Chapter 13

Integer Compostion Formulas

There are several ways to construct sums of squares formulas, and most of them use integer coefficients. In fact the bilinear forms involved have coefficients in $\{0, 1, -1\}$ and the constructions are combinatorial in nature. The most fruitful method for these constructions is to use the theory of "intercalate matrices" to restate the composition problem, then to apply various ways of gluing such matrices together. This approach to compositions was pioneered by Yuzvinsky (1981) and considerably extended in the works of Yiu.

After the quaternions and octonions were discovered in the 1840s several mathematicians searched for generalizations. It was around this time that several mathematicians became convinced of the impossibility of a 16-square identity, but no convincing proof was available until much later. In 1848 Kirkman obtained composition formulas of various sizes, including [10, 10, 16] and [12, 12, 26]. He was also aware of the simple construction of a [16, 16, 32] formula obtained from the 8-square identity[1]. The work of Kirkman was not widely known, and those formulas were re-discovered and generalized by K. Y. Lam (1966) and others.

To clarify the ideas, we extend the notations set up in Chapter 12 and define

$$r *_{\mathbb{Z}} s = \min\{n : \text{there is a composition formula of size } [r, s, n] \text{ over } \mathbb{Z}\}.$$

The values of $r *_{\mathbb{Z}} s$ are already known when $r \le 9$. In fact we mentioned in (12.13) that

$$\text{if } r \le 9 \text{ then } r *_{\mathbb{Z}} s = r * s = r \circ s.$$

Lam exhibited several formulas, including a [10, 10, 16], in his 1966 thesis. Subsequently Adem (1975) discovered numerous new formulas derived from the Cayley–Dickson algebras. Based on this experience, Adem conjectured that

$$r *_F r = \begin{cases} 26 & \text{if } r = 11, 12 \\ 28 & \text{if } r = 13 \\ 32 & \text{if } r = 14, 15, 16 \end{cases}$$

1 Kirkman attributes this to J. R. Young, who first observed that if $k = 2, 4$, or 8 then there is a $[km, kn, kmn]$ formula. Further historical information is presented in Dickson (1919).

for any field F of characteristic not 2. Constructions of formulas of those sizes are described below, but it is unknown whether these sizes are best possible, even if real coefficients are used. However using the discrete nature of integer compositions, Yiu has succeeded in proving that Adem's bounds are best possible over $\mathbb{Z}$. Recently Yiu obtained sharp upper bounds for $r *_{\mathbb{Z}} s$ for every $r, s \le 16$. In listing his results we may assume $r, s \ge 10$, since we already know the values when $r \le 9$ or $s \le 9$.

13.1 Theorem (Yiu) The values of $r *_{\mathbb{Z}} s$ for $10 \le r, s \le 16$ are listed in the following table:

$r \backslash s$	10	11	12	13	14	15	16
10	16	26	26	27	27	28	28
11	26	26	26	28	28	30	30
12	26	26	26	28	30	32	32
13	27	28	28	28	32	32	32
14	27	28	30	32	32	32	32
15	28	30	32	32	32	32	32
16	28	30	32	32	32	32	32

Yiu's early work on integer compositions involved a mixture of topological and combinatorial methods to find lower bounds for $r *_{\mathbb{Z}} s$. *However the theorem above was proved by elementary (but intricate) combinatorial methods, avoiding the use of topology.*

We will present the details for the construction of some of these formulas, but we give only brief hints about Yiu's non-existence proofs. Constructions of composition formulas beyond the range $r, s \le 16$ have been considered by several authors. Their results are presented in Appendix C below.

To begin the analysis let us recall three formulations of the problem of integer compositions.

13.2 Lemma. *The following statements are equivalent.*

(1) *There exists an* $[r, s, n]_{\mathbb{Z}}$ *formula*

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where each $z_k = z_k(X, Y)$ *is a bilinear form in* X *and* Y *with coefficients in* $\mathbb{Z}$.

(2) *There is a set of* $n \times s$ *matrices* $A_1, \dots, A_r$ *with coefficients in* $\mathbb{Z}$ *and satisfying the Hurwitz equations* $A_i^\top \cdot A_j + A_j^\top \cdot A_i = 2\delta_{ij} I_s$ *for* $1 \le i, j \le r$.

(3) *There is a bilinear map* $f : \mathbb{Z}^r \times \mathbb{Z}^s \to \mathbb{Z}^n$ *satisfying the norm condition*

$$|f(x, y)| = |x| \cdot |y|.$$

Proof. This is a special case of Proposition 1.9. We could allow each z_k in (1) to be a polynomial in $\mathbb{Z}[X, Y]$. A degree argument implies that z_k must be a bilinear form. □

Let A, B, C be the standard (orthonormal) bases for $\mathbb{Z}^r, \mathbb{Z}^s, \mathbb{Z}^n$, respectively. Then for instance, every $x \in \mathbb{Z}^r$ can be uniquely expressed as $x = \sum_{a \in A} x_a a$, for $x_a \in \mathbb{Z}$. If $a \in A$ and $b \in B$, then $f(a, b) = \sum_c \gamma_c^{a,b} c$ and $1 = |a|^2 \cdot |b|^2 = |f(a, b)|^2 = \sum_c (\gamma_c^{a,b})^2$. Since these are integers, there is exactly one c for which $\gamma_c^{a,b} = \pm 1$, while all the other terms are 0. That choice of c depends only on a, b and we write it as $c = \varphi(a, b)$. Then φ is a well-defined function on $A \times B$ and $f(a, b) = \pm\varphi(a, b)$. Letting $\varepsilon(a, b)$ be that sign, we obtain functions

$$\varphi : A \times B \to C$$
$$\varepsilon : A \times B \to \{1, -1\}$$

such that $f(a, b) = \varepsilon(a, b) \cdot \varphi(a, b)$ for every $a \in A$ and $b \in B$.

We will translate the norm condition on f to statements about these new functions. As before, $\langle u, v\rangle$ denotes the inner product. Using indeterminates x_a for $a \in A$ and y_b for $b \in B$, we obtain:

$$\sum_{a,b} x_a^2 y_b^2 = \Big|\sum_a x_a a\Big|^2 \cdot \Big|\sum_b y_b b\Big|^2 = \Big|\sum_{a,b} x_a y_b f(a, b)\Big|^2$$
$$= \sum_{a,b} \sum_{a',b'} x_a x_{a'} y_b y_{b'} \langle f(a, b), f(a', b')\rangle.$$

Comparing coefficients of x_a^2 we find that if $b \neq b'$ then $\langle f(a, b), f(a, b')\rangle = 0$. Since C is an orthonormal set, this condition says: if $b \neq b'$ then $\varphi(a, b) \neq \varphi(a, b')$, an injectivity condition on φ.

Similarly the coefficients of y_b^2 show that $a \neq a'$ implies $\varphi(a, b) \neq \varphi(a', b)$. Fixing the subscripts $a \neq a'$ and $b \neq b'$ and comparing coefficients, we find:

$$0 = \langle f(a, b), f(a', b')\rangle + \langle f(a, b'), f(a', b)\rangle.$$

Therefore: $\varphi(a, b) = \varphi(a', b')$ if and only if $\varphi(a, b') = \varphi(a', b)$. Moreover if these equalities hold for given indices a, b, a', b', then by computing the signs we find:

$$\varepsilon(a, b) \cdot \varepsilon(a', b') = -\varepsilon(a, b') \cdot \varepsilon(a', b).$$

The function φ can be tabulated as an $r \times s$ matrix M (with rows indexed by A and columns indexed by B) with entries in C. Following Yiu's terminology, the entries of M are called *colors* and $n(M)$ denotes the number of distinct colors in M. If $n = n(M)$ we usually take the set of colors to be $\{1, 2, \ldots, n\}$ or $\{0, 1, \ldots, n-1\}$.

13.3 Definition. Suppose M is an $r \times s$ matrix with entries taken from a set of "colors". Let $M(i, j)$ be the (i, j)-entry of M.

(a) M is an *intercalate*[2] matrix if:

(1) The colors along each row (resp. column) are distinct.

(2) If $M(i, j) = M(i', j')$ then $M(i, j') = M(i', j)$. (intercalacy)

An intercalate matrix M has *type* (r, s, n) if it is an $r \times s$ matrix at most n colors: $n(M) \leq n$.

(b) An intercalate matrix M is *signed consistently* if there exist $\varepsilon_{ij} = \pm 1$ such that

$$\varepsilon_{ij}\varepsilon_{ij'}\varepsilon_{i'j}\varepsilon_{i'j'} = -1 \text{ whenever } M(i, j) = M(i', j') \text{ and } i \neq i' \text{ and } j \neq j'.$$

The intercalacy condition says that every 2×2 submatrix of M involves an even number of distinct colors. The consistency condition says that every 2×2 submatrix with only two distinct colors must have an odd number of minus signs.

13.4 Lemma. *There is an* $[r, s, n]_{\mathbb{Z}}$ *formula if and only if there is a consistently signed intercalate matrix of type* (r, s, n).

Proof. This equivalence is explained in the preceding discussion. Note that if $x = \sum_i x_i a_i \in \mathbb{Z}^r$ and $y = \sum_j y_i b_j \in \mathbb{Z}^s$ then $f(x, y) = \sum_k z_k c_k$ where $z_k = \sum \varepsilon_{ij} x_i y_j$ summed over all i, j such that $M(i, j) = k$. Then the terms in z_k correspond to occurrences of the color k in the intercalate matrix. $\square$

These matrices and their signings were first studied by Yuzvinsky (1981) who used the term "monomial pairings". He noted that with this formulation the problem of $[r, s, n]_{\mathbb{Z}}$ formulas separates into two questions:

(1) For which values r, s, n is there an intercalate matrix of type (r, s, n)?

(2) Given an intercalate matrix, does it have a consistent signing?

The reader is invited to verify that the following 3×5 matrix is intercalate, to find a consistent signing and to write out the corresponding composition formula of size [3, 5, 7].

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 6 \\ 3 & 4 & 1 & 2 & 7 \end{pmatrix}$$

Two intercalate matrices A, B of type (r, s, n) are defined to be *equivalent* if A can be brought to B by permutation of rows, permutation of columns, and relabelling of colors. Up to equivalence $D_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the unique intercalate matrix of type

2 Pronounced with the accent on the syllable "ter". The word "intercalate" was introduced in this context by Yiu, following some related usage in combinatorics.

$(2, 2, 2)$. One consistent signing of D_1 is $\begin{pmatrix} +0 & +1 \\ +1 & -0 \end{pmatrix}$. Of course these signed values, like $+1$ and -0, should be interpreted formally as a sign and a color, certainly not as a real number. This signed matrix can easily be re-written as a composition formula using the expression for z_k given in the proof of (13.4). With the colors $\{0, 1\}$ here it is convenient to number the rows and columns of D_1 by the indices $\{0, 1\}$ as well. In this case we find $z_0 = +x_0y_0 - x_1y_1$ and $z_1 = +x_0y_1 + x_1y_0$.

If an intercalate matrix is consistently signed then that signing can be carried over to any equivalent matrix. Moreover on a given intercalate matrix M there may be several consistent signings. Starting from one such signing, changing all the signs in any row (or column) yields another consistent signing. Similarly changing the signs of all occurrences of a single color yields another consistent signing. If one signing of M can be transformed to another by some sequence of these three types of changes we say the signings are *equivalent*. Any signing is equivalent ot a "standard" signing: all " $+$" signs in the first row and first column.

There are several methods for constructing new intercalate matrices from old ones. In some cases these methods provide consistent signings as well. For example suppose M is an intercalate matrix of type (r, s, n). Then any submatrix M' of M is also intercalate, and if M is consistently signed then so is M'. In this case M' is called a *restriction* of M. On the level of sums of squares formulas this construction is the same as setting a subset of the x's and a subset of the y's equal to zero.

Another construction is the *direct sum*. Suppose A, A' are intercalate matrices of types (r, s, n) and (r, s', n'), respectively. Replace A' by an equivalent matrix if necessary to assume that A and A' involve disjoint sets of colors, and define

$$M = (\,A \quad A'\,)\,.$$

Then M is an intercalate matrix of type $(r, s + s', n + n')$. If A and A' are consistently signed then so is M. On the level of normed mappings this direct sum construction was mentioned in the proof of (12.12). (What is the corresponding construction for composition formulas?)

Of course the construction may be done with the roles of r and s reversed: $(r, s, n) \oplus (r', s, n') \;\Rightarrow\; (r + r', s, n + n')$. Let us apply these ideas to the standard consistently signed intercalate matrix A of type $(8, 8, 8)$. Define A', A", A''' to be copies of A, using disjoint sets of 8 colors. Then the matrix

$$M = \begin{pmatrix} A & A' \\ A'' & A''' \end{pmatrix}$$

is the double direct sum of four copies of A. It is a consistently signed intercalate matrix of type (16, 16, 32). The corresponding composition formula was mentioned earlier.

Perhaps the simplest intercalate matrices are of the type (r, s, rs) in which all entries of the matrix are distinct. Every signing of this matrix is consistent and the corresponding sums of squares formula is the trivial one in which all the terms are multiplied out. This example can be built by a sequence of direct sum operations applied to the 1×1 matrix $D_0 = [0]$.

A third construction is the *tensor product* (Kronecker product) of matrices. Suppose $A = (a_{ij})$ and $B = (b_{kl})$ are intercalate matrices of types (r_1, s_1, n_1) and (r_2, s_2, n_2), respectively. Then $A \otimes B = (c_{ik,jl})$ is an intercalate matrix of type (r_1r_2, s_1s_2, n_1n_2). Here the color $c_{ik,jl}$ is the ordered pair (a_{ij}, b_{kl}) and the row-indices (i, k) and the column-indices (j, l) must each be listed in some definite order. The matrix $A \otimes B$ is intercalate if and only if A and B are intercalate. In writing out a tensor product we re-write the colors as integers from 0 to $n-1$. For example starting from $D_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we obtain

$$D_2 = D_1 \otimes D_1 = \begin{pmatrix} 00 & 01 & 10 & 11 \\ 01 & 00 & 11 & 10 \\ 10 & 11 & 00 & 01 \\ 11 & 10 & 01 & 00 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}.$$

(The translation from bit-strings to integers uses the standard base 2, or dyadic, notation.) This tensoring process can be repeated to obtain intercalate matrices D_t of type $(2^t, 2^t, 2^t)$. These matrices D_t may also be defined inductively, without explicit mention of tensor products, as follows:

$$D_0 = (\,0\,) \quad \text{and} \quad D_{t+1} = \begin{pmatrix} D_t & 2^t + D_t \\ 2^t + D_t & D_t \end{pmatrix}.$$

Here D_t is a matrix of integers and $2^t + D_t$ is obtained by adding 2^t to each entry of D_t. Another step of this process yields the 8×8 matrix D_3:

$$\left(\begin{array}{cccc|cccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 0 & 3 & 2 & 5 & 4 & 7 & 6 \\ 2 & 3 & 0 & 1 & 6 & 7 & 4 & 5 \\ 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\ \hline 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\ 5 & 4 & 7 & 6 & 1 & 0 & 3 & 2 \\ 6 & 7 & 4 & 5 & 2 & 3 & 0 & 1 \\ 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \end{array}\right)$$

It is not hard to check from this definition that every D_t is intercalate. However, D_t cannot be consistently signed when $t > 3$, by the original 1, 2, 4, 8 Theorem.

This matrix D_t can also be viewed as the table of a binary operation on the interval $[0, 2^t) = \{0, 1, 2, \ldots, 2^t - 1\}$. If $m, n \in [0, 2^t)$ define

$$m \boxplus n = \text{the } (m, n)\text{-entry of } D_t,$$

where the rows and columns of D_t are indexed by the values $0, 1, 2, \ldots, 2t - 1$. This operation is the well-known "Nim-addition" studied in the analysis of the game of Nim. (For further information on Nim and related games see books on recreational mathematics. A good example is Berlekamp–Conway–Guy (1982).) The Nim-sum $m \boxplus n$ is easily described using the dyadic expansions of m, n: express m, n as bit-strings of length t, add them as t-tuples in the group $(\mathbb{Z}/2\mathbb{Z})^t$, transform the resulting bit-string back to an integer. For example $3 = (011)$ and $6 = (110)$ in dyadic expansion and $3 \boxplus 6 = (101) = 5$. Certainly the Nim sum makes the nonnegative integer into a group, such that $n \boxplus n = 0$ for every n.

Therefore the matrix D_t is just the addition table for the group $(\mathbb{Z}/2\mathbb{Z})^t$, re-written with the labels $0, 1, 2, \dots, 2t - 1$ in place of bit-strings. With this interpretation the intercalacy condition is obvious:

$$i \boxplus j = i' \boxplus j' \quad \text{implies} \quad i \boxplus j' = i' \boxplus j.$$

Certainly every submatrix of D_t is intercalate. Define an intercalate matrix to be *dyadic* if it is equivalent to a submatrix of some D_t. The standard dyadic $r \times s$ intercalate matrix is $D_{r,s}$, defined to be the upper left $r \times s$ corner of D_t (where t is chosen so that $r, s \leq 2^t$). For instance the 3×5 matrix mentioned after (13.4) is exactly the matrix $D_{3,5}$ with each entry increased by 1. That matrix $D_{3,5}$ involves 7 of the 8 colors of D_3. How many colors are involved in $D_{r,s}$? Surprisingly the answer is provided by the Stiefel–Hopf function $r \circ$ defined in (12.5).

13.5 Lemma. *$D_{r,s}$ involves exactly $r \circ s$ colors.*

Proof. Let $r \bullet s = n(D_{r,s})$, the number of colors in $D_{r,s}$. Certainly $r \bullet s = s \bullet r$; $1 \bullet s = s$; $2^m \bullet 2^m = 2^m$; and if $r \leq r'$ then $r \bullet s \leq r' \bullet s$. Using the inductive definition of D_{t+1} check that $2^m \bullet (2^m + 1) = 2^{m+1}$ and that if $r, s \leq 2^m$ then $r \bullet (s + 2^m) = (r \bullet s) + 2^m$. These properties suffice to determine all values $r \bullet s$, and these match the values $r \circ s$ by (12.10). ◻

This property of $D_{r,s}$ was first noted by Yuzvinsky (1981). He conjectured that every $r \times s$ intercalate matrix contains at least $r \circ s$ colors, and he proved this conjecture for dyadic matrices, (that is for submatrices of some D_t.) An elegant new proof of this result has been recently discovered by Eliahou and Kervaire, using polynomial method popularized by Alon and Tarsi. See Appendix A below. Yuzvinsky's conjecture remains open for non-dyadic intercalate matrices, although Yiu has proved the conjecture whenever $r, s \leq 16$.

The classical n-square identities arise from the Cayley–Dickson doubling process, as described in the appendix to Chapter 1. Using a standard basis of the Cayley–Dickson algebra A_t, the multiplication table turns out to be a signed version of the matrix D_t. The signs are not hard to work out (Exercise 5) using the inductive definition of "doubling". For later reference we display here the signing of D_4 which arises from the Cayley–Dickson algebra A_4.

$$
\left(\begin{array}{cccccccc|cccccccc}
+0 & +1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +9 & +10 & +11 & +12 & +13 & +14 & +15 \\
+1 & -0 & +3 & -2 & +5 & -4 & -7 & +6 & +9 & -8 & -11 & +10 & -13 & +12 & +15 & -14 \\
+2 & -3 & -0 & +1 & +6 & +7 & -4 & -5 & +10 & +11 & -8 & -9 & -14 & -15 & +12 & +13 \\
+3 & +2 & -1 & -0 & +7 & -6 & +5 & -4 & +11 & -10 & +9 & -8 & -15 & +14 & -13 & +12 \\
+5 & +4 & -7 & +6 & -1 & -0 & -3 & +2 & +13 & -12 & +15 & -14 & +9 & -8 & +11 & -10 \\
+6 & +7 & +4 & -5 & -2 & +3 & -0 & -1 & +14 & -15 & -12 & +13 & +10 & -11 & -8 & +9 \\
+7 & -6 & +5 & +4 & -3 & -2 & +1 & -0 & +15 & +14 & -13\bullet & -12 & +11 & +10 & -9 & -8 \\
\hline
+8 & -9 & -10 & -11 & -12 & -13 & -14 & -15 & -0 & +1 & +2 & +3 & +4 & +5 & +6 & +7 \\
+9 & +8 & -11 & +10 & -13\bullet & +12 & +15 & -14 & -1 & -0 & -3\bullet & +2 & -5 & +4 & +7 & -6 \\
+10 & +11 & +8 & -9 & -14 & -15 & +12 & +13 & -2 & +3 & -0 & -1 & -6 & -7 & +4 & +5 \\
+11 & -10 & +9 & +8 & -15 & +14 & -13 & +12 & -3 & -2 & +1 & -0 & -7 & +6 & -5 & +4 \\
+12 & +13 & +14 & +15 & +8 & -9 & -10 & -11 & -4 & +5 & +6 & +7 & -0 & -1 & -2 & -3 \\
+13 & -12 & +15 & -14 & +9 & +8 & +11 & -10 & -5 & -4 & +7 & -6 & +1 & -0 & +3 & -2 \\
+14 & -15 & -12 & +13 & +10 & -11 & +8 & +9 & -6 & -7 & -4 & +5 & +2 & -3 & -0 & +1 \\
+15 & +14 & -13 & -12 & +11 & +10 & -9 & +8 & -7 & +6 & -5 & -4 & +1 & +2 & -1 & -0
\end{array}\right)
$$

Observe that this signing is not consistent: for example the signs of colors 3 and 13 in rows 7, 9 and columns 4, 10 do not satisfy the condition for consistent signs. Those entries are marked with bullets "$\bullet$". However there are some interesting submatrices which are consistently signed. We will analyze $D_{9,16}$ and $D_{10,10}$.

One can verify directly that the signings of these submatrices are consistent. For a more conceptual method, recall that the upper left 8×8 block D_3 is consistently signed since it arises from the standard 8-square identity. Now examine the larger 9×16 block. This provides an example of the following "doubling construction".

13.6 Proposition. *Any consistently signed intercalate matrix of type* (r, s, n) *can be enlarged to one of type* $(r + 1, 2s, 2n)$.

Proof. Let A be the given intercalate matrix with sign matrix S. We may assume that the top row of A is $v = (0, 1, 2, \ldots, s - 1)$ and the top row of S is all " $+$" signs. Let A' be the intercalate matrix obtained from A by replacing every color c by a new color c'. Then the top row of A' is $v' = (0', 1', 2', \ldots)$. Define $M = \begin{pmatrix} A & A' \\ v & v' \end{pmatrix}$. Since M is a submatrix of the tensor product $A \otimes D_1$, it is intercalate of type $(r + 1, 2s, 2n)$. It remains to show that M can be consistently signed.

Use the given signs $S = (\varepsilon_{ij})$ for the submatrix A, " $+$" signs on the top row of A' and attach arbitrary signs $(\alpha_0, \alpha_1, \alpha_2, \ldots)$ for the v' in the bottom row of M.

Claim. There is a unique way to attach signs to v and to the rest of A' to produce a consistent signing of M.

The sign condition for the top and bottom rows and for columns j and $s+j$ forces the signs attached to the row v to be $(-\alpha_0, -\alpha_1, -\alpha_2, \dots)$. For given i, j with $0 < i \le r$, we will determine the sign ε'_{ij} attached to the entry $A'(i, j)$. Let $A'(i, j) = k'$ so that $A(i, j) = k$. The intercalacy for the rows 0 and i and for columns j and k shows that $A(i, k) = j$. The sign condition for this rectangle implies that $\varepsilon_{ij}\varepsilon_{ik} = -1$, as well. The following picture of the matrix M may help to clarify this argument.

$$
\begin{array}{cc}
 & \begin{array}{ccccccc|ccccc} & & & j & & k & & & & & s+j & \end{array} \\
\begin{array}{r} \\ \\ i \\ \\ r+1 \end{array} & \left(\begin{array}{ccccccc|ccccc}
+0 & +1 & \dots & +j & \dots & +k & \dots & +0' & +1' & \dots & +j' & \cdots \\
 & & & & & & & & & & & \\
\dots & \dots & \dots & \epsilon_{ij}k & \dots & \epsilon_{ij}j & \dots & \dots & \dots & \dots & \epsilon_{ij}k' & \dots \\
\hline
\alpha_0 0' & \alpha_1 1' & \dots & \alpha_j j' & \dots & \alpha_k k' & \dots & -\alpha_0 0 & -\alpha_1 1' & \dots & -\alpha_j j & \dots
\end{array}\right)
\end{array}
$$

Now examine the rectangle with opposite corners $M(i, k) = j$ and $M(r+1, s+j) = j$ to see that $\varepsilon_{ik}\varepsilon'_{ij}\alpha_k(-\alpha_j) = -1$. Since $\varepsilon_{ik} = -\varepsilon_{ij}$ we conclude:

$$\varepsilon'_{ij} - \alpha_k\alpha_j\varepsilon_{ij} \quad \text{where } k = A(i, j).$$

We must verify that this signing is consistent. By construction all the sign conditions involving the bottom row of M are consistent. Since A and A' have no colors in common, it remains only to check the submatrix A'. The signs ε'_{ij} of A' are obtained from the signs S as follows: multiply the jth column by the sign $-\alpha_j$ and multiply every occurrence of the color k by the sign α_k. Therefore the signing of A' is equivalent to the consistent signing of A. ◻

On the level of sums of squares formulas it was mentioned in Exercise 0.2. Now let us return to the multiplication table for A_4 displayed earlier. It is not hard to verify that the first 9 rows are obtained by this doubling construction applied to the standard consistent signing of D_3. Therefore that 9×16 block is consistently signed and we have a sums of squares formula of size [9, 16, 16]. (Of course we already constructed such a formula in the proof of the Hurwitz–Radon Theorem.) Another application of the doubling process, this time with the roles of r and s reversed, yields a formula of size [18, 17, 32], improving on the earlier [16, 16, 32].

Repeated application of the doubling process starting from [8, 8, 8] produces formulas of sizes $[t+5, 2^t, 2^t]_{\mathbb{Z}}$. In fact the corresponding signed intercalate matrix can be found inside the multiplication table of A_t by choosing the columns $0, 1, 2, \dots, 7$ and 2^k for $k = 3, \dots, t-1$. On the other hand, Khalil (1993) proved no subset of

$t+6$ columns of the multiplication table of A_t is consistently signed. In particular it is not possible to find a [12, 64, 64] formula inside A_6. We can also use that matrix $[9, 16, 16]_{\mathbb{Z}}$ to give another proof of (12.13):

13.7 Corollary. *If $r \leq 9$ then $r *_{\mathbb{Z}} s = r \circ s$.*

Proof. We know generally that $r \circ s \leq r * s \leq r *_{\mathbb{Z}} s$ from the results of Stiefel and Hopf discussed in Chapter 12. Equality holds if there exists an $[r, s, r \circ s]_{\mathbb{Z}}$ formula. Suppose $t \geq 4$ and consider $D_{9,2^t}$. This matrix can be consistently signed by viewing it as the direct sum of 2^{t-4} copies of $D_{9,16}$. Then if $r \leq 9$ and $s \leq 2^t$ the submatrix $D_{r,s}$ is consistently signed and involves exactly $r \circ s$ colors by (13.5). □

The matrices D_t are examples of intercalate matrices of type (n, n, n). Are there any other examples? Consider more generally a square intercalate matrix M of type (r, r, n). A color is called *ubiquitous* in the $r \times r$ matrix M if it appears in every row and every column. If M has a ubiquitous color then M is equivalent to a symmetric matrix, with the ubiquitous color along the diagonal. (This follows from the intercalacy condition.)

13.8 Lemma. *Suppose the intercalate matrix M of type (r, r, n) has two ubiquitous colors. Then r and n are even and M is equivalent to a tensor product $D_1 \otimes M'$.*

Proof. We may assume M is symmetric with one color along the diagonal. Permute the rows and columns to arrange the second ubiquitous color along the principal 2×2 blocks. From this it follows that r is even. Partition M into 2×2 blocks and use the intercalacy condition with the diagonal blocks to deduce that each block is of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. Then n must be even and the tensor decomposition follows. □

One can now check (as in Exercise 3) that the D_t's are the only intercalate matrices of type (n, n, n). Most of our examples are signings of various submatrices of D_t. However there exist intercalate matrices which are not equivalent to a submatrix of any D_t. (See Exercise 1.)

Suppose M is a symmetric intercalate matrix of type (r, r, n), so that the diagonal of M contains a single (ubiquitous) color. We can enlarge M to a matrix $\sum M$ which is symmetric intercalate of type $(r+1, r+1, r+n)$ by appending a new row and column to the bottom and right of M using r new colors (symmetrically) for that row and column, and assigning the diagonal color of M to the lower right corner. For example starting with $L_1 = (\,0\,)$ of type $(1, 1, 1)$ we obtain inductively $L_{r+1} = \sum L_r$, a symmetric intercalate matrix of type $(r, r, 1 + \binom{r}{2})$. We may choose the colors successively from

$\{0, 1, 2, 3, \ldots\}$ to obtain

$$\begin{pmatrix} 0 & 1 & 2 & 4 & 7 & \vdots & \vdots \\ 1 & 0 & 3 & 5 & 8 & \vdots & \vdots \\ 2 & 3 & 0 & 6 & 9 & \vdots & \vdots \\ 4 & 5 & 6 & 0 & 10 & \vdots & \vdots \\ 7 & 8 & 9 & 10 & 0 & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & 0 & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \end{pmatrix}$$

Each of these matrices L_r can be consistently signed: endow each color in the upper triangle, including the diagonal, of L_r with " $+$" and each color in the lower triangle with " $-$". The corresponding sums of squares identity is the Lagrange identity:

$$(x_1^2+x_2^2+\cdots+x_r^2)\cdot(y_1^2+y_2^2+\cdots+y_r^2) = (x_1y_1+\cdots+x_ry_r)^2+\sum_{i<j}(x_iy_j-x_jy_i)^2,$$

of type $[r, r, 1+\binom{r}{2}]_{\mathbb{Z}}$. This identity provides one proof of the Cauchy–Schwartz inequality.

Now let us re-examine the 10×10 submatrix of the Cayley–Dickson signing of D_4. That matrix decomposes into 2×2 blocks corresponding to the two ubiquitous colors 0, 1. The basic 8×8 matrix is expanded to the 10×0 using an analog of the $\sum$ construction as follows.

13.9 Lemma. *Suppose M is a consistently signed intercalate matrix of type (r, r, n) with two ubiquitous colors. Then M can be expanded to a consistently signed intercalate matrix M' of type $(r+2, r+2, r+n)$. The same two colors are ubiquitous in M'.*

Proof. Replacing M by an equivalent matrix we may assume that M is decomposed into 2×2 blocks with first diagonal block $\begin{pmatrix} +0 & +1 \\ +1 & -0 \end{pmatrix}$ and subsequent diagonal blocks block $\begin{pmatrix} -0 & +1 \\ -1 & -0 \end{pmatrix}$. Construct the matrix M' by appending a new row and column of 2×2 blocks to M. The first $r/2$ blocks in the new column are of the form block $\begin{pmatrix} +a & +b \\ +b & -a \end{pmatrix}$, involving r new colors, and the lower right corner block is assigned the diagonal value block $\begin{pmatrix} -0 & +1 \\ -1 & -0 \end{pmatrix}$. The entries along the bottom row are determined by the intercalacy and sign conditions, and involve the same r new colors. This matrix M' is intercalate of type $(r+2, r+2, r+n)$ and is consistently signed. □

Applying this construction to the standard (8, 8, 8), we get a consistently signed intercalate matrix of type $(10, 10, 16)$. This matrix appears as the upper left 10×10 submatrix of the signed D_4 displayed earlier. Repeating this construction we obtain a consistently signed intercalate matrix of type $(12, 12, 26)$. Consequently there are integer composition formulas of types $[10, 10, 16]$ and $[12, 12, 26]$. Another repetition does not yield an interesting result since we already know a formula of type $[16, 16, 32]$.

We saw in Chapters 1 and 2 that for any n there exists a composition formula of size $[\rho(n), n, n]$. In fact we gave an explicit construction for such formulas: First build an $(m+1, m+1)$-family, either by the Construction Lemma (2.7) or by using the trace form on a Clifford algebra (Exercise 3.15). Then apply the Shift Lemma (2.6) and Expansion Lemma (2.5). If the underlying quadratic form is the sum of squares then all entries of the matrices are in $\mathbb{Z}$ and there must be a corresponding signed intercalate matrix. Can it be constructed directly using the combinatorial methods here?

There are two constructions in the literature for explicit signed intercalate matrices which realize the Hurwitz–Radon formulas. There are given in Yiu (1985), and in Yuzvinsky (1984) as corrected by Lam–Smith (1993). Both of these constructions are obtained by consistently signing a suitably chosen $\rho(t) \times 2^t$ submatrix of D_t. These two constructions do not yield equivalent formulas, even though we proved in Chapter 7 that any two formulas of size $[\rho(n), n, n]$ are equivalent, over any field F. The point here is that the notion of equivalence of composition formulas over $\mathbb{Z}$ (i.e. of signed intercalate matrices) is much more restrictive than equivalence over a field.

We will outline (without proofs) some of the underlying ideas involved in the Yuzvinsky–Lam–Smith construction, since that method leads to infinite sequences of new composition formulas. Recall from the discussion before (13.3) that an $[r, s, n]_{\mathbb{Z}}$ formula is determined by two mappings φ and θ where $\varphi : A \times B \to C$ and $\theta : A \times B \to \{1, -1\}$. Here A, B, C are sets of cardinalities r, s, n, respectively. With this notation the three conditions in (13.3) become:

(i) If $a \in A$ the map $\varphi|a \times B$ is injective. If $b \in B$ the map $\varphi|_{A\times b}$ is injective.

(ii) If $a_i \in A$ and $b_i \in B$ and $\varphi(a_1, b_1) = \varphi(a_2, b_2)$ then $\varphi(a_1, b_2) = \varphi(a_2, b_1)$.

(iii) If $a_1 \neq a_2$ and $\varphi(a_1, b_1) = \varphi(a_2, b_2)$ then

$$\theta(a_1, b_1) \cdot \theta(a_1, b_2) \cdot \theta(a_2, b_1) \cdot \theta(a_2, b_2) = -1.$$

To construct maps φ and θ satisfying these conditions consider a normal subgroup H of some finite group G. Left multiplications induces a permutation action $G \times G/H \to G/H$. Choose subsets $A \subseteq G$ and $B \subseteq G/H$, use the map $\varphi : A \times B \to G/H$ given by restriction, and try to find a signing map $\theta : A \times B \to \{\pm 1\}$ so that the three conditions are satisfied. To define θ choose a homomorphism $\chi : H \to \{\pm 1\}$, and a set $\{d_1, \dots, d_n\}$ of coset representatives of H in G. For any d_i and any $g \in G$ then $gd_iH = d_jH$ for some d_j. Define θ by setting $\theta(g, d_iH) = \chi(d_j^{-1}gd_i)$. If φ and θ are constructed this way the three conditions above become the following:

(i′) $g^{-1}g' \notin H$ whenever $g, g' \in A$ and $g \neq g'$.

Suppose $g \neq g'$ in A and there exist $d_i, d_j \in B$ such that $gd_iH = g'd_jH$.
(ii′) $(g^{-1}g')^2 \in H$.
(iii′) $\chi(d_i^{-1}(g^{-1}g')^2 d_i) = -1$.

To apply this criterion we use the group G_r defined as follows by generators and relations:

$$G_r = \langle \varepsilon, g_1, \ldots, g_r \mid \varepsilon^2 = 1,\ g_i^2 = \varepsilon,\ g_i g_j = \varepsilon g_j g_i \rangle.$$

This is the group employed by Eckmann (1942/43) in his proof of the Hurwitz–Radon Theorem using group representations. In fact this approach was motivated directly by Eckmann's work. If V is an F-vector space and $\pi : G_r \to \mathrm{GL}(V)$ is a group homomorphism with $\pi(\varepsilon) = -1$, then the elements $f_i = \pi(g_i)$ generate a Clifford algebra (they anticommute and have squares equal to -1). Now suppose $G = G_r$, H is a normal subgroup containing ε, and $\chi : H \to \{\pm 1\}$ is a homomorphism with $\chi(\varepsilon) = -1$. Then the three conditions above boil down to one requirement:

$$(g^{-1}g')^2 = \varepsilon \text{ whenever } g, g' \in A \text{ and } g \neq g'.$$

For example, these conditions hold if H is a maximal elementary abelian 2-subgroup of G_r, $A = \{1, g_1, g_2, \ldots, g_r\}$ and $B = G/H$. If $|G/H| = 2^m$ this provides an $[r+1, 2^m, 2^m]_{\mathbb{Z}}$ formula. It turns out that this value 2^m is exactly the value needed for a formula of Hurwitz–Radon type; that is, $\rho(2^m) = r + 1$.

Yuzvinsky's idea is to construct new examples by modifying the pairings derived in this way. He found a way to enlarge the set A while decreasing the set B, keeping C the same. He obtained various formulas of size $(2m+2, 2^m - p(m), 2^m)$ where $p(m)$ represents the number of elements in B which must be excluded to accommodate the increase of 1 or 2 elements in A. There are a number of errors and gaps in Yuzvinsky's paper but these have been carefully corrected and clarified in the work of Lam–Smith (1993). Here are the two families of formulas which follow from these methods.

13.10 Proposition. *Suppose* $m > 1$. *Then there exists a* $[2m+2, 2^m - p(m), 2^m]_{\mathbb{Z}}$ *formula in the following two cases:*
(1) $m \equiv 0 \pmod 4$ *and* $p(m) = \binom{m}{m/2}$.
(2) $m \equiv 1 \pmod 4$ *and* $p(m) = 2\binom{m-1}{(m-1)/2}$.

We omit further details. Applying this calculation when $m = 4, 5$ provides $[10, 10, 16]_{\mathbb{Z}}$ and $[12, 20, 32]_{\mathbb{Z}}$ formulas. This last example is important for us since it can be modified to yield some of the values appearing in Theorem 13.1.

13.11 Corollary. *There exist formulas of sizes* $[10, 16, 28]$ *and* $[12, 14, 30]$.

Outline. These formulas arise as restrictions of the explicit $[12, 20, 32]$ constructed by the group-theoretic method above. Signed intercalate matrices of these sizes are displayed in the Appendix to Lam–Smith (1993). These formulas are also mentioned in Smith–Yiu (1994). ◻

There are several formulas still to construct in order to realize all the values of $r *_{\mathbb{Z}} s$ listed in Theorem 13.1. As in Smith–Yiu (1994) we derive these formulas by explicitly displaying various signed intercalate matrices. The consistently signed intercalate matrix given below, of type $(17, 17, 32)$, is obtained as follows: from the $[18, 17, 32]_{\mathbb{Z}}$ constructed by the doubling process (13.6), delete the bottom row and move the rightmost column to the middle of the matrix. For $12 \le r \le s \le 16$ the $r \times s$ submatrix in the upper left corner contains exactly $24 + (r-9) \circ (s-9)$ colors. Therefore this matrix furnishes formulas for all the entries of the table in Theorem 13.1 for the cases $12 \le r \le s$, except for the cases $(r, s) = (12, 12)$ and $(12, 14)$. Since those two sizes were constructed earlier, only the cases $r = 10$ and 11 remain to be verified. In this display we follow the convention of Yiu and use the colors $\{1, 2, \ldots, 32\}$ (rather than $\{0, 1, \ldots, 31\}$).

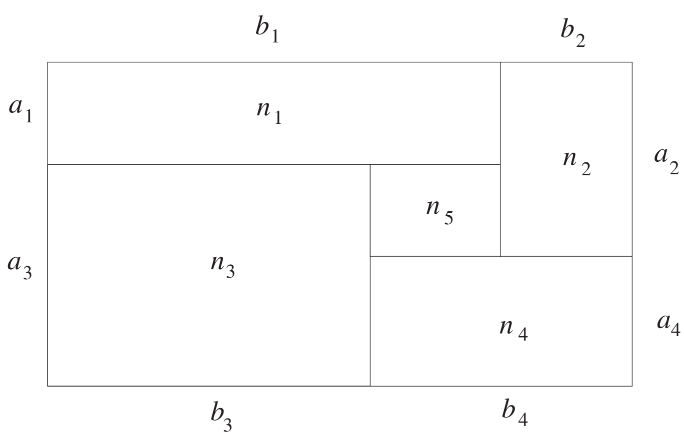
$$
\left(\begin{array}{cccccccc|c|cccccccc}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +17 & +9 & +10 & +11 & +12 & +13 & +14 & +15 & +16 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 & +18 & -10 & +9 & -12 & +11 & -14 & +13 & +16 & -15 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 & +19 & -11 & +12 & +9 & -10 & -15 & -16 & +13 & +14 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 & +20 & -12 & -11 & +10 & +9 & -16 & +15 & -14 & +13 \\
+5 & -6 & -7 & -8 & -1 & -2 & +3 & +4 & +21 & -13 & +14 & +15 & +16 & +9 & -10 & -11 & -12 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 & +22 & -14 & -13 & +16 & -15 & +10 & +9 & +12 & -11 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 & +23 & -15 & -16 & +13 & +14 & +11 & -12 & +9 & +10 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 & +24 & -16 & +15 & -14 & -13 & +12 & +11 & -10 & +9 \\
\hline
+9 & +10 & +11 & +12 & +13 & +14 & +15 & +16 & +25 & -1 & -2 & -3 & -4 & -5 & -6 & -7 & -8 \\
\hline
+17 & +18 & -19 & -20 & -21 & -22 & -23 & -24 & -1 & -25 & -26 & -27 & -28 & -29 & -30 & -31 & -32 \\
+18 & +17 & -20 & +19 & -22 & -21 & +24 & -23 & -2 & +26 & -25 & +28 & -27 & +30 & -29 & -32 & +31 \\
+19 & +20 & +17 & -18 & -23 & -24 & +21 & +22 & -3 & +27 & -28 & -25 & +26 & +31 & +32 & -29 & -30 \\
+20 & -19 & +18 & +17 & -24 & +23 & -22 & +21 & -4 & +28 & +27 & -26 & -25 & +32 & -31 & +30 & -29 \\
+21 & +22 & +23 & +24 & +17 & -18 & -19 & -20 & -5 & +29 & -30 & -31 & -32 & -25 & +26 & +27 & +28 \\
+22 & -21 & +24 & -23 & +18 & +17 & +20 & -19 & -6 & +30 & +29 & -32 & +31 & -26 & -25 & -28 & +27 \\
+23 & -24 & -21 & +22 & +19 & -20 & +17 & +18 & -7 & +31 & +32 & +29 & -30 & -27 & +28 & -25 & -26 \\
+24 & +23 & -22 & -21 & +20 & +19 & -18 & +17 & -8 & +32 & -31 & +30 & +29 & -28 & -27 & +26 & -25
\end{array}\right)
$$

Finally we present below a consistently signed matrix of type $(11, 18, 32)$. It contains a submatrix of type $(9, 16, 16)$ by using the first 9 rows and deleting columns 9, 10. The signing of this $(9, 16, 16)$ matches the first 9 rows of the Cayley–Dickson signing of D_4 listed earlier (renumbering the colors by adding 1). The matrix below also contains a $(10, 10, 16)$ by using the first 10 columns and deleting row 9. Given these two consistently signed parts it is not hard to sign the remaining colors 25, 26, $\ldots$, 32 consistently. Now if $11 \le s \le 16$, the first s columns contain exactly $24 + 2 \circ (s - 10)$ colors. This verifies the entries for $11 *_{\mathbb{Z}} s$ in Theorem 3.1.

The verification of the existence of formulas listed in (13.1) is now complete, except for the case $(10, 14, 27)$. We will skip that case, referring the reader to Smith–Yiu (1994).

$$
\left(\begin{array}{cccccccc|cc|cccccccc}
+1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 & +17 & +18 & +9 & +10 & +11 & +12 & +13 & +14 & +15 & +16 \\
+2 & -1 & +4 & -3 & +6 & -5 & -8 & +7 & +18 & -17 & +10 & -9 & -12 & +11 & -14 & +13 & +16 & -15 \\
+3 & -4 & -1 & +2 & +7 & +8 & -5 & -6 & +19 & +20 & +11 & +12 & -9 & -10 & -15 & -16 & +13 & +14 \\
+4 & +3 & -2 & -1 & +8 & -7 & +6 & -5 & +20 & -19 & +12 & -11 & +10 & -9 & -16 & +15 & -14 & +13 \\
+5 & -6 & -7 & -8 & -1 & +2 & +3 & +4 & +21 & +22 & +13 & +14 & +15 & +16 & -9 & -10 & -11 & -12 \\
+6 & +5 & -8 & +7 & -2 & -1 & -4 & +3 & +22 & -21 & +14 & -13 & +16 & -15 & +10 & -9 & +12 & -11 \\
+7 & +8 & +5 & -6 & -3 & +4 & -1 & -2 & +23 & -24 & +15 & -16 & -13 & +14 & +11 & -12 & -9 & +10 \\
+8 & -7 & +6 & +5 & -4 & -3 & +2 & -1 & +24 & +23 & +16 & +15 & -14 & -13 & +12 & +11 & -10 & -9 \\
\hline
+9 & -10 & -11 & -12 & -13 & -14 & -15 & -16 & -25 & -26 & -1 & +2 & +3 & +4 & +5 & +6 & +7 & +8 \\
\hline
+17 & -18 & -19 & -20 & -21 & -22 & -23 & -24 & -1 & +2 & +25 & +26 & +27 & +28 & +29 & +30 & +31 & +32 \\
+18 & +17 & -20 & +19 & -22 & +21 & +24 & -23 & -2 & -1 & +26 & -25 & +28 & -27 & +30 & -29 & +32 & -31
\end{array}\right)
$$

There is one more construction technique of interest for larger matrices. The idea, due to Romero, is to glue together several smaller matrices. An $r \times s$ matrix can be partitioned into five smaller matrices in the following pattern.



Here we have $r = a_1 + a_3 = a_2 + a_4$ and $s = b_1 + b_2 = b_3 + b_4$, etc. If each subrectangle represents a consistently signed intercalate matrix with dimensions and numbers of colors as indicated, no two of them sharing common colors, then this construction shows that

$$r *_{\mathbb{Z}} s \le n_1 + n_2 + n_3 + n_4 + n_5.$$

For example using two copies of a $[9, 13, 16]_{\mathbb{Z}}$, two copies of a $[13, 9, 16]$, and one $[4, 4, 4]_{\mathbb{Z}}$ then this construction produces a $[22, 22, 68]_{\mathbb{Z}}$. Therefore $22 *_{\mathbb{Z}} 22 \le 68$. Using $[9, 16, 16]$'s on the outside yields similarly that $25 *_{\mathbb{Z}} 25 \le 72$. For further information and extensions of this idea see Romero (1995), Yiu (1996) and Sánches–Flores (1996).

Of course it is far more difficult to prove that the values given in Theorem 13.1 are best possible. Yiu's 1990 paper is devoted to a detailed analysis of small intercalate matrices, culminating in a proof that a $[16, 16, 31]_{\mathbb{Z}}$ formula is impossible. The full result is proved in Yiu (1993) by modifying and considerably expanding his earlier ideas. The arguments are too intricate to present here, even in outline form. However we will mention one of the simplest tricks that lead toward Yiu's non-existence results.

If M is an intercalate matrix of type (r, s, n), define a *partial signing* of M to be an $r \times s$ matrix S some of whose entries might be undefined, but such that each defined entry is either $+1$ or -1. Each entry of S is viewed as a sign or a blank attached to the corresponding entry of M. A partial signing is *complete* if every entry is defined.

There is a straightforward algorithm to check whether M admits a consistent signing. (Actually it produces all possible consistent signings of M.) First write in "$+$" signs along the first row and first column. Then attach a "$+$" to one occurrence of any color which does not appear in the first row or column. Now use the consistency condition to deduce all possible consequences of this partial signing S. More precisely, suppose $M(i, j)$ is an unsigned entry. If it is possible to find indices i', j' such that

$$M(i, j) = M(i', j') \text{ and } S(i', j), S(i, j'), S(i', j') \text{ are all defined,}$$

then endow $M(i, j)$ with the sign $S(i, j) = -S(i', j) \cdot S(i, j') \cdot S(i', j')$. Repeat this procedure as long as possible to obtain a maximal signing matrix S_0. There may be an inconsistency of the signs at this point (a submatrix of type $(2, 2, 2)$ violating the sign condition). If that does not occur then S_0 is consistent. If S_0 is also complete we are done. Otherwise choose an unsigned entry of M, give it an indeterminate sign ε, and repeat the process of deducing all possible consequences. Eventually we will get either an inconsistency or a complete consistent signing. Here is one application of this algorithm.

13.12 Lemma. *The following intercalate matrix M' of type $(7, 7, 15)$ cannot be consistently signed.*

$$\left(\begin{array}{cccc:c:c:c}
1 & 2 & 3 & 4 & 5 & 9 & 13 \\
2 & 1 & 4 & 3 & 6 & 10 & 14 \\
3 & 4 & 1 & 2 & 7 & 11 & 15 \\
\hdashline
5 & 6 & 7 & 8 & 1 & 13 & 9 \\
6 & 5 & 8 & 7 & 2 & 14 & 10 \\
\hdashline
9 & 10 & 11 & 12 & 13 & 1 & 5 \\
11 & 12 & 9 & 10 & 15 & 3 & 7
\end{array}\right)$$

Proof. We begin with "$+$" signs along the first row and column and a "$+$" for one occurrence of each of the colors 7, 8, 10, 12, 14, 15. Deriving all the consequences we obtain the following partially signed matrix:

$$\begin{pmatrix} +1 & +2 & +3 & +4 & +5 & +9 & +13 \\ +2 & -1 & 4 & 3 & 6 & +10 & +14 \\ +3 & 4 & -1 & 2 & +7 & 11 & +15 \\ +5 & 6 & -7 & +8 & -1 & 13 & 9 \\ +6 & 5 & 8 & 7 & 2 & 14 & 10 \\ +9 & -10 & 11 & +12 & 13 & -1 & 5 \\ +11 & 12 & 9 & 10 & 15 & 3 & 7 \end{pmatrix}$$

Following the algorithm, we next attach indeterminate signs α, β, γ to the unsigned colors 4 in $M(2,3)$; 6 in $M(2,5)$; and 3 in $M(7,6)$. Deducing all the consequences yields:

$$\begin{pmatrix} +1 & +2 & +3 & +4 & +5 & +9 & +13 \\ +2 & -1 & \alpha 4 & -\alpha 3 & \beta 6 & +10 & +14 \\ +3 & -\alpha 4 & -1 & \alpha 2 & +7 & -\gamma 11 & +15 \\ +5 & -\beta 6 & -7 & +8 & -1 & 13 & 9 \\ +6 & \beta 5 & \alpha\beta 8 & \alpha\beta 7 & -\beta 2 & 14 & 10 \\ +9 & -10 & \gamma 11 & +12 & 13 & -1 & 5 \\ +11 & \alpha\gamma 12 & -\gamma 9 & \alpha\gamma 10 & 15 & \gamma 3 & 7 \end{pmatrix}$$

This partial signing is consistent, but now let us consider a sign δ attached to color 5 in $M(6,7)$. This implies:

$-\delta$ for color 9 in $M(4,7)$,

$\beta\delta$ for color 10 in $M(5,7)$,

$\gamma\delta$ for color 7 in $M(7,7)$,

$-(\alpha\beta)(\beta\delta)(\alpha\gamma) = -\gamma\delta$ also for color 7 in $M(7,7)$, which is impossible.

This completes the proof. Compare Exercise 7(b). □

The matrix M' above is an example of an intercalate matrix partitioned into blocks in the following way:

$$M = \begin{pmatrix} A_0 & * & * & * \\ * & A_1 & * & * \\ * & * & A_2 & * \\ * & * & * & * \end{pmatrix}$$

such that no colors in A_0 appear in any of the blocks marked with $*$, and every color in A_1 and in A_2 does appear in A_0. We continue to follow Yiu's notation here, using colors $\{1, 2, 3, \ldots\}$.

13.13 Corollary. *Suppose M is an $r \times s$ intercalate matrix, with $r, s \geq 7$, which is partitioned into blocks as above such that:*

$$A_0 = D_{3,4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

If $n(M) \leq 16$ then M cannot be consistently signed.

Proof. We relate M with the matrix M' of (13.12). Since the four colors in the row directly below A_0 must involve colors not in $\{1, 2, 3, 4\}$ we may number them 5, 6, 7, 8 and use the intercalacy condition to see that the submatrix $\begin{pmatrix} A_0 & * \\ * & A_1 \end{pmatrix}$ must equal

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 6 \\ 3 & 4 & 1 & 2 & 7 \\ 5 & 6 & 7 & 8 & 1 \\ 6 & 5 & 8 & 7 & 2 \end{pmatrix}.$$

A similar analysis with A_2 yields all of M' except the last column. Since none of the colors in that column can be in $\{1, 2, 3, 4\}$ the intercalacy implies that the top entry must be 13, 14, 15 or 16, and that each of those choices determines the rest of the entries in the column. If that top entry is 13 then $M = M'$ and (13.12) applies. In each of the other cases the proof of (13.12) can be modified to prove that there is no consistent signing. $\square$

Yiu establishes this Corollary and similar results as the first steps toward proving the impossibility of various integer composition formulas, eventually leading to a proof of Theorem 13.1.

We end this chapter with a remark on the interesting structure of a $[10, 10, 16]_{\mathbb{Z}}$ formula.

13.14 Theorem. *Every* $[10, 10, 16]_{\mathbb{Z}}$ *formula is obtained by signing* $D_{10,10}$.

Proof outline. This was proved by Yiu (1987) using topology (namely the homotopy groups of certain Stiefel manifolds, the J-homomorphism and the technique of "hidden formulas" described in Chapter 15), as well as some combinatorial arguments. Yiu reports that this result can also be proved by replacing the topology by the elaborate combinatorics developed in his later papers. $\square$

As mentioned earlier this formula is the smallest one not obviously obtainable as a restriction of one of the Hurwitz–Radon formulas.

13.15 Conjecture. No $[10, 10, 16]_{\mathbb{Z}}$ formula can be a restriction of an $[r, n, n]_{\mathbb{Z}}$ formula. Possibly no [10.10, 16] is a restriction of any $[r, n, n]$ over $\mathbb{R}$ as well.

Yiu reports that he has a proof of the first statement, but I have not seen the details. The idea is to note that any $[10, n, n]_{\mathbb{Z}}$ is an orthogonal sum of $[10, 32, 32]_{\mathbb{Z}}$ formulas. A dimension count should show that the $[10, 10, 16]_{\mathbb{Z}}$ is embedded in some $[10, 32, 32]_{\mathbb{Z}}$, but the corresponding intercalate matrix does not contain a submatrix equivalent to $D_{10,10}$.

Appendix A to Chapter 13. A new proof of Yuzvinsky's Theorem

In 1981 Yuzvinsky showed that the number of colors involved in the intercalate matrix $D_{r,s}$ is exactly the Stiefel–Hopf number $r \circ s$. He conjectured that every $r \times s$ intercalate matrix must involve at least $r \circ s$ colors. He proved this conjecture for dyadic intercalate matrices M, that is, for M which are equivalent to a submatrix of some D_t. Replacing M by an equivalent matrix, we may view it as a submatrix of the addition table of an $\mathbb{F}_2$-vector space V. The entries (colors) in M then arise as the set of values obtained by adding elements of certain subsets $A, B \subseteq V$ with $|A| = r$ and $|B| = s$. Yuzvinsky's theorem about dyadic matrices becomes the following counting result.

A.1 Yuzvinsky's Theorem. *If V is an $\mathbb{F}_2$-vector space and $A, B \subseteq V$, then* $|A + B| \geq |A| \circ |B|$.

Of course (13.5) shows that this lower bound cannot be improved. We present here the elegant new proof due to Eliahou and Kervaire (1998).

We work with the polynomial ring $F[x, y]$ over a field F. If g, h are polynomials, then (g, h) is the ideal in $F[x, y]$ generated by g and h. If $A \subseteq F$ is a finite subset, define $g_A(t)$ to be the polynomial in $F[t]$ which vanishes exactly on A. That is, $g_A(t) = \prod_{a \in A}(t - a)$.

A.2 Lemma. *Suppose $A, B \subseteq F$ are finite subsets and $f(x, y) \in F[x, y]$. Then: $f(x, y)$ vanishes on $A \times B$ if and only if $f(x, y) \in (g_A(x), g_B(y))$.*

Proof. Divide $f(x, y)$ by $g_A(x)$ and $g_B(y)$ to determine that $f(x, y) = g_A(x) \cdot u(x, y) + g_B(y) \cdot v(x, y) + h(x, y)$, where $h(x, y)$ vanishes on $A \times B$ and has x-degree $< |A|$ and y-degree $< |B|$. Then for each $a \in A$, the polynomial $h(a, y)$ is identically zero, since it has more zeros than its degree. A similar argument applied to the coefficients of $h(a, y)$ shows that $h(x, y) = 0$. □

The statement of the next lemma uses the idea of the leading form, or top term, of a polynomial. Any $f \in F[x, y]$ can be uniquely expressed $f = f_0 + f_1 + \cdots + f_d$ where f_j is a form (homogeneous polynomial) of degree j. If $f_d \neq 0$ then d is the (total) degree of f and f_d is the *top form* of f. In this case, define $\mathrm{top}(f) = f_d$. (Also define $\mathrm{top}(0) = 0$.) Certainly $\mathrm{top}(g \cdot h) = \mathrm{top}(g) \cdot \mathrm{top}(h)$, but $\mathrm{top}(g + h)$ does not necessarily belong to the ideal $(\mathrm{top}(f), \mathrm{top}(g))$. However is some special cases this property does hold.

A.3 Lemma. *Suppose $g(x), h(y) \in F[x, y]$ are polynomials in one variable, with* $\deg(g) = r$ *and* $\deg(h) = s$. *If $f \in (g(x), h(y))$ then* $\mathrm{top}(f) \in (x^r, y^s)$.

Proof outline. Suppose $\mathrm{top}(f) \notin (x^r, y^s)$. Then there exists some monomial $M = c \cdot x^i y^j$ occurring in $\mathrm{top}(f)$ satisfying $i < r$ and $j < s$. Reduce f first modulo $g(x)$, and then modulo $h(y)$. If a monomial $ax^u y^v$ occurs in f and $u \geq r$ or $v \geq s$ then during this reduction that monomial is replaced by a polynomial with smaller total degree. This process cannot produce terms cancelling M, since every monomial in f has total degree $\leq i + j$. Consequently f cannot be reduced to zero by that reduction process. This means that $f \notin (g(x), h(y))$. Contradiction. □

We can now describe how to use these simple polynomial lemmas to prove the result.

Proof of Yuzvinsky's Theorem. Suppose $A, B \subseteq V$ are finite subsets with $|A| = r$ and $|B| = s$, and let $C = A + B$. We may assume V is finite, say with 2^n elements. Identifying V with the field F of 2^n elements, define $f(x, y) = \prod_{c \in C}(x + y - c) \in F[x, y]$. Then f vanishes on $A \times B$ and (A.2) implies $f(x, y) \in (g_A(x), g_B(y))$. Then (A.3) implies $\mathrm{top}(f) = (x + y)^{|C|} \in (x^r, y^s)$ in $F[x, y]$. Choosing an $\mathbb{F}_2$-basis of F and comparing coefficients, we find that this relation holds in $\mathbb{F}_2[x, y]$ as well. Then by (12.6) we conclude that $|C| \geq r \circ s$. □

The Nim sum is also closely related to the "circle function" $r \circ s$. This observation (due to Eliahou–Kervaire) provides yet another aspect of $r \circ s$. Recall that the Nim sum $a \boxplus b$ is defined as the sum in $(\mathbb{Z}/2\mathbb{Z})^t$ of the bit strings determined by the dyadic expansions of a and b. As in Exercise 12.3 let $\mathrm{Bit}(n)$ be the indices of the bits involved in n. For example $10 = 2^1 + 2^3$ so $\mathrm{Bit}(10) = \{1, 3\}$. Integers a, b are "bit-disjoint" if $\mathrm{Bit}(a) \cap \mathrm{Bit}(b) = \emptyset$.

A.4 Lemma. (i) *$a \boxplus b \leq a + b$, with equality iff a, b are bit-disjoint.*

(ii) *If $a, b < 2m$ then $a \boxplus b < 2m$.*

(iii) *If $a < 2^m$ then $a \boxplus (2^m + b) = 2^m + (a \boxplus b)$.*

(iv) *If $a \boxplus b = n > 0$ then $n - 1 = a' \boxplus b'$ for some $a' \leq a$ and $b' \leq b$.*

Proof. See Exercise 4. □

A.5 Proposition. $r \circ s = 1 + \max\{a \boxplus b : 0 \leq a < r \text{ and } 0 \leq b < s\}$.

Proof. Let $r \bullet s$ be the quantity on the right. Then certainly $r \bullet s = s \bullet r$ and $1 \bullet s = s$, and also: $s \leq s'$ implies $r \bullet s' \leq r \bullet s$. In particular, $\max\{r, s\} \leq r \bullet s$. By (A.4) (ii) we find: $r, s \leq 2^m$ implies $r \bullet s \leq 2^m$. Consequently, if $r \leq 2^m$ then $r \bullet 2^m = 2^m$. These observations and the following fact suffice to show that $r \bullet s$ and $r \circ s$ coincide, as hoped.

Claim. If $r \leq 2^m$ then $r \bullet (2^m + s) = 2^m + (r \bullet s)$.

Proof. We may assume $s \geq 1$. Suppose $r \bullet s = 1 + (a \boxplus b)$ for some $a < r$ and $b < s$. Then by (A.4) (iii), $2^m + (r \bullet s) = 2^m + 1 + (a \boxplus b) = 1 + a \boxplus (2^m + b) \leq r \bullet (2^m + s)$. Conversely suppose $r \bullet (2^m + s) = 1 + a \boxplus b'$ for some $a < r$ and $b' < 2^m + s$. If $b' < 2^m$ then $a \boxplus b' < 2^m$ and the inequality follows easily. Otherwise $b' \geq 2^m$

so that $b' = 2^m + b$ where $0 \le b < s$. Then $r \bullet (2^m + s) = 1 + a \boxplus (2^m + b) = 1 + 2^m + (a \boxplus b) \le 2^m + r \bullet s$, again using (A.4) (iii). □

With this interpretation of $r \circ s$ we obtain another proof of (13.5). See Exercise 4. Eliahou and Kervaire (1998) generalize all of this to subsets of an $\mathbb{F}_p$-vector space. If $A, B \subseteq V$ are subsets of cardinality r, s, respectively, they prove $|A + B| \ge \beta_p(r, s)$. This $\beta_p(r, s)$ is the p-analog of $r \circ s$ as defined in Exercise 12.25.

As mentioned earlier, Yuzvinsky conjectured that any intercalate matrix of type (r, s, n) must have $n \ge r \circ s$. Theorem A.1 above proves this for dyadic intercalate matrices. For the non-dyadic cases Yiu reports that this conjecture can be proved when $r, s \le 16$ by invoking the complete characterization of small intercalate matrices given in Yiu (1990a) and (1994).

Appendix B to Chapter 13. Monomial Compositions

Let us now consider compositions of more general quadratic forms, not just sums of squares. Let F be a field (or ring) with characteristic $\neq 2$. Suppose α, β, γ are regular quadratic forms over F with dimensions r, s, n, respectively. A *composition* for this triple of forms is a formula

$$\alpha(X) \cdot \beta(Y) = \gamma(Z)$$

where each z_k is bilinear in the systems $X = (x_1, \dots, x_r)$ and $Y = (y_1, \dots, y_s)$, with coefficients in F. Let (U, α), (V, β), (W, γ) be the corresponding quadratic spaces over F. A composition for α, β, γ becomes a bilinear map $f : U \times V \to W$ satisfying the norm property:

$$\gamma(f(u, v)) = \alpha(u) \cdot \beta(v) \quad \text{for every } u \in U \text{ and } v \in V.$$

Choose orthogonal bases $A = \{u_1, \dots, u_r\}$, $B = \{v_1, \dots, v_s\}$ and $C = \{w_1, \dots, w_n\}$. The quadratic forms are then diagonalized: $\alpha \simeq \langle a_1, \dots, a_r\rangle$, $\beta \simeq \langle b_1, \dots, b_s\rangle$ and $\gamma \simeq \langle c_1, \simeq, c_n\rangle$. Each vector $f(u_i, v_j)$ is expressible as a linear combination of $w_1, \dots, w_n$ and we can represent the pairing f as a system of r matrices of size $n \times s$. Motivated by the integer case above, we consider here the pairings where each $f(u_i, v_j)$ involves only one of the basis vectors w_k.

B.1 Definition. A bilinear pairing $f : F^r \times F^s \to F^n$ is *monomial* if for every i, j there exists k such that $f(u_i, v_j) \in F \cdot w_k$.

A monomial pairing f is equivalent to two maps

$$\varphi : A \times B \to C \quad \text{and} \quad \varepsilon : A \times B \to F$$

such that $f(u_i, v_j) = \varepsilon(u_i, v_j) \cdot \varphi(u_i, v_j)$. To shorten notations let us write this as $f(u_i, v_j) = \varepsilon_{ij} \cdot \varphi_{ij}$. Suppose the composition formula above for the forms α, β, γ is monomial. In the case of sums of squares ($a_i = b_j = c_k = 1$) the map φ is tabulated by an intercalate matrix while ε provides a consistent signing. Extending the analysis used in deriving (13.3) we find:

$$\sum_{i=1}^{r}\sum_{j=1}^{s} a_i b_j x_i^2 y_j^2 = \sum_{i_1,i_2}\sum_{j_1,j_2} \langle f(u_{i_1}, v_{j_1}),\, f(u_{i_2}, v_{j_2})\rangle \cdot x_{i_1}x_{i_2}y_{j_1}y_{j_2}.$$

The same argument shows that $M = (\varphi_{ij})$ is an intercalate matrix of type (r, s, n). As before we use k in place of w_k when writing entries of M. The analog of the "consistent signing" is more complicated. Each row-index i has an assigned scalar a_i and each column-index j has as assigned scalar b_j. These values will be listed to the left and above the matrix M. Comparing the $x_i^2 y_j^2$ coefficients above we find:

$$\text{if } M(i, j) = k \text{ then } a_i b_j = \varepsilon_{ij}^2 \cdot c_k. \tag{$*$}$$

In particular these square classes are equal: $\langle a_i b_j\rangle = \langle c_k\rangle$ in $F^\bullet/F^{\bullet 2}$. To keep track of these scalars we label the color k in position $M(i, j)$ by the length c_k of the corresponding vector w_k. This label is written as an exponent. This condition alone puts some restrictions on the forms. For example if $r = s$ and the intercalate matrix M has a ubiquitous color (see (13.8)) then $\alpha = \beta$.

This labeled intercalate matrix M still needs the associated "signs" ε_{ij}. These scalars ε_{ij} are not necessarily ± 1, but the equation $(*)$ shows that given M, each ε_{ij} is determined up to a sign. The analog of the consistent signing condition turns out to be:

If $M(i, j) = M(i', j') = k$ where $i \neq i'$ and $j \neq j'$ then $M(i, j') = M(i', j) = k'$ for some color k' and:

$$(\varepsilon_{ij}\varepsilon_{i'j'}) \cdot c_k = -(\varepsilon_{i'j}\varepsilon_{ij'})) \cdot c_{k'}.$$

This "sign" ε_{ij} is written in parentheses to the left of the entry $M(i, j)$. Then there is a monomial composition for the forms α, β, γ if and only if there exists a consistently signed intercalate matrix of this type.

An example should help clarify these conditions. The standard [4, 4, 4] composition formula for the form $\alpha = \beta = \gamma = \langle 1, a, b, ab\rangle$ comes from the quaternions, where $i^2 = -a$, $j^2 = -b$ and $k^2 = -ab$. Multiplying $x_0 + x_1 i + x_2 j + x_3 k$ and $y_0 + y_1 i + y_2 j + y_3 k$ yields the coefficients:

$$\begin{aligned}
z_0 &= x_0y_0 - ax_1y_1 - bx_2y_2 - abx_3y_3\\
z_1 &= x_1y_0 + x_0y_1 - bx_3y_2 + bx_2y_3\\
z_2 &= x_2y_0 + ax_3y_1 + x_0y_2 - ax_1y_3\\
z_3 &= x_3y_0 - x_2y_1 + x_1y_2 + x_0y_3
\end{aligned}$$

The corresponding labeled intercalate matrix is:

$$
\begin{array}{cc}
 & \begin{array}{cccc} 1 & a & b & ab \end{array} \\
\begin{array}{c} 1 \\ a \\ b \\ ab \end{array} &
\begin{pmatrix}
(1)\mathbf{0}^1 & (1)\mathbf{1}^a & (1)\mathbf{2}^b & (1)\mathbf{3}^{ab} \\
(1)\mathbf{1}^a & (-a)\mathbf{0}^1 & (1)\mathbf{3}^{ab} & (-a)\mathbf{2}^b \\
(1)\mathbf{2}^b & (-1)\mathbf{3}^{ab} & (-b)\mathbf{0}^1 & (b)\mathbf{1}^a \\
(1)\mathbf{3}^{ab} & (a)\mathbf{2}^b & (-b)\mathbf{1}^a & (-ab)\mathbf{0}^1
\end{pmatrix}
\end{array}
$$

The "colors" here are the indices $\{0, 1, 2, 3\}$. Suppose color k occurs at position (i, j) in the matrix Then the term $x_i y_j$ occurs in z_k and the coefficient is given by the matrix entry in parentheses. For instance, the labeled color $(-a)2^b$ occurs in the matrix above in position (1, 3). This says that the term $-ax_1y_3$ occurs in z_2. The superscripts for those entries could have been omitted, but they are useful for checking the consistency conditions.

Starting from an intercalate matrix, what further information is needed to build a monomial pairing? First choose consistent labels a_i, b_j along the top and side. These determine the quadratic forms α and β. Once those row and columns labels are decided, the color labels (written as exponents) are easily decided, determining the quadratic form γ. Every occurrence of a color in this matrix must have the same color label. The unsigned coefficient (in parentheses) for each entry is given by the overlaps in the row and column labels for that position. The signs of those coefficients now form a consistent signing in the original sense of (13.3).

As before we may freely change a "signing" of an intercalate matrix M by altering the signs of an entire row, of an entire column, or of all occurrences of a single color. Any sequence of such moves yield equivalent signings.

The constructions done earlier can be generalized to these monomial compositions. For example the standard [8, 8, 8] formula for the quadratic form $\langle\langle a, b, c\rangle\rangle$ can be expanded to a [10, 10, 16] formula for the quadratic forms α, β, γ where

$$\alpha = \beta = \langle\langle a, b, c\rangle\rangle \perp d\langle\langle a\rangle\rangle \quad \text{and} \quad \gamma = \langle\langle a, b, c, d\rangle\rangle.$$

Here is the 10×10 matrix which tabulates these formulas.

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & a & b & ab & c & ac & bc & abc & d & ad \end{array} \\
\begin{array}{c} 1 \\ a \\ b \\ ab \\ c \\ ac \\ bc \\ abc \\ d \\ ad \end{array} &
\left(\begin{array}{cccc|cccc|cc}
(1)\mathbf{0}^1 & (1)\mathbf{1}^a & (1)\mathbf{2}^b & (1)\mathbf{3}^{ab} & (1)\mathbf{4}^c & (1)\mathbf{5}^{ac} & (1)\mathbf{6}^{bc} & (1)\mathbf{7}^{abc} & (1)\mathbf{8}^d & (1)\mathbf{9}^{ad} \\
(1)\mathbf{1}^a & (-a)\mathbf{0}^1 & (1)\mathbf{3}^{ab} & (-a)\mathbf{2}^b & (1)\mathbf{5}^{ac} & (-a)\mathbf{4}^c & (-1)\mathbf{7}^{abc} & (a)\mathbf{6}^{bc} & (1)\mathbf{9}^{ad} & (-a)\mathbf{8}^d \\
(1)\mathbf{2}^b & (-1)\mathbf{3}^{ab} & (-b)\mathbf{0}^1 & (b)\mathbf{1}^a & (1)\mathbf{6}^{bc} & (1)\mathbf{7}^{abc} & (-b)\mathbf{4}^c & (-b)\mathbf{5}^{ac} & (1)\mathbf{10}^{bd} & (1)\mathbf{11}^{abd} \\
(1)\mathbf{3}^{ab} & (a)\mathbf{2}^b & (-b)\mathbf{1}^a & (-ab)\mathbf{0}^1 & (1)\mathbf{7}^{abc} & (-a)\mathbf{6}^{bc} & (b)\mathbf{5}^{ac} & (-ab)\mathbf{4}^c & (1)\mathbf{11}^{abd} & (-a)\mathbf{10}^{bd} \\
\hline
(1)\mathbf{4}^c & (-1)\mathbf{5}^{ac} & (-1)\mathbf{6}^{bc} & (-1)\mathbf{7}^{abc} & (-c)\mathbf{0}^1 & (c)\mathbf{1}^a & (c)\mathbf{2}^b & (c)\mathbf{3}^{ab} & (1)\mathbf{12}^{cd} & (1)\mathbf{13}^{acd} \\
(1)\mathbf{5}^{ac} & (a)\mathbf{4}^c & (-1)\mathbf{7}^{abc} & (a)\mathbf{6}^{bc} & (-c)\mathbf{1}^a & (-ac)\mathbf{0}^1 & (-c)\mathbf{3}^{ab} & (ac)\mathbf{2}^b & (1)\mathbf{13}^{acd} & (-a)\mathbf{12}^{cd} \\
(1)\mathbf{6}^{bc} & (1)\mathbf{7}^{abc} & (b)\mathbf{4}^c & (-b)\mathbf{5}^{ac} & (-c)\mathbf{2}^b & (c)\mathbf{3}^{ab} & (-bc)\mathbf{0}^1 & (-bc)\mathbf{1}^a & (1)\mathbf{14}^{bcd} & (-1)\mathbf{15}^{abcd} \\
(1)\mathbf{7}^{abc} & (-a)\mathbf{6}^{bc} & (b)\mathbf{5}^{ac} & (ab)\mathbf{4}^c & (-c)\mathbf{3}^{ab} & (-ac)\mathbf{2}^b & (bc)\mathbf{1}^a & (-abc)\mathbf{0}^1 & (1)\mathbf{15}^{abcd} & (a)\mathbf{14}^{bcd} \\
\hline
(1)\mathbf{8}^d & (-1)\mathbf{9}^{ad} & (-1)\mathbf{10}^{bd} & (-1)\mathbf{11}^{abd} & (-1)\mathbf{12}^{cd} & (-1)\mathbf{13}^{acd} & (-1)\mathbf{14}^{bcd} & (-1)\mathbf{15}^{abcd} & (-d)\mathbf{0}^1 & (d)\mathbf{1}^a \\
(1)\mathbf{9}^{ad} & (a)\mathbf{8}^d & (-1)\mathbf{11}^{abd} & (a)\mathbf{10}^{bd} & (-1)\mathbf{13}^{acd} & (a)\mathbf{12}^{cd} & (1)\mathbf{15}^{abcd} & (-a)\mathbf{14}^{bcd} & (-d)\mathbf{1}^a & (-ad)\mathbf{0}^1
\end{array}\right)
\end{array}
$$

Is every monomial composition of size [10, 10, 16] of this type? That seems to be a difficult question. Over $\mathbb{R}$ we obtain the positive definite example, along with several examples where $\gamma = 8\mathbb{H} = 8\langle 1\rangle \perp 8\langle -1\rangle$ is hyperbolic. In that hyperbolic case we can arrange $\alpha = \beta$ to be one of the forms

$$8\langle 1\rangle \perp 2\langle -1\rangle, \quad 6\langle 1\rangle \perp 4\langle -1\rangle, \quad 5\langle 1\rangle \perp 5\langle -1\rangle.$$

(Since α and β may scaled by -1 in such a composition we may assume α has signature ≥ 0.)

Another case of interest is the formula of type [12, 12, 26]. The same procedure yields formulas where

$$\alpha = \beta = \langle\langle a, b, c\rangle\rangle \perp \langle\langle a\rangle\rangle \otimes \langle d, e\rangle$$

and

$$\gamma = \langle\langle a, b, c, d\rangle\rangle \perp e\langle\langle a, b, c\rangle\rangle \perp de\langle\langle a\rangle\rangle = \langle\langle a, b, c\rangle\rangle \otimes \langle 1, d, e\rangle \perp de\langle\langle a\rangle\rangle.$$

Over $\mathbb{R}$ we obtain examples where

$\alpha = \beta = 10\langle 1\rangle \perp 2\langle -1\rangle$	and		$\gamma = 16\langle 1\rangle \perp 10\langle -1\rangle$.
$\alpha = \beta = 8\langle 1\rangle \perp 4\langle -1\rangle$	and	either	$\gamma = 14\langle 1\rangle \perp 12\langle -1\rangle$;
		or	$\gamma = 12\langle 1\rangle \perp 14\langle -1\rangle$;
		or	$\gamma = 10\langle 1\rangle \perp 16\langle -1\rangle$.
$\alpha = \beta = 6\langle 1\rangle \perp 6\langle -1\rangle$	and	either	$\gamma = 14\langle 1\rangle \perp 12\langle -1\rangle$;
		or	$\gamma = 13\langle 1\rangle \perp 13\langle -1\rangle$;
		or	$\gamma = 12\langle 1\rangle \perp 14\langle -1\rangle$.

It would be interesting to get some further information about the composition of indefinite quadratic forms over $\mathbb{R}$. Some restrictions on the sizes of such compositions are obtained by lifting to the complex field (see (14.1)). But for an allowable size like [10, 10, 16] it remains unclear what signatures are possible for the three forms.

Appendix C to Chapter 13. Known upper bounds for $r * s$

Upper bounds are provided by constructions. The bound $r * s \leq n$ means that there exists a normed bilinear map (over $\mathbb{R}$) of size $[r, s, n]$. All the known constructions can be done with integer coefficients, and hence with intercalate matrices. Much of this chapter was spent describing methods for constructing signed intercalate matrices and showing that the values listed in Theorem 13.1 are upper bounds. (Less space was spent on the much harder task of proving that those values are best possible.)

What about larger values for r, s ? In this appendix we list the known upper bounds for $r *_{\mathbb{Z}} s$, following the work of Adem (1975), Yuzvinsky (1984), Lam–Smith (1993), Smith–Yiu (1994), Romero (1995), Yiu (1996), Sánchez–Flores (1996). We list here a table of upper bounds, as presented in Yiu (1996).

To list upper bounds for $r *_{\mathbb{Z}} s$ we may assume $r \leq s$. If $r \leq 9$ then $r *_{\mathbb{Z}} s$ is known (see(12.13) or (13.7)). If $r, s \leq 16$ then Yiu's Theorem 13.1 provides the exact value of $r * \mathbb{Z}s$. Let us consider the next block of values:

$$r \leq s, \quad 10 \leq r \leq 32 \quad \text{and} \quad 17 \leq s \leq 32.$$

In the following table of upper bounds for $r *_{\mathbb{Z}} s$, each underlined entry is known to be the exact value for $r *_{\mathbb{Z}} s$.

$r \backslash s$	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32
10	29	29	30	30	$\underline{30}$	$\underline{30}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$
11	32	32	32	32	42	44	44	44	46	48	48	48	48	52	52	52
12	$\underline{32}$	$\underline{32}$	$\underline{32}$	$\underline{32}$	42	44	44	44	48	48	48	48	48	52	52	52
13	$\underline{32}$	$\underline{32}$	43	44	44	44	48	48	48	48	48	58	58	58	58	58
14	$\underline{32}$	$\underline{32}$	43	44	46	48	48	48	48	48	48	58	58	58	58	58
15	$\underline{32}$	$\underline{32}$	44	46	48	48	48	48	48	48	48	60	62	63	64	64
16	$\underline{32}$	$\underline{32}$	44	46	48	48	48	48	48	48	48	60	62	64	64	64
17	$\underline{32}$	$\underline{32}$	49	50	51	52	53	54	55	56	57	61	64	64	64	64
18		50	50	52	52	54	54	56	56	57	57	64	64	64	64	64
19			56	56	59	60	60	64	64	64	64	64	64	64	64	64
20				56	60	60	60	64	64	64	64	64	64	64	64	64
21					64	64	64	64	72	76	77	80	80	84	84	84
22						68	72	72	72	78	80	80	80	84	84	84
23							72	72	72	78	80	84	88	90	90	90
24								72	72	80	80	88	88	90	90	90
25									72	80	80	88	94	95	96	96
26										80	80	89	94	96	96	96
27											89	89	96	96	96	96
28												96	96	96	96	96
29													96	96	96	96
30														96	96	96
31															116	116
32																116

We conclude with a table of upper bounds for $r * s$ in the range $32 \leq r \leq 64$ and $10 \leq s \leq 16$. (Here we use $s \leq r$ for typographical reasons). Next to that table appear the known upper bounds for $r * r$ in that range.

$r \backslash s$	10	11	12	13	14	15	16
33	42	56	56	63	64	64	64
34	42	56	56	64	64	64	64
35	44	57	58	64	64	64	64
36	44	58	58	64	64	64	64
37	46	58	58	64	64	76	76
38	46	58	58	64	64	78	78
39	48	59	60	64	64	79	80
40	48	59	60	64	64	80	80
41	48	59	60	74	74	80	80
42	48	60	60	78	78	80	80
43	58	60	60	79	80	80	80
44	58	60	60	80	80	80	80
45	59	61	62	80	80	80	80
46	59	61	62	80	80	92	92
47	60	61	62	80	80	94	94
48	60	62	62	80	80	95	96
49	61	62	62	80	80	96	96
50	61	62	62	90	90	96	96
51	62	62	62	92	92	96	96
52	62	62	62	92	94	96	96
53	62	63	64	92	96	96	96
54	62	64	64	96	96	96	96
55	64	64	64	96	96	108	108
56	64	64	64	96	96	108	108
57	64	64	64	96	96	111	112

r	$r * r$
33	127
34	128
35	128
36	128
37	160
38	168
39	168
40	168
41	187
42	188
43	208
44	214
45	216
46	222
47	233
48	240
49	254
50	256
51	273
52	274
53	283
54	304
55	312
56	312
57	320

$r\backslash s$	10	11	12	13	14	15	16
58	64	64	64	96	96	112	112
59	64	64	64	104	106	112	112
60	64	64	64	104	108	112	112
61	64	64	64	104	110	112	112
62	64	64	64	104	112	112	112
63	64	64	64	104	112	112	112
64	64	64	64	104	112	112	112

r	$r * r$
8	320
59	320
60	320
61	360
62	368
63	368
64	368

These two tables and further details of the required constructions were compiled by Yiu (1996). Further tables of upper bounds for all values where $r, s \le 64$ are presented in Sanchez– Flores (1996).

Exercises for Chapter 13

1. The matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 5 & 6 & 7 & 8 \\ 6 & 5 & 9 & 10 \end{pmatrix}$$

is a non-dyadic intercalate matrix. That is, it is an intercalate matrix but is not equivalent to a submatrix of any D_t.

2. Let $\mathcal{N}(r, s) = \{n :$ there exists an $r \times s$ intercalate matrix with exactly n colors$\}$. Certainly $r \circ s$ and $rs \in \mathcal{N}(r, s)$, but values in between might not occur. For example:

There exists a $2 \times s$ intercalate with n colors $\Leftrightarrow$ $s \le n \le 2s$ and n is even.

$\mathcal{N}(3, 3) = \{4, 7, 9\}$,

$\mathcal{N}(3, 4) = \{4, 6, 7, 8, 10, 12\}$,

$\mathcal{N}(3, 5) = \{7, 8, 10, 11, 13, 15\}$,

$\mathcal{N}(4, 4) = \{4, 7, 8, 10, 12, 14, 16\}$.

For the dyadic case we ask: If V is an $\mathbb{F}_2$-vector space and $A, B \subseteq V$ with $|A| = r$ and $|B| = s$, then what sizes are possible for $A + B$?

3. **Ubiquitous colors.** Let M be a symmetric intercalate matrix of type (r, r, n). If $r = 2^b \cdot$ (odd) then the number of ubiquitous colors of M is 2^t for some $t \le b$. Let $r = 2^t \cdot r_1$ and $n = 2^t \cdot n_1$. Then M is equivalent to $D_t \otimes N$where N is symmetric intercalate of type (r_1, r_1, n_1).

Corollary. *If M is intercalate of type (n, n, n) then M is equivalent to D_t for some t.*

(Hint. As in (13.8) we get $M = D_1 \otimes M'$. Each ubiquitous color of M' corresponds to two ubiquitous colors of M.)

4. **Nim sum.** (1) Prove the four parts of (A.4).

(2) Does $2^m(a \boxplus b) = (2^m a) \boxplus (2^m b)$?

(3) Writing $[0, m)$ for the interval, then: $[0, r) \boxplus [0, s) = [0, r \circ s)$. This is a restatement of Lemma 13.5.

(Hint. (1) (ii) Note that $a < 2^m \Leftrightarrow \mathrm{Bit}(a) \subseteq [0, m) = \{0, 1, \dots, m-1\}$.

(iii) Express $b = b_0 + b_1$ where $\mathrm{Bit}(b_0) \subseteq [0, m)$ and $\mathrm{Bit}(b_1) \subseteq [m, \infty)$. Then $a \boxplus (2^m + b) = (a \boxplus b_0) + 2^m + b_1 = 2^m + (a \boxplus b)$.

(iv) Assume a, b are bit-disjoint. If $n = 2^k +$ (higher terms) then none of 1, 2, $2^2, \dots, 2^{k-1}$ occur in a or b. If 2^k occurs in a then $n - 1 = (a-1) \boxplus b$.

(3) Use (A.4) (iv).)

5.**Cayley–Dickson.** Suppose $e_0, e_1, \dots, e_{2^m-1}$ is the standard basis of the Cayley–Dickson algebra A_m as described in Exercise 1.24. The product is given by:

$$e_i \cdot e_j = \varepsilon_{ij} e_k \text{ where } k = i \boxplus j \text{ is the Nim-sum}$$

and the signs $\varepsilon_{ij} = \pm 1$ are determined inductively as follows. Given the signs ε_{ij} for $0 \le i, j < 2^m$, the remaining signs ε_{hk} for $0 \le h, k < 2^{m+1}$ are given by the formulas:

$$\varepsilon_{i,2^m+j} = \begin{cases} +1 & \text{if } i = 1 \text{ or } j = 1, \\ -1 & \text{if } i = j \neq 1, \\ -\epsilon_{ij} & \text{otherwise;} \end{cases}$$

$$\varepsilon_{2^m+i,j} = \begin{cases} +1 & \text{if } j = 1 \text{ or } i = j, \\ -1 & \text{if } i = 1 \text{ ans } j \neq i, \\ -\epsilon_{ij} & \text{otherwise;} \end{cases}$$

$$\varepsilon_{2^m+i,2^m+j} = \begin{cases} +1 & \text{if } i = 1 \text{ or } j \neq 1, \\ -1 & \text{if } j = 1 \text{ 0r } i = j, \\ -\epsilon_{ij} & \text{otherwise;} \end{cases}$$

(Hint. Express $A_{m+1} = A_m \oplus A_m$ and for $0 \le i < 2^m$ identify e_i with $(e_i, 0)$ and e_{i+2^m} with $(0, e_i)$. Work out the products using the “doubling process” stated in (1.A6).)

6. Let D_t be signed according to the Cayley–Dickson process.

(a) If $t \ge 4$, the first 9 rows of D_t are consistently signed. Note that these do *not* form the direct sum of several copies of the upper left 9×16 block.

(b) Examining the displayed signs for D_4 arising from A_4, are the signings of the four 8×8 blocks equivalent?

7. **Equivalence of signs.** (a) All the consistent signings of the intercalate matrix D_4 are equivalent. Similarly for D_8 and D_{10} and for the matrix of type $(12, 12, 26)$ constructed in (13.9).

(b) In the proof of (13.12) we may assume that the new signs α, β, γ are all " $+$" .

(Hint. For example in D_4 assume that the first row and column are all signed with " $+$", so the three diagonal zeros have sign " $-$". To alter the signs of the middle 3s, change the signs of all the 3's, and change the sign of the last row and last column. We may assume the 3 in the second row has sign " $+$". The remaining signs are determined. Similar moves prove the uniqueness for all these examples.)

8. The matrix $D_{10,10}$ can be extended to an intercalate matrix of type $(10, 11, 16)$ in several ways. None of these extensions can be consistently signed. (This follows from $10 \# 11 = 17$, mentioned near the end of Chapter 12. However that proof is difficult.)

(Hint. By Exercise 7 we may assume D_{10} has the standard signing coming from A_4 displayed before (13.6). The 11th column of the extension must match one of the remaining columns of A_4. Each case yields a sign inconsistency.)

9. **Hidden formulas.** Let M be an intercalate matrix of type (r, s, n) and suppose the color c has frequency k in M. Permute the rows and columns of M to assume that these k occurrences of c appear along the main diagonal, yielding a partition $M = M_{(c)} = \begin{pmatrix} A & C \\ B & A'' \end{pmatrix}$, where A is a $k \times k$ matrix with color c along the diagonal, and color c does not appear in C, B or A".

Lemma. *The matrix* $M_{(c)} = \begin{pmatrix} A & C & B^\top \end{pmatrix}$ *is also intercalate, of type* $(k, r + s - k, m)$ *for some* $m \le n$. *Furthermore, if* M *is consistently signed so that each occurrence of color* c *has a " $+$", then* $M_{(c)}$ *is also consistently signed.*

This $M_{(c)}$ is called the intercalate matrix "hidden behind c".

(Hint. Note that $A^\top$ is the same as $-A$, except for the diagonal. Checking the $\begin{pmatrix} A & B^\top \end{pmatrix}$ part is easy. For $\begin{pmatrix} C & B^\top \end{pmatrix}$ suppose color a occurs in C and in B. Permuting rows and columns yields a submatrix of M of the type: $M' = \begin{pmatrix} c & x & -y \\ -x & c & a \\ a & y & x \end{pmatrix}$, where x, y are some other colors. Examine the corresponding parts of $M_{(c)}$.)

10. There exist formulas of the following sizes:

$$[17, 17, 32] \quad [18, 18, 50] \quad [20, 20, 56] \quad [21, 21, 64] \quad [22, 22, 68]$$
$$[25, 25, 72] \quad [26, 26, 80] \quad [27, 27, 89] \quad [30, 30, 96]$$

These provide some of the upper bounds listed in the first table of Appendix C.

(Hint. A $[12, 20, 32]_{\mathbb{Z}}$ formula was constructed by Lam–Smith (1993). Use this, earlier formulas, and the techniques of restriction, direct sums and doubling. Examples:

$[18, 19, 50] = [18, 17, 32] \oplus [18, 2, 18]$; $[26, 27, 80] = 3 \cdot [16, 9, 16] \oplus [10, 27, 32]$; and $[27, 27, 89] = [17, 18, 32] \oplus [17, 9, 25] \oplus [10, 27, 32]$. For 22 and 25 recall Romero's construction.)

11. **Generalizing Yuzvinsky.** (1) Generalize (A.2) and (A.3) to n variable polynomials. If V is an $\mathbb{F}_2$-vector space and $A_1, \dots, A_k \subseteq V$, with $|A_j| = r_j$, what is the minimal value for $|A_1 + \cdots + A_k|$?

(2) Suppose V is an $\mathbb{F}_p$-vector space and $A, B \subseteq V$ with cardinalities $|A| = r$ and $|B| = s$. What is the smallest possibility for $|A + B|$?

12. Generalize the constructions in this chapter to the monomial pairings of Appendix B. For example, what is the analog of the doubling process (13.6) for a composition of α, β, γ? How does (13.9) generalize?

Notes on Chapter 13.

In writing this chapter I closely followed the presentations in Lam–Smith (1993), Smith–Yiu (1994), Yiu (1990a) and Yiu (1993).

Integer composition formulas were analyzed by several of 19th century mathematicians who were seeking to generalize the 8-square identity. Proofs that 16-square identities (over $\mathbb{Z}$) are impossible were given (with various levels of rigor) by several mathematicians, including Young, Cayley, Kirkman and Roberts. For instance, Cayley (1881), using more clumsy terminology, seems to provide a complete list of intercalate matrices of size [16, 16, 16] and shows that none of them has a consistent signing. For further information and references see Dickson (1919).

The work of Kirkman (1848) was tracked down by Yiu, following a reference in Dickson (1919), and reported in Yiu (1990a). Kirkman obtained formulas of types $[2k, 2k, k^2 - 3k + b]$ where $b = 8, 4, 6$ according as $k \equiv 0, 1, 2 \pmod 3$.

Composition formulas of size $[\rho(n), n, n]$ appear implicitly in the works of Hurwitz, Radon and Eckmann. They have been given in more explicit form by a number of authors, including: Wong (1961), K. Y. Lam (1966), Zvengrowski (1968), Geramita–Pullman (1974), Gabel (1974), Shapiro (1977), Adem (1978b), Yuzvinsky (1981), Bier (1984), K. Y. Lam (1984), Lam–Yiu (1987), Au-Yeung and Cheng (1993). The two methods of constructing signed intercalate matrices of size $(\rho(n), n, n)$ mentioned after (13.9) are also outlined in Smith–Yiu (1993).

This doubling construction of (13.6) is a variation of the one given by Lam–Smith (1993).

Lemma 13.12 and Corollary 13.13 appear in Yiu (1993) in Example 4.10 and Lemma 5.3.

Yuzvinsky (1981) p. 143 mentions Conjecture 13.15 (without proof).

Appendix A. Theorem A.1 is a major result in Yuzvinsky (1981). Our proof closely follows the presentation in Eliahou–Kervaire (1998). They use this polynomial method to prove several related results, including those asked in Exercise 11. I am grateful to Eliahou and Kervaire for sending me a preliminary version of their paper.

Appendix B. The term "monomial pairing" was used by Yuzvinsky (1981) when he introduced what we call intercalate matrices. The calculations using general quadratic forms here seem to be new.

Exercise 1. See Yiu (1990a), p. 466. Further information on determining whether an intercalate matrix is dyadic see Calvillo, Gitler, Martínez–Bernal (1996).

Exercise 2. A consistently signed intercalate $r \times s$ matrix with exactly n colors leads to a full composition. See Chapter 14. I believe that these sets $\mathcal{N}(r, s)$ have not been investigated elsewhere.

Exercise 3. See Yiu (1990a) Prop. 2.11. Recall that if $t > 3$ then D_t cannot be consistently signed. It turns out that if a consistently signed intercalate matrix has more than 2 ubiquitous colors then it must have type [4, 4, 4] or [8, 8, 8]. See (15.30) and Exercise 15.16.

Exercise 5. These formulas are also stated in Yiu (1993) §2.

Exercise 6. See Yiu (1993) Prop. 2.8.

Exercise 8. Yiu (1987) Prop. 1.3.

Exercise 9. The hidden formulas were first discovered in the general context of quadratic forms between euclidean spheres in Yiu (1986) and Lam–Yiu (1987). See Chapter 15. They were translated into this intercalate matrix version in Lam–Yiu (1988). Also see Yiu (1990a) Theorem 8 and Yiu (1993) Proposition 14.2. Those hidden formulas play an important role in the proof of Theorem 13.1.

Exercise 10. Smith–Yiu (1994).

Exercise 11. See Eliahou–Kervaire (1998).