Dedicated to Amanda, Becky and Jacob

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Introduction

This book addresses basic questions about compositions of quadratic forms in the sense of Hurwitz and Radon. The initial question is: For what dimensions can they exist? Subsequent questions involve classification and analysis of the quadratic forms which can occur in a composition.

This topic originated with the "1, 2, 4, 8 Theorem" concerning formulas for a product of two sums of squares. That theorem, proved by Adolf Hurwitz in 1898, was generalized in various ways during the folowing century, leading to the theories discussed here. This area is worth studying because it is so centrally located in mathematics: these compositions have close connections with mathematical history, algebra, combinatorics, geometry, and topology.

Compositions have deep historical roots: the 1, 2, 4, 8 Theorem settled a long standing question about the existence of "n-square identities" and exhibited some of the power of linear algebra. Compositions are also entwined with the nineteenth century development of quaternions, octonions and Clifford algebras.

Another attraction of this subject is its fascinating relationship with Clifford algebras and the algebraic theory of quadratic forms. A general composition formula involves arbitrary quadratic forms over a field, not just the classical sums of squares. Such compositions can be reformulated in terms of Clifford algebras and their involutions. There is also a close connection between the forms involved in compositions and the multiplicative quadratic forms introduced by Pfister in the 1960s.

All the known constructions of composition formulas for sums of squares can be achieved using integer coefficients. A composition formula with integer coefficients can be recast as a combinatorial object: a special sort of matrix of symbols and signs. These "intercalate" matrices have been studied intensively, leading to a classification of the integer compositions which involve at most 16 squares.

Finally this topic is connected with certain deep questions in geometry. For instance, composition formulas provide examples of vector bundles on projective spaces, of independent vector fields on spheres, of immersions of projective spaces into euclidean spaces, and of Hopf maps between euclidean spheres. The topological tools developed to analyze these topics also yield results about real compositions.

Let us now describe the original question with more precision: A composition formula of size $[r, s, n]$ is a sum of squares formula of the type

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_1, y_2, \ldots, y_s)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y. Such a formula can be viewed in several different ways, with each version providing different insights and techniques. Hurwitz restated the formula as a system of r different $n \times s$ matrices. More geometrically (assuming that the z_k's have real coefficients), the formula becomes a bilinear pairing

$$f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$$

which satisfies the norm condition: $|f(x, y)| = |x| \cdot |y|$ for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. For example the usual multiplication of complex numbers provides a formula of size $[2, 2, 2]$. In the original sums-of-squares language, this bilinear pairing becomes the formula:

$$(x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2) = z_1^2 + z_2^2 \quad \text{where} \quad z_1 = x_1y_1 - x_2y_2 \ \text{ and } \ z_2 = x_1y_2 + x_2y_1.$$

The quaternion and octonion algebras, discovered in the 1840s, provide similar formulas of sizes $[4, 4, 4]$ and $[8, 8, 8]$. Using his matrix formulation Hurwitz (1898) proved that a formula of size $[n, n, n]$ exists if and only if n is 1, 2, 4 or 8. Hurwitz and Radon used similar techniques to determine exactly when formulas of size $[r, n, n]$ can exist. It is far more difficult to analyze compositions of sizes $[r, s, n]$ when $r, s < n$.

These ideas have been generalized in two main directions, determining the contents of the two parts of this book.

Part I: If the composition involves general quadratic forms over a field in place of the sums of squares, what can be said about those forms? Interesting results have been obtained for the classical sizes $[r, n, n]$.

Part II: What sizes r, s, n are possible in the general case? Does the answer depend on the field of coefficients? Many partial results have been obtained using methods of algebraic topology, combinatorics, linear algebra and geometry.

Further descriptions of the historical background and the contents of this work appear in Chapter 0 and in the Introduction to Part II.

Readers of this work are expected to have knowledge of some abstract algebra. The first two chapters assume familiarity with only the basic properties of linear algebra and inner product spaces. The next five chapters require quadratic forms, Clifford algebras, central simple algebras and involutions, although many of those concepts are developed in the text. For example, Clifford algebras are defined and their basic properties are established in Chapter 3. Later chapters assume further background. For example Chapter 11 uses algebraic number theory and Chapter 12 employs algebraic topology.

Each chapter begins with a brief statement of its content and ends with some exercises, usually involving alternative methods or related results. In fact many related topics and open questions have been converted to exercises. This practice lengthens the exercise sections, but adds some further depth to the book. The Notes at the end of each chapter provide additional comments, historical remarks and references. At the end of the book there is a fairly extensive bibliography, arranged alphabetically by first author.

Most of the material described in this book has already appeared in the mathematical literature, usually in research papers. However there are many items that have not been previously published. These include:

- an improved version of the Eigenspace Lemma (2.10);
- a discussion of anti-commuting skew-symmetric matrices, Exercise 2.13;
- the trace methods used to analyze (2, 2)-families, Chapter 5;
- the treatment of composition algebras, Chapter 1.A (due to Conway);
- the analysis of "minimal" pairs, Chapter 7;
- properties of the topological space of all compositions, Chapter 8;
- monotopies and isotopies, Chapter 8 (due to Conway);
- the matrix approach to Pfaffians, Chapter 10;
- Hasse principle for divisibility, Chapter 11.A (due to Wadsworth);
- general monomial compositions, Chapter 13.B;
- the characterization of all compositions of codimension 2, (14.18);
- nonsingular and surjective bilinear pairings over fields, Exercises 14.16–19.

This book evolved over many years, starting from series of lectures I gave on this subject at the Universität Regensburg (Germany) in 1977, at the Universidad de Chile in 1981, at the University of California-Berkeley in 1983, at the Universität Dortmund (Germany) in 1991, at the Universidad de Talca (Chile) in 1999 and several times at the Ohio State University. I am grateful to these institutions, to the National Science Foundation, to the Alexander von Humboldt Stiftung and to the Fundación Andes for their generous support. It is also a pleasure to thank many friends and colleagues for their interest in this work and their encouragement over the years. Special thanks are due to several colleagues who have made observations directly affecting this book. These include J. Adem, R. Baeza, E. Becker, A. Geramita, J. Hsia, I. Kaplansky, M. Knebusch, K. Y. Lam, T. Y. Lam, D. Leep, T. Smith, M. Szyjewski, J.-P. Tignol, A. Wadsworth, P. Yiu, and S. Yuzvinsky. Extra thanks are due to Adrian Wadsworth for providing great help and support in the early years of my mathematical career.

I am also grateful to those colleagues and students who have proofread sections of this book, finding errors and making worthwhile suggestions. However I take full responsibility for the remaining grammatical and mathematical errors, the incorrect cross references, the inconsistencies of notation and the gaps in understanding.

As mentioned above, this book has been in progress for many years. In fact it is hard for me to believe how long it has been. The writing was finally finished in 1998, barely in time to celebrate the centennial of the Hurwitz 1, 2, 4, 8 Theorem.