The usual 5-pointed star ✷ is a wonderful figure. It was the mystic symbol, called a pentagram, for the “Pythagorean” cult of ancient Greece. They investigated many of its mathematical and magical properties. The 6-pointed star ☿, called a hexagram or Star of David, also has a long history as a religious symbol. What other stars can you draw?

Let’s look at the pentagram more closely. Start with 5 dots, equally spaced around a circle. At each dot we place an imaginary jumping spider which leaves a straight web-trail wherever it jumps. Suppose each spider jumps directly to the second dot to its right, that is: to the dot which is two steps away, clockwise around the circle. Here is what that traced figure might look like:

The usual way to create this star on paper is to draw one segment after another, without lifting the pencil from the paper.

Now use the same 5 dots, but let the spiders jump to the dot which is just one step away. (We use the word “step” to mean the motion from one dot the next one, clockwise around the circle.) This produces a different figure, the regular pentagon:

This “star” can also be drawn one segment after another, without lifting the pencil from the paper. What if you use those 5 dots, but now each spider jumps to the dot which is 3 steps away? What figure do you get? What happens when they jump 4 steps each time? What about 5 steps?

Let’s try 6 dots now, rather than 5. We can have each of the 6 spiders jump to the next dot clockwise (1 step). The resulting figure is a regular hexagon □. If each spider jumps to the dot 2 steps away, we get the hexagram ☿. This one cannot be drawn without lifting the pencil, since it is made of 2 overlapping triangles. If each of the 6 spiders jumps to the dot which is 3 steps away we get an “asterisk” ✻. It is made of 3 separate line segments.
After pondering these examples you know we will go on to discuss more general stars. For whole numbers \( n \) and \( d \) let’s construct the “\( n \)-sided star with step size \( d \)”’. More briefly let’s call it an “\( n \)-star with \( d \) steps”. This is built using \( n \) equally spaced dots on a circle and \( n \) spiders, each jumping to the dot which is \( d \) steps away, clockwise around the circle. To avoid such a long description of this figure we’ll refer to it by the compact symbol: \( \{n/d\} \).

We already mentioned some 5-stars and 6-stars. A \( \{5/2\} \) is a 5-star with 2 steps, which is just a pentagram \( \star \). Similarly a \( \{5/1\} \) is a pentagon \( \bigcirc \), a \( \{6/1\} \) is a hexagon \( \bigcirc \) , and a \( \{6/2\} \) is a hexagram \( \bigtriangleup \). Remember that this \( \{6/2\} \) is formed from 2 overlapping \( \{3/1\} \)’s, which are the triangles \( \triangleleft \) and \( \triangleright \). Also the asterisk \( \{6/3\} \bigstar \) is built from 3 overlapping \( \{2/1\} \)’s, which are the segments \( \backslash \), \( \| \) and \( / \). What about “degenerate” cases like \( \{5/0\} \)? That 5-star has 5 dots but no segments (the spiders stay in place without jumping), a picture that doesn’t look much like a traditional star. It is built from 5 separate pieces, each one a single dot (which would be simply 5 copies of the 1-star \( \{1/0\} \)).

Let’s move up to the 7-stars. The \( \{7/1\} \) is a regular heptagon \( \bigcirc \), but the \( \{7/2\} \) is more interesting. It is constructed by placing 7 dots around a circle and drawing all the 2-step segments. Once you see what it looks like you can practice drawing it freehand, without lifting the pencil from the paper. Some people think that a \( \{7/3\} \) looks nicer but others find it too pointy.

![\{7/2\}](image1)

![\{7/3\}](image2)

Some artistic students decorate their notebooks and papers with these stars. (But others think that sort of thing is very annoying. There’s no accounting for tastes!) What would the stars \( \{7/4\} \), \( \{7/5\} \) and \( \{7/6\} \) look like? How about \( \{7/0\} \) and \( \{7/7\} \) ?

The 8-stars provide more examples of “non-connected” behavior. For example an \( \{8/2\} \bigcirc \) is built from 2 overlapping squares or \( \{4/1\} \)’s, and an \( \{8/4\} \bigstar \) consists of 4 overlapping line segments or \( \{2/1\} \)’s. More interesting to draw is the \( \{8/3\} \), which seems somewhere between the two 7-stars illustrated above.

![\{8/3\}](image3)
Stars with more sides become harder to draw freehand. Here are some 9-stars to practice on. Which of these stars is connected?

{9/2}  
{9/3}  
{9/4}

Here are a few questions that might inspire you to think further about this subject.

1. The stars \{5/2\} and \{5/3\} look the same, but they can be distinguished by the direction they are drawn, clockwise or counterclockwise. But \{5/6\} is identical with \{5/1\}, and \{5/7\} is identical with \{5/2\}. How about \{5/5\} and \{5/0\}? What is the general rule here?

2. Which stars are connected? How many separate pieces does the star \{n/d\} have? For example \{6/2\} is made from 2 smaller stars so it has 2 pieces, \{6/3\} has 3 pieces, \{6/4\} has 2 pieces, and \{5/0\} has 5 pieces. What's the pattern?

3. How is the star \{n/d\} related to the fraction \(\frac{n}{d}\)? How is “reducing the fraction to lowest terms” related to drawing the associated star? For example, the fractions \(\frac{6}{2}\) and \(\frac{3}{1}\) are equal. How are the stars \{6/2\} and \{3/1\} related?

4. Suppose a line is drawn through the center of the n-star \{n/d\}. How many of the sides of the star does that line cross? (To get a good count, avoid lines passing through a corner of the star.)

5. How many connected n-stars are there for given \(n\)? For example there is one 1-star (a dot), there is one connected 2-star (a segment), there are two connected 3-stars (triangles traced in opposite directions), and there are two connected 4-stars (squares traced in opposite directions). If we write \(\Phi(n)\) to represent this number of connected stars (that \(\Phi\) is the Greek letter phi), then:

\[
\begin{align*}
\Phi(1) &= 1 \\
\Phi(2) &= 1 \\
\Phi(3) &= 2 \\
\Phi(4) &= 2 \\
\Phi(5) &= 4 \\
\Phi(6) &= 2 \\
\Phi(7) &= 6 \\
\Phi(8) &= 4 \\
\Phi(9) &= 6.
\end{align*}
\]

6. What other mysteries are hidden in the stars?