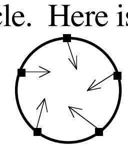
Drawing Stars

by Daniel B. Shapiro 9/99

The usual 5-pointed star is a wonderful figure. It was the mystic symbol, called a pentagram, for the "Pythagorean" cult of ancient Greece. They investigated many of its mathematical and magical properties. The 6-pointed star, called a hexagram or Star of David, also has a long history as a religious symbol. What other stars can you draw?

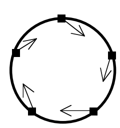
Let's look at the pentagram more closely. Start with 5 dots, equally spaced around a circle. At each dot we place an imaginary jumping spider which leaves a straight web-trail wherever it jumps. Suppose each spider jumps directly to the second dot to its right, that is: to the dot which is two steps away, clockwise around the circle. Here is what that traced figure might look like:


One-third done


Finished

The usual way to create this star on paper is to draw one segment after another, without lifting the pencil from the paper.

Now use the same 5 dots, but let the spiders jump to the dot which is just *one* step away. (We use the word "step" to mean the motion from one dot the next one, clockwise around the circle.) This produces a different figure, the regular pentagon: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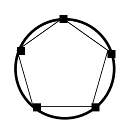
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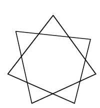
This "star" can also be drawn one segment after another, without lifting the pencil from the paper. What if you use those 5 dots, but now each spider jumps to the dot which is 3 steps away? What figure do you get? What happens when they jump 4 steps each time? What about 5 steps?

Let's try 6 dots now, rather than 5. We can have each of the 6 spiders jump to the next dot clockwise (1 step). The resulting figure is a regular hexagon. If each spider jumps to the dot 2 steps away, we get the hexagram. This one cannot be drawn without lifting the pencil, since it is made of 2 overlapping triangles. If each of the 6 spiders jumps to the dot which is 3 steps away we get an "asterisk". It is made of 3 separate line segments.

After pondering these examples you know we will go on to discuss more general stars. For whole numbers n and d let's construct the "n-sided star with step size d". More briefly let's call it an "n-star with d steps". This is built using n equally spaced dots on a circle and n spiders, each jumping to the dot which is d steps away, clockwise around the circle. To avoid such a long description of this figure we'll refer to it by the compact symbol: {n/d}.

We already mentioned some 5-stars and 6-stars. A {5/2} is a 5-star with 2 steps, which is just a pentagram ☆. Similarly a {5/1} is a pentagon ⬠, a {6/1} is a hexagon ⬡, and a {6/2} is a hexagram ✡. Remember that this {6/2} is formed from 2 overlapping {3/1}'s, which are the triangles ◁ and ▷. Also the asterisk {6/3} ✳ is built from 3 overlapping {2/1}'s, which are the segments ╲, — and ╱. What about "degenerate" cases like {5/0}? That 5-star has 5 dots but no segments (the spiders stay in place without jumping), a picture that doesn't look much like a traditional star. It is built from 5 separate pieces, each one a single dot (which would be simply 5 copies of the 1-star {1/0}).

Let's move up to the 7-stars. The {7/1} is a regular heptagon ⬡, but the {7/2} is more interesting. It is constructed by placing 7 dots around a circle and drawing all the 2-step segments. Once you see what it looks like you can practice drawing it freehand, without lifting the pencil from the paper. Some people think that a {7/3} looks nicer but others find it too pointy.



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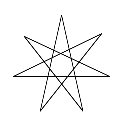

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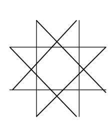
Some artistic students decorate their notebooks and papers with these stars. (But others think that sort of thing is very annoying. There's no accounting for tastes!) What would the stars {7/4}, {7/5} and {7/6} look like? How about {7/0} and {7/7} ?

The 8-stars provide more examples of "non-connected" behavior. For example an {8/2} ✴ is built from 2 overlapping squares or {4/1}'s, and an {8/4} ✳ consists of 4 overlapping line segments or {2/1}'s. More interesting to draw is the {8/3}, which seems somewhere between the two 7-stars illustrated above.



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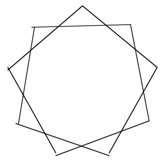
Stars with more sides become harder to draw freehand. Here are some 9-stars to practice on. Which of these stars is connected?



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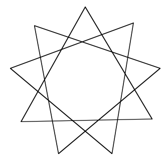

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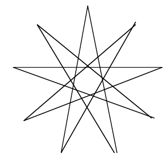

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Here are a few questions that might inspire you to think further about this subject.

1. The stars {5/2} and {5/3} look the same, but they can be distinguished by the *direction* they are drawn, clockwise or counterclockwise. But {5/6} is identical with {5/1}, and {5/7} is identical with {5/2}. How about {5/5} and {5/0} ? What is the general rule here?

2. Which stars are connected? How many separate pieces does the star {n/d} have?
For example {6/2} is made from 2 smaller stars so it has 2 pieces, {6/3} has 3 pieces, {6/4} has 2 pieces, and {5/0} has 5 pieces. What's the pattern?

3. How is the star {n/d} related to the fraction $\frac{n}{d}$? How is "reducing the fraction to lowest terms" related to drawing the associated star? For example, the fractions $\frac{6}{2}$ and $\frac{3}{1}$ are equal. How are the stars {6/2} and {3/1} related?

4. Suppose a line is drawn through the center of the n-star {n/d}. How many of the sides of the star does that line cross? (To get a good count, avoid lines passing through a corner of the star.)

5. How many connected n-stars are there for given n ? For example there is one 1-star (a dot), there is one connected 2-star (a segment), there are two connected 3-stars (triangles traced in opposite directions), and there are two connected 4-stars (squares traced in opposite directions). If we write $\phi(n)$ to represent this number of connected stars (that ϕ is the Greek letter phi), then:

$\phi(1) = 1$	$\phi(4) = 2$	$\phi(7) = 6$
$\phi(2) = 1$	$\phi(5) = 4$	$\phi(8) = 4$
$\phi(3) = 2$	$\phi(6) = 2$	$\phi(9) = 6.$

6. What other mysteries are hidden in the stars?