G_J , the general result easily follows from the induction hypothesis.

Now we assume that $F \not\supset G_j$ for each j . For any $i = 1, \dots, n$, let $q_i = G \rightarrow G_i$ and $\bar{q}_i : G/\Gamma \rightarrow G_{\{i\}}/\Gamma_i$ be the natural projection maps for $i = 1, \dots, n$. Then from the case of $n = 1$ we deduce that

$$\bar{q}_i(Fx) = q_i(F)\bar{q}_i(x) = G_i/\Gamma_i.$$

Hence $q_i(F) = G_i$.

Let $F_1 = G_{\{1\}}F$. Since $G_{\{1\}}y$ is compact for all $y \in G/\Gamma$, we have that F_1x is compact. Therefore by what we have proved above there exists $w \in W$ and $\mathcal{J} \in \mathcal{C}$ such that

$$wH_{\mathcal{J}}w^{-1} \subset F_1 \subset Z(G)wH_{\mathcal{J}}w^{-1}.$$

If $H_{\mathcal{J}} \neq G$ then $wH_{\mathcal{J}}w^{-1} \cong \mathrm{SL}_2(K)^k$ for some $1 \leq k \leq n-1$. Since

$$wH_{\mathcal{J}}w^{-1}/(wH_{\mathcal{J}}w^{-1})_x = \prod_{J \in \mathcal{J}} wH_Jw^{-1}/(wH_Jw^{-1})_x,$$

we conclude the result from the induction hypothesis.

Therefore we can assume that $G_{\{1\}}F = F_1 = G$. Since $F \cap \ker(q_1) = F \cap G_{\{2, \dots, n\}}$ is a normal subgroup of F , and it commutes with $G_{\{1\}}$, we have that $F \cap \ker(q_1)$ is normal in G and in particular it is a normal subgroup of $G_{\{2, \dots, n\}}$. Since we have assumed that F does not contain $G_{\{j\}}$ for any j , we have that $F \cap \ker(q_1) \subset Z(G)$. Since $q_1(F) = G_{\{1\}}$, we have $n = 2$. Thus $G = \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$, and $\mathrm{Lie}(F) \cong \mathrm{Lie}(\mathrm{SL}_2(K))$. By the same argument as above $F \cap \ker(q_2) \subset Z(G)$. Since projection of F on each of the factors is surjective, there exists $g \in G_2$ such that

$$\mathrm{Lie}(F) = \{(X, \mathrm{Ad}(g)X) : X \in \mathrm{Lie}(\mathrm{SL}_2(K))\}$$

Since $U \subset F$, we have that $\mathrm{Ad}(g)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore $g \in Z(G)W \cap G_2$. Therefore we can choose $w \in W$ such that $wH_{\{1,2\}}w^{-1} \subset F \subset Z(G)wH_{\{1,2\}}w^{-1}$. This proves the proposition in all the cases. \square

Now from the above result it is straightforward to deduce the following:

Corollary 10.2. — *The singular set $\mathcal{S}(U, \Gamma)$ consists of those $x \in G/\Gamma$ such that Fx is compact for some proper closed subgroup F of G containing U .* \square

10.2. Ergodic U -invariant measures on G/Γ . — The following description of U -ergodic measures was obtained in [12, 17, 19]

Theorem 10.3 (Ratner, Margulis-Tomanov). — *Let λ be a U -invariant U -ergodic probability measure on G/Γ . Then there exists a closed subgroup F of G containing U and a point $x \in G/\Gamma$ such that $Fx \cong F/F_x$ is compact and λ is the unique F -invariant probability measure supported on Fx .*

In particular, by Corollary 10.2, if $\lambda(\mathcal{S}(U, \Gamma)) = 0$ then λ is G -invariant. □

Proof of Theorem 6.3. — We intend to prove this result by induction on n .

If $x \in \mathcal{S}(U, \Gamma)$ then there exists $w \in W$ and $J \in \mathcal{C}$ such that if we put $F = wH_Jw^{-1}$ then $Ux \subset Fx$, Fx is compact, and $F \cong \mathrm{SL}_2(K)^k$, where $k = |J| \leq n - 1$. Since

$$Fx \cong F/F_x = \prod_{J \in \mathcal{J}} wH_Jw^{-1}/(wH_Jw^{-1})_x$$

and $wH_Jw^{-1} \cong \mathrm{SL}_2(K)$, we can replace G by F and the result follows from the induction hypothesis.

Therefore now we can assume that $x \in G/\Gamma \setminus \mathcal{S}(U, \Gamma)$. We put $x_i = x$ for all i . Choose any sequence $T_i \rightarrow \infty$ in K . Then by (43) and (44) we have that $\mu_i = \mu_{T_i}$ for all i . Now by passing to a subsequence we may assume that $\mu_i \rightarrow \mu$ for some $\mu \in \mathcal{M}$; that is, for any $f \in C(G/\Gamma)$,

$$\lim_{i \rightarrow \infty} \int_{G/\Gamma} f d\mu_i = \int_{G/\Gamma} f d\mu.$$

By Lemma 6.2 we have that μ is U -invariant. By Theorem 6.1 we have that $\mu(\mathcal{S}(U, \Gamma)) = 0$. Therefore in view of the decomposition of an invariant measure into its ergodic components, we have that $\lambda(\mathcal{S}(U, \Gamma)) = 0$ for almost all U -ergodic components λ of μ . Therefore by Theorem 10.3 almost all U -ergodic components of μ are G -invariant. Hence μ is G -invariant. □

References

- [1] M. B. BEKKA & M. MAYER – *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, London Mathematical Society Lecture Note Series, vol. 269, Cambridge University Press, 2000.
- [2] C. CORNUT – “Mazur’s conjecture on higher Heegner points”, *Invent. Math.* **148** (2002), no. 3, p. 495–523.
- [3] C. CORNUT & V. VATSAL – “CM points and quaternion algebras”, *Doc. Math.* **10** (2005), p. 263–309.
- [4] S. G. DANI – “A simple proof of Borel’s density theorem”, *Math. Z.* **174** (1980), no. 1, p. 81–94.
- [5] ———, “Dynamical properties of linear and projective transformations and their applications”, *Indian J. Pure Appl. Math.* **35** (2004), no. 12, p. 1365–1394.
- [6] S. G. DANI & G. A. MARGULIS – “Values of quadratic forms at primitive integral points”, *Invent. Math.* **98** (1989), no. 2, p. 405–424.

- [7] ———, “Limit distributions of orbits of unipotent flows and values of quadratic forms”, in *I. M. Gelfand Seminar*, Amer. Math. Soc., 1993, p. 91–137.
- [8] S. G. DANI & S. RAGHAVAN – “Orbits of Euclidean frames under discrete linear groups”, *Israel J. Math.* **36** (1980), no. 3-4, p. 300–320.
- [9] G. A. MARGULIS – “On the action of unipotent groups in the space of lattices”, in *Lie groups and their representations (Proc. Summer School, Bolyai, János Math. Soc., Budapest, 1971)*, Halsted, New York, 1975, p. 365–370.
- [10] ———, “Discrete subgroups and ergodic theory”, in *Number theory, trace formulas and discrete groups (Oslo, 1987)*, Academic Press, 1989, p. 377–398.
- [11] ———, “Orbits of group actions and values of quadratic forms at integral points”, in *Festschrift in honor of I. I. Piatetski-Shapiro on the occasion of his sixtieth birthday, Part II (Ramat Aviv, 1989)*, Israel Math. Conf. Proc., vol. 3, Weizmann, 1990, p. 127–150.
- [12] G. A. MARGULIS & G. M. TOMANOV – “Invariant measures for actions of unipotent groups over local fields on homogeneous spaces”, *Invent. Math.* **116** (1994), no. 1-3, p. 347–392.
- [13] F. I. MAUTNER – “Geodesic flows and unitary representations”, *Proc. Nat. Acad. Sci. U. S. A.* **40** (1954), p. 33–36.
- [14] M. S. RAGHUNATHAN – *Discrete subgroups of Lie groups*, Springer, 1972.
- [15] M. RATNER – “On measure rigidity of unipotent subgroups of semisimple groups”, *Acta Math.* **165** (1990), no. 3-4, p. 229–309.
- [16] ———, “Strict measure rigidity for unipotent subgroups of solvable groups”, *Invent. Math.* **101** (1990), no. 2, p. 449–482.
- [17] ———, “On Raghunathan’s measure conjecture”, *Ann. of Math.* **134** (1991), no. 3, p. 545–607.
- [18] ———, “Raghunathan’s topological conjecture and distributions of unipotent flows”, *Duke Math. J.* **63** (1991), no. 1, p. 235–280.
- [19] ———, “Raghunathan’s conjectures for Cartesian products of real and p -adic Lie groups”, *Duke Math. J.* **77** (1995), no. 2, p. 275–382.
- [20] W. RUDIN – *Real and complex analysis*, McGraw-Hill Book Co., 1966.
- [21] N. A. SHAH – “Uniformly distributed orbits of certain flows on homogeneous spaces”, *Math. Ann.* **289** (1991), no. 2, p. 315–334.
- [22] ———, *Unipotent flows on homogeneous spaces of $SL(2, \mathbb{C})$* , Master’s Thesis, Tata Institute of Fundamental Research, Mumbai, India, 1992.
- [23] ———, “Invariant measures and orbit closures on homogeneous spaces for actions of subgroups generated by unipotent elements”, in *Lie groups and ergodic theory (Mumbai, 1996)*, Tata Inst. Fund. Res. Stud. Math., vol. 14, Tata Inst. Fund. Res., 1998, p. 229–271.
- [24] V. VATSAL – “Uniform distribution of Heegner points”, *Invent. Math.* **148** (2002), no. 1, p. 1–46.
- [25] ———, “Special values of anticyclotomic L -functions”, *Duke Math. J.* **116** (2003), no. 2, p. 219–261.

NIMISH A. SHAH, Tata Institute of Fundamental Research, Mumbai 40005, India
E-mail : nimish@math.tifr.res.in