Problems with Measuring

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Two men broke into a wizard's workshop and stole a crystal decanter filled with 8 ounces of powerful magic potion. In their hideout they want to split the prize, but they can locate only two other containers: a flask that holds 5 ounces and a vase that holds 3 ounces. Can they split the potion equally using only those containers?

This is a classic measuring problem. To model it mathematically we should consider only the key information, ignoring the rest. There are three containers: a decanter, a flask, and a vase, with capacities 8, 5, and 3 ounces. Starting with 8 ounces in the decanter, the men pour the liquid back and forth, trying to end up somehow with 4 ounces in one of the containers. Here's one way they could start: First they fill the flask, so the flask has 5 ounces of potion while 3 ounces remain in the decanter. For their second move they fill the vase by pouring from the flask. At that point there are 3 ounces in the vase, 2 ounces in the flask, with 3 ounces remaining in the decanter.

Explaining the steps with English words soon becomes confusing. To keep track of the moves let's write the symbol (x, y, z) to mean that there are x ounces in the decanter, y ounces in the flask, and z ounces in the vase. The story begins at (8, 0, 0). The first move above leads to (3, 5, 0) and after the second move they have (3, 2, 3). From there they could move to (0, 5, 3), or to (6, 2, 0), or back to (3, 5, 0). Are there other possibilities? There seem to be several ways to move after each step. The problem is to find a sequence of moves that ends with a 4 appearing somewhere. Take a minute now to see if you can discover the moves they need.[1]

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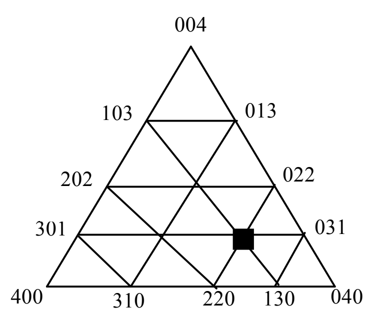
A position (x, y, z) in this game involves numbers x, y, z whose values are somewhat restricted. For instance, x represents the number of ounces in the decanter so it must be between 0 and 8. In symbols we write that as: $0 \le x \le 8$. Similarly there are y ounces in the flask, so y must lie between 0 and 5. That is: $0 \le y \le 5$. For the vase we find $0 \le z \le 3$. Finally since no spills are allowed (the potion is too valuable) there must be a total of 8 ounces at all times. In symbols: $x + y + z = 8$.



Here's a somewhat surprising connection with geometry. The points in a triangle are labeled so that each of those triples (x, y, z) corresponds to a point. For simplicity we start with a smaller example: Triples (x, y, z) where the entries are non-negative and $x + y + z = 4$. In the triangular grid pictured here, the labels are written in for the grid points around the outside. For example, the label "400" is shorthand for the triple (4, 0, 0) and represents the lower-left corner point. Can you see the pattern and figure out the labels for the unmarked interior points?

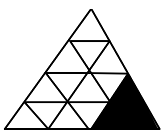
One explanation of this labeling pattern is to erase the "x" component and look only at the pair "yz". The labels along the bottom row become 00, 10, 20, 30, 40, and the labels along the left

[1] Nearly the same problem appeared in the movie "Die Hard 3". A bomb will explode if the hero (played by Bruce Willis) cannot solve that problem within a couple of minutes.

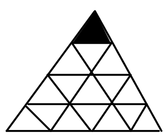
edge become 00, 01, 02, 03, 04. These behave like coordinates on the axes in standard plane geometry. For example the marked interior point is 2 steps to the right and 1 up, so its "*yz*" coordinates are "21". To find the missing "*x*" recall that $x + y + z = 4$. For this point $x + 2 + 1 = 4$ so that $x = 1$ and the triple representing that point is: 121.

Alternatively, erase the "*z*" component and examine only "*xy*" to get a similar situation with the origin 00 at the top. The labels down-left are 00, 10, 20, ... while the labels down-right are 00, 01, 02, ... The marked interior point is then 1 step down-left and 2 steps down-right, so its "*xy*" coordinates are 12. Filling in the missing "*z*" we find the full "*xyz*" coordinates to be 121, as before.

The equation $y = 2$ represents all points labeled $x2z$ (where $x + 2 + z = 4$). From the triangle picture we see this is the diagonal line containing the grid points 220, 121, 022. Which points in the triangle satisfy the inequality $y \leq 2$? That set of those points is white in triangle A below. Similarly the solution set for the inequality $z \leq 3$ is white in the triangle B.



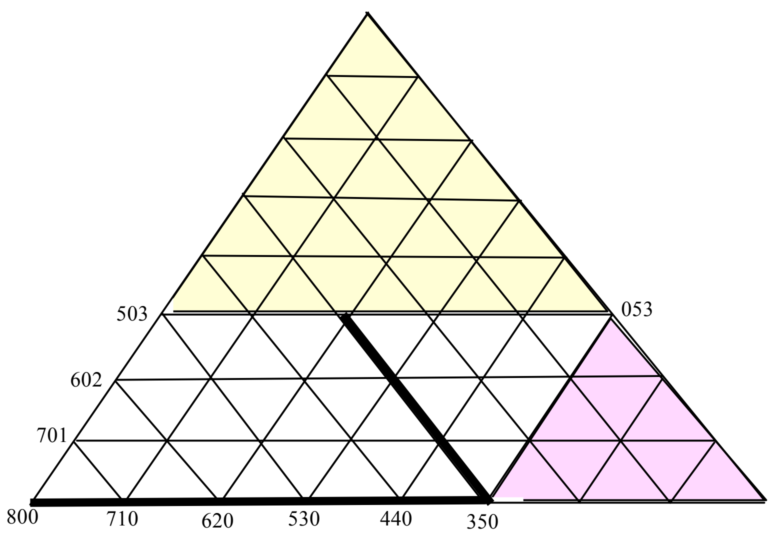
A



B

Can these coordinates be extended to points outside the triangle? Such an extension can be done provided negative coordinates are allowed. But we don't need those cases here.

Now let's return to the original problem with 8 ounces of potion. This situation can be drawn on a larger labeled triangle. Here every point in the triangle is represented by a triple *(x, y, z)* satisfying $x \geq 0$, $y \geq 0$, $z \geq 0$, and $x + y + z = 8$. Once that triangle is drawn and labeled we restrict attention to the region of all points satisfying $y \leq 5$ and $z \leq 3$, arising from the capacities of the flask and vase. (Why is the condition $x \leq 8$ automatically satisfied?) In the picture, the points satisfying those conditions lie in the white region of the triangle. The initial pour from the decanter to the flask is represented by the thickened segment from 800 to 350. The second move becomes the shaded segment from 350 to 323. Each step of pouring from one container to another appears in the picture as a line segment along the grid in the white region. The endpoint of each segment must always lie on the boundary of the white region. Those boundary points represent the situations where one (or more) of the containers is empty or full. At each step we have a choice of directions: we can move along any grid line as long as we stay within the white region and stop when we reach the boundary. After experimenting with ways to bounce around within that region, we do obtain a solution to the original problem: 800 → 350 → 323 → 620 → 602 → 152 → 143 → 440. Check that each step there represents a valid move, pouring the potion from one container to



another. Here's a different solution: 800 → 503 → 530 → 233 → 251 → 701 → 710 → 134 → 440. Can you trace that path within the white region above?

That problem was pretty easy! Now that you understand this geometric method you can solve much more complicated measuring problems. The hardest part is usually the artwork: drawing and labeling the triangle picture.

Note: In three-bottle problems the restrictions on x, y and z might lead to a *hexagonal* shape within the triangular grid. We had a parallelogram shape in the example above because all the liquid fit into the largest container.

Here are a few puzzlers you can use to test your new skills.

1. The robbers return to the wizard's stronghold and to steal a decanter filled with 12 ounces of water from the Fountain of Youth. They have two other containers: a flask that can hold 9 ounces and a vase that can hold 5 ounces. Can they equally split their ill-gotten gains, using only those containers and not spilling any of the precious liquid?

2. Bossie the cow has been biologically engineered to produce milk containing an antibiotic effective against the blue plague. The hospital needs exactly 10 pints of fresh milk for immediate use. Bossie's owner rushes out to the barn, but soon discovers that the only containers she has are three cans holding 19 pints, 13 pints and 7 pints. Can she milk Bossie and measure 10 pints using only those containers and without spilling any of the precious milk?

3. Three children have a large baggie containing 24 ounces of melted popsicles. They want to divide up the liquid but they have only three jars, which hold 13, 11 and 5 ounces. Can they divide that delicious concoction equally using only the items just mentioned?
[Hint: Measure 8 ounces first and put it back in the baggie. There remain 16 ounces to divide in two.]

4. Pierre has three crystal bottles with sizes 8, 6 and 4 ounces. The first one is full of expensive French perfume and the others are empty. He needs to measure 5 ounces. Explain why this cannot be done with only those tools. What is the general rule here?

5. This time Pierre has bottles of sizes 8, 6 and 3 ounces. He needs to get 4 ounces of perfume from the large container in the store room. Pierre has mathematical training and writes out the triangular grid for this problem. However, if he fills the 8 ounce bottle and pours perfume back and forth, he cannot figure out how to measure out exactly 4 ounces! Can you help him?
[Hint. He might fill the other bottles instead.]

Make up some similar problems of your own!

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