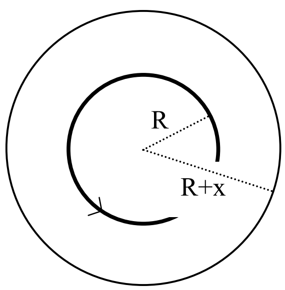
Walking the dog

by Daniel Shapiro, 2004

I like to walk with my dog Phido on the path that goes around the nearby park. That dog is so well trained that he stays exactly one yard to my right at all times. Since I walk with the center of the park to my left, Phido's path is somewhat longer than mine. How much longer is it?

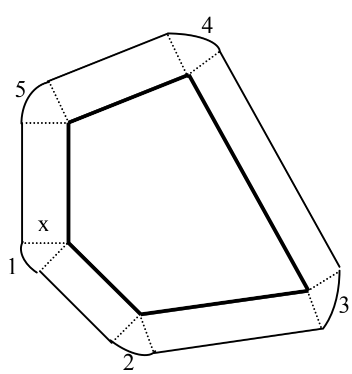
Let's think about the problem in a simplified (mathematical) world. Phido and I are represented by moving points and **P** is my path. For instance, if the path is a circle with a 100 yard radius, the total length L that I walk is the circumference $L = 2\pi\cdot 100$ or about 628.32 yards. Phido's path is a circle with a 101 yard radius, with total length $2\pi\cdot 101$ which is about 634.60. Then the difference in path lengths is about $634.60 - 628.32 = 6.28$ yards.

What about leashes of different length? For instance if Phido walks x yards to my right, then Phido's path is a circle with radius $100 + x$ yards, and the distance he walks is the circumference $2\pi\cdot(100 + x)$, which equals $2\pi\cdot 100 + 2\pi\cdot x$. Since my path length is $L = 2\pi\cdot 100$, we find that Phido walks exactly $2\pi\cdot x$ yards farther than I do.

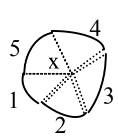


I was surprised to notice that the answer remains the same for circular paths of any size. If I go along a circle of radius R then Phido's path is a circle of radius $R + x$. Compare my path length $L = 2\pi\cdot R$ with Phido's path length $2\pi\cdot(R + x) = 2\pi\cdot R + 2\pi\cdot x$. The formula shows that Phido walks $2\pi\cdot x$ yards farther than I do. The answer doesn't depend on the size of the circle.

How about paths of other shapes, like a rectangle or hexagon? More generally suppose my path **P** is a polygon, composed of several straight segments arranged in a roughly round shape. I walk with the center of the park to my left and Phido is always x yards to my right. When I go along a straight segment, the dog's path is a parallel segment of the same length. When I get to a corner, I stop and turn to face a new direction while Phido walks through an arc of a circle. Phido's path is made of straight sections (matching the straight sections of my path) together with several circular arcs of radius x.



For example, if my path is a square, those arcs make 4 quarter circles, which together form one full circle. For the pentagon in the picture, those 5 circular arcs also seem to fit together exactly to make a full circle.



Do the circular arcs always fit together perfectly?
[In the picture those arcs make a somewhat lumpy circle, but I'm no draftsman.]
To explain why they do fit, suppose I walk holding a big arrow which points to my right at every instant. After going

once around the park, the arrow ends up pointing in the same direction it did at the start (since I went in a loop). During that trip the arrow turns through one full circle. As I walk on the straight sections, the arrow's direction doesn't change, but at each corner the arrow sweeps through the angle of the circular sector at that corner. Therefore the sum of the angle-measures at all the corners must equal the total amount of turning done by the arrow. This says that those circular sectors do join together exactly to make one full circle of radius x.

Consequently, the distance Phido walks equals the sum of the lengths of the straight segments and the lengths of those arcs. That equals my distance (from the straight segments) plus the circumference of that full circle of radius x, where x is the length of the leash. That is,

Phido walks $2\pi x$ yards farther than I do.

Whoa! That is the same result as before! The answers for a polygonal path around the park are the same as for a circular path.

Care has to be taken with paths that aren't nearly round. If the path zig-zags (with some left turns and some right turns) then Phido might trace some circular arcs *backwards*, and those arcs would have to be counted as "negative distance". Let's avoid those cases and stick to paths with only left turns.

What about smooth paths like ellipses and more general ovals? Replace the given smooth path with a nearby polygonal one built from a large number of very short, straight steps (with a slight left turn at the end of each one). For that path, and leash length x, Phido walks $2\pi\cdot x$ yards farther than I do. Since we can make a polygonal path extremely close to the given smooth path, that formula holds for the smooth path as well[1].

For *any* path around the park involving only left turns,
Phido walks $2\pi\cdot x$ yards farther than I do.

This really is amazing: the difference in distances doesn't depend on the shape of the path. The answer depends only on the leash length x, and on the fact that I walk once around, making only left turns as I travel.

What if Phido stays on my *left* as I walk? In that case, his path is inside mine and his distance should be shorter than mine. How much shorter will it be? Let's reverse the roles and think of Phido walking around the park with me always x yards to his right. Then the discussion above (switching Phido and me) shows that my path is $2\pi\cdot x$ yards longer than his. That is, Phido's path is exactly $2\pi\cdot x$ yards *shorter* than mine.

For you negative number fans, we can re-think this situation. Having Phido x yards to my left is the same having him $-x$ yards to my right. Also, Phido's path is $2\pi\cdot x$ yards shorter, which is that same as $-2\pi\cdot x$ longer, than mine. That results in the same formula as before:

For my path as above, if Phido stays x yards to my right
then he travels $2\pi\cdot x$ yards farther than I do, even if x is negative!

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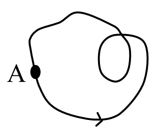
[1] This is a "limit" process as done in calculus.

Our formula can be generalized even more. For instance, if I walk only part of the way around the park, how much farther does Phido travel? If I go halfway around I hope Phido travels $\pi \cdot x$ yards farther. But what does "halfway" mean here: is it half the distance, or half the angle turned?

Here are a few questions that come to mind when thinking about my faithful pooch:

1. The inner lane of a running track around a football field is one-quarter mile in length. The next lane is one yard further out. If the finish lines are the same, how should the starting line on the outer lane be adjusted to make the two lanes have equal length?

2. If I walk twice around the park, how much farther does Phido travel? How about n times around? What happens if I walk once around the park in the reverse direction? (Is that -1 times around?) What if I walk along the path pictured here, starting and ending at point A?



3. When I walk once around the park my path encloses a certain area A. Phido's path also encloses a larger area. For a circle my area is $A = \pi R^2$ and Phido's area is $\pi(R+x)^2$. Expand this to $\pi(R^2 + 2Rx + x^2)$. Multiply through by π and recall earlier formulas for A and L, to deduce: Phido's area = $A + Lx + \pi x^2$.
Does this area formula also work for polygonal paths going around the outside of a region? How about smooth paths?