Nonstandard Digits
by Daniel Shapiro

The ten digits from 0 to 9 are used in the standard base ten system to represent all whole numbers. For instance, 8572 represents the number $8 \cdot 1000 + 5 \cdot 100 + 7 \cdot 10 + 2$. [*] This "place value" method has each digit count a different value depending on its place in the number. So 2 is the units digit, 7 is the tens digit, 5 is the hundreds digit, and so on. As we move left in the number, each digit counts quantities ten times as large as in the previous step. Mathematicians usually use exponents to represent those quantities, with 10^2 standing for one-hundred ($10 \cdot 10$) and 10^3 standing for one-thousand ($10 \cdot 10 \cdot 10$). Then our example number is $8572 = 8 \cdot 10^3 + 5 \cdot 10^2 + 7 \cdot 10 + 2$.

The usual ten digits represent the **first** ten numbers, counting whole numbers in order from zero up to nine. With those ten digits (0, 1, 2, 3, 4, 5, 6, 7, . . . , 9) and the place-value system we can write down any whole number. Kids learn this in pre-school.

But wait a minute! Do we have to pick that particular set of digits to make this work? What if we introduce a new digit X which stands for ten. This extra digit provides two representations for some numbers (after all, X = 10). To balance this let's throw out the digit 0. Then the *only* way to represent ten is by the single digit X. This X-system has the ten digits: 1, 2, 3, 4, 5, 6, 7, 8, 9, X.

Remember, we're still using the usual base ten system, but counting becomes a bit different since no digit "zero" is allowed. The number after 19 cannot be 20 since that uses a zero: instead we write it as 1X (after nine-teen comes ten-teen?). Of course after 1X come 21, 22, 23, and so on. Similarly we count ... , 27, 28, 29, 2X, 31, 32, ... , avoiding the digit zero.

What about one-hundred, which cannot be 100 any more since zero is forbidden? That becomes "ninety-ten" or 9X. The next numbers after that are X1, X2, . . . The largest 2-digit number will be XX (one-hundred-ten) followed by the first 3-digit number, 111.

The usual methods of addition and multiplication (with carrying) work as usual in this X-system. For instance:

$$\begin{array}{r} {\scriptstyle 1\ 1} \\ 2X5 \\ +\ \ 98 \\ \hline 3X3 \end{array} \qquad\qquad \begin{array}{r} {\scriptstyle 1\ 6} \\ 52X \\ \cdot\ \ \ 7 \\ \hline 36XX \end{array}$$

Here's some explanation of the multiplication problem above on the right.[**] Multiply 7 by the unit digit X to get 6X (sixty-ten, or seventy). Write down the X and carry the 6. Multiply 7 by the tens digit 2 and add the carried 6 to get 1X (ten-teen, or twenty). Write down the X and carry the 1. Multiply 7 by the hundreds digit 5 and add the

---

[*] The raised dot (·) stands for multiplication. The crossed lines (+) stand for addition (in case you didn't know).

[**] We'll leave the addition problem on the left for you to figure out.

carried 1 to get 36. Then the answer is 36XX. With standard notation this multiplication problem becomes $530 \cdot 7 = 3710$.

That worked pretty well! Let's begin again, trying a different set of digits. Starting from the standard digits let's introduce the new digit J to represent the number −1 (negative one). Then for example 2J6 represents $2 \cdot 10^2 + J \cdot 10 + 6$, which equals $200 - 10 + 6$, which is 196 in the standard system. Similarly $1J = 1 \cdot 10 + J = 10 - 1 = 9$. To avoid two representations of the same number, let's drop the digit 9 from the system. Then in this "J-system" we use the ten digits:

$$J, \ 0, \ 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 7, \ 8.$$

In this J-system we can still write down every whole number. For instance 19 is now represented as 2J. How can we write eighty-nine or ninety, knowing that the digit 9 is forbidden? Well, since nine is expressed as 1J, ninety should be 1J0. (To multiply a number by 10 just adjoin a zero on the right).[***] To get eighty-nine, the number just before this one, we can just subtract 1 (which is the same as adding J):

$$89 \ = \ 1J0 + J \ = \ 1JJ.$$

To check, calculate $1JJ = 1 \cdot 10^2 + J \cdot 10 + J \ = \ 100 - 10 - 1 \ = \ 89$. To verify that addition methods still work, let's add 1 to ninety-eight. Since 98 equals 1J8, we obtain:

$$\overset{\scriptstyle 1}{} $$
$$1J8$$
$$\underline{+ \ 1}$$
$$10J$$

Here we first add units digits: $8 + 1 = 1J$, write down J and carry 1. In the tens place, since $1 + J = 0$, there are no further carries and we get the answer written above. To check this is the correct value, ninety-nine, calculate:

$$10J = \ 1 \cdot 100 + 0 \cdot 10 + J \ = \ 100 + 0 - 1 \ = \ 99.$$

Can you write down the numbers, in order, from 85 to 105 in this J-system?

Numbers starting with J are negative. For instance: $J7 = J \cdot 10 + 7 = -10 + 7 \ = \ -3$. Similarly, $J4 = -6$ and $JJ = -11$. For a larger number let's try J43J. This equals $J \cdot 10^3 + 4 \cdot 10^2 + 3 \cdot 10 + J \ = \ -1000 + 400 + 30 - 1 \ = \ -571$. I was surprised to realize that every negative number can be expressed in the J-system *without using any minus signs*. For instance, let's try −284. Since 284 goes to the hundreds place, the trick is to go one step higher (one-thousand) and compute the difference: $1000 - 284 \ = \ 716$. Subtracting 1000 from both sides we find: $-284 \ = \ -1000 + 716$. Since J is −1, this becomes: $-284 \ = \ J000 + 716 \ = \ J716$. This expresses the negative number −284 without any minus sign.

Let try −371. With the same method, we first find $1000 - 371 \ = \ 629$. Then $-371 \ = \ -1000 + 629$. Converting those two terms to the J-system (where 9 isn't a digit), we get J000 and 63J, and we obtain the conversion: $-371 \ = \ J000 + 63J \ = \ J63J$.

[***] Why does that work?

This method will work for every negative number! That is, every integer (positive or negative) can be represented in exactly one way in the J-system, without using any minus signs. This is certainly a nicer property than the old system that uses ten digits along with an extra "minus" symbol.

Wouldn't life be simpler if everyone converted to the J-system?

Exercises:

0. How is −12 represented in the J-system? One method is to start with the expression −12 = −100 + 88. Another method is to subtract 1 from −11. That is the same as adding J to JJ. Do this addition, remembering that J + J = −2 = J8.

1. Check the following additions and multiplications in the J-system.
[Recall that J + J = J8, 3·J = J7, and J·8 = −8 = J2.]

	J		1 J		2		J
	4J		64J		1J8		J8
+	6J	·	3	·	3	·	J
	1J8		2J17		3J4		02

That last multiplication shows that (−1)·(−2) = 2.

2. Can you show that (−2)·(−3) = 6 by doing multiplication entirely in the J-system? The product starts out as:

$$\begin{array}{r} J8 \\ \cdot\ \underline{J7} \end{array}$$

3. We have looked at the X-system and the J-system. Think of another set of ten digits that could be the basis of another system. How hard is it to count, add, and multiply in your new system?

4. All examples in this article have been in base ten. Do these ideas work for other bases? For instance, it's interesting to investigate the *base three* system with digits J, 0, 1. Starting from zero and counting upwards we get

0, 1, 1J, 10, 11, 1JJ, 1J0, 1J1, 10J, 100, …