Big Numbers

by Daniel B. Shapiro

What is the largest number you can think of? (Here I am asking about actual counting numbers, not “infinity”.) People answer this question in different ways depending on how many numbers they know. Preschoolers might answer “one hundred” while others might answer “a zillion” or “a skillion” or (my favorite) “an impossibillion”. But these terms are not really the standard names for any numbers.

When I first learned about large numbers (a large number of years ago), I read about a “googol”. This nonsense name was made up around 1940 by authors Kasner and Newman to represent the number

10000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000.

This is a one followed by one hundred zeros. It is a big number. There are fewer than a googol atoms in the entire visible universe (even if all that space was packed solid with atoms). A googolplex was then defined to equal a one followed by a googol zeros. Writing out a googolplex in the usual base ten notation would take up a lot of paper!

Here is a more compact way to write such large numbers. If b and n are whole numbers then b^n (read “b to the n”) is the product of n copies of b. For example,

$$5^2 = 5 \times 5 = 25 \qquad\qquad 3^4 = 3 \times 3 \times 3 \times 3 = 81,$$

$$10^2 = 10 \times 10 = 100, \qquad 10^3 = 10 \times 10 \times 10 = 1000, \qquad 10^4 = 10000.$$

From these examples you can see that 10^n is the number which is a one followed by n zeros. In particular a googol is 10^{100}. Similarly a googolplex equals $10^{\text{googol}} = 10^{(10^{100})}$. This is certainly a shorter way to write that number! Since stacks of exponents are cumbersome to type, we’ll use $b\uparrow n$ instead of b^n. Then a googol equals $10\uparrow 100 = 10\uparrow(10\uparrow 2)$ and a googolplex equals $10\uparrow(\text{googol}) = 10\uparrow(10\uparrow(10\uparrow 2))$.

With this arrow notation the placement of parentheses can make a big difference. For example the number $2\uparrow 3\uparrow 4$ can be interpreted in two ways:

either $(2\uparrow 3)\uparrow 4 = (2^3)^4 = 8^4 = 4096$

or $2\uparrow(3\uparrow 4) = 2^{(3^4)} = 2^{81} = 2417851639229258349412352.$

In 1976 the computer scientist D. Knuth introduced a double arrow notation. He defined $b\uparrow\uparrow n$ to be the number obtained by "arrowing" n copies of b. For example, $3\uparrow\uparrow 4 = 3\uparrow 3\uparrow 3\uparrow 3$. To make the meaning clear, we always perform these arrow operations from right to left. That is, the parentheses are nested to the right. For example

$$2\uparrow\uparrow 4 = 2\uparrow(2\uparrow(2\uparrow 2)) = 2\uparrow(2\uparrow 4) = 2\uparrow 16 = 65536.$$
$$3\uparrow\uparrow 3 = 3\uparrow(3\uparrow 3) = 3\uparrow 27 = 7625597484987.$$
$$3\uparrow\uparrow 4 = 3\uparrow(3\uparrow(3\uparrow 3)) = 3\uparrow(3\uparrow 27) = 3\uparrow(7625597484987).$$

That last number has more than three trillion digits, if written out as a base ten numeral. Do you see that a googolplex is far smaller than $10\uparrow\uparrow 4$?

Of course Knuth didn't stop there. He went on to define triple arrows: $b\uparrow\uparrow\uparrow n$ is the number obtained by "double-arrowing" n copies of b. For instance, $3\uparrow\uparrow\uparrow 3 = 3\uparrow\uparrow(3\uparrow\uparrow 3) = 3\uparrow\uparrow(7625597484987)$. Now this is getting large, making a googolplex look tiny. But, of course in comparison to $10\uparrow\uparrow\uparrow 10$ that previous number is so minuscule it seems nearly equal to zero. Numbers like these are ridiculously large, in any real sense. But in mathematics we don't let reality put limits on our imagination. No matter how far we go there are more numbers, incredibly larger than any number thought of earlier. Nevertheless, all of these numbers are ordinary whole numbers. They are all finite. In fact, *most* whole numbers are larger than $10\uparrow\uparrow\uparrow 10$. (To me that is logical but seems hard to truly understand.)

You might want to think about $10\uparrow\uparrow\uparrow\uparrow 10$ or further generalizations. These ideas provide one example of the power of good notation. This new way of expressing large numbers not only simplifies the writing but it allows us to describe things that we couldn't even think about earlier.

Numbers like $10\uparrow\uparrow 8$ and $10\uparrow\uparrow\uparrow 3$ consist of a 1 followed by a long list of zeros (if it is written out as a base ten numeral). Other numbers, like $2\uparrow\uparrow 5$ and $3\uparrow\uparrow 4$, are smaller but their digits are far more complicated. It seems difficult to investigate all of those digits, but with some thought we can determine their *units* digits. To clarify the terms, note that 7318 has units digit 8.

What is the units digit of $2\uparrow\uparrow 5$? This number is $2\uparrow(2\uparrow(2\uparrow(2\uparrow 2)))$ so it can be expressed as $2\uparrow m = 2^m$ where the exponent m equals $2\uparrow(2\uparrow(2\uparrow 2)) = 2^{16} = 65536$. Let's start small, with a list of the first few powers of 2:

$2^1 = 2$	$2^2 = 4$	$2^3 = 8$	$2^4 = 16$	$2^5 = 32$	$2^6 = 64$
$2^7 = 128$	$2^8 = 256$	$2^9 = 512$	$2^{10} = 1024$	$2^{11} = 2048$	$2^{12} = 4096$

There is a simple pattern for the units digits there: 2, 4, 8, 6, 2, 4, 8, 6, 2, 4, . . . (Why will this digit pattern continue forever, repeating in blocks of four?) To find the units digit of 2^m we must determine the m-th term in this digit pattern. Since $m = 2^{16}$ is a multiple of 4 and every fourth term in the digit pattern is a 6, we conclude that $2\uparrow\uparrow 5$ has units digit equal to 6.

What is the units digit of $3\uparrow\uparrow 4$? This number equals 3^n where $n = 7625597484987$. Easy calculations show that the units digits for the sequence 3^1, 3^2, 3^3, . . . also repeat in blocks of four, in the pattern: 3, 9, 7, 1, 3, 9, 7, 1, 3, . . . Since n is odd the n-th term in that digit pattern must be 3 or 7. The numbers k where 3^k has units digit equal to 3 are $k = 1, 5, 9, 13, 17, \ldots$ The other odd values of k (where 3^k has units digit 7) are $k = 3, 7, 11, 15, \ldots$ Add 1 to all these k values and notice that the second list is the sequence of numbers k for which $k + 1$ is a multiple of 4. For our number n observe that $n + 1 = 7625597484988$ is a multiple of 4. Then n belongs to the second list and we deduce that $3\uparrow\uparrow 4 = 3^n$ has units digit equal to 7.

You might try to determine the units digit for other numbers like $4\uparrow\uparrow\uparrow 6$ or $7\uparrow\uparrow 4$. These "units digit" questions form a part of elementary number theory. With improved notations and a few simple ideas, many problems of this sort can be solved. But not surprisingly, we quickly find many further questions which are deeper and harder to analyze. Like the large numbers mentioned earlier, the unanswered questions in mathematics and science go on forever.

References

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D. E. Knuth, "Coping with finiteness", *Science* **194** (1976) 1235-1242