Fractal Dimension and Lower Bounds for Geometric Problems

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The curse of dimensionality

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- Many types of dimension, e.g. Euclidean dimension, doubling dimension etc.

Fractals

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- Fractals are ubiquitous in nature.

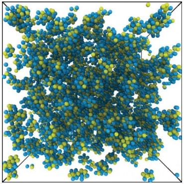


Figure: Fractal arrangement of atoms in $Cu_{46}Zr_{54}$ (left), and lightning bolts (right).

Fractal Dimension

Several notions of fractal dimension:

- Hausdorff dimension
- Box-counting dimension
- Information dimension
- ...

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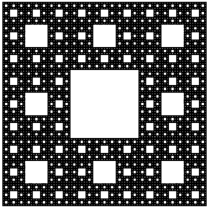
- Sierpiński carpet has fractal dimension $\log_3 8$, since scaling by a factor of 3 increases the volume by a factor of 8.



Fractal dimension of discrete sets

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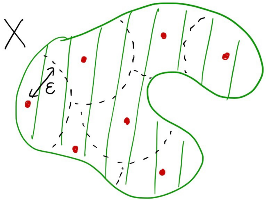
- E.g. the Hausdorff dimension of any countable set is zero.

A definition of fractal dimension for discrete sets

Given a pointset $X \subset \mathbb{R}^d$, the fractal dimension of X, denoted by $\dim_f(X)$, is the infimum over all $\delta > 0$ such that for all $x \in \mathbb{R}^d$, for all $\epsilon > 0$, $r \geq 2\epsilon$ and for all ϵ-nets N of X, we have

$$|N \cap \mathrm{ball}(x, r)| = O((r/\epsilon)^\delta)$$

ϵ-**net** - Maximal $N \subseteq X$ such that for all $x, x' \in N$, $x \neq x'$, $d_X(x, x') > \epsilon$.



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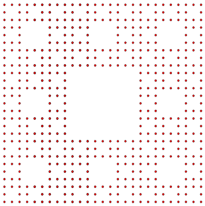
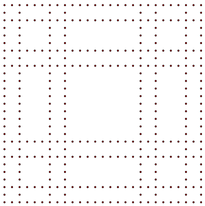
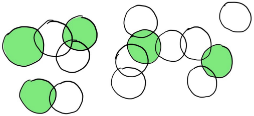
- $\dim_{\mathrm{f}}(\{1, \ldots, n^{1/d}\}^d) = d$.
- Fractal dimension of discrete Sierpiński carpet is $\log_3 8$ (left below), and of discrete Cantor crossbar is $\log_3 6$ (right below).





k-Independent Set of Unit Balls



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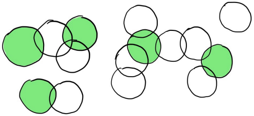

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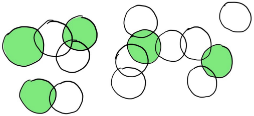

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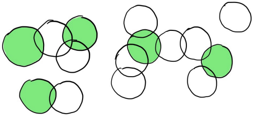

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- If the set of centers has fractal dimension $\delta > 1$, assuming ETH, no $f(k)n^{o(k^{1-1/(\delta-\epsilon)})}$ algorithm exists, for all $\epsilon > 0$ [Sidiropoulos, S., Sridhar'18].

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Problem - Given a set of n points in $\mathbb{R}^d$, find a closed tour of shortest length that visits all the points.

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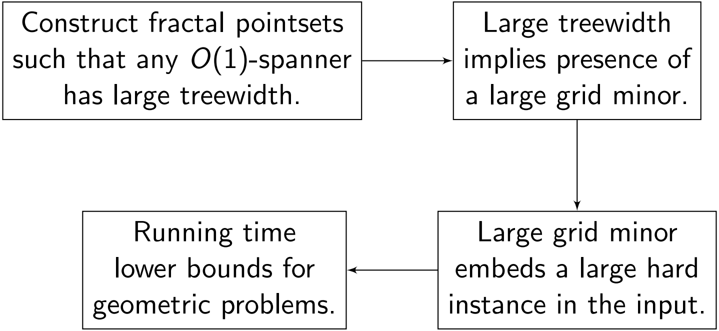
Central Idea



Lower bound on treewidth of spanners

For a metric space (X, ρ), a c-spanner is a graph $G = (X, E)$ such that for all $x, x' \in X$,

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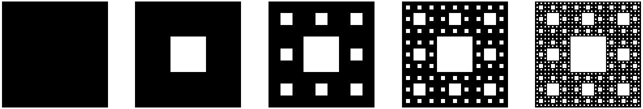
- For every $\epsilon > 0$ and $\delta > 1$, there exists $X \subset \mathbb{R}^d$ with $|X| = n$ such that $\dim_{\mathrm{f}}(X) \leq \delta$, and any c-spanner of X has treewidth $\Omega\left(\frac{n^{1-1/(\delta-\epsilon)}}{c^{d-1}}\right)$ [Sidiropoulos, S., Sridhar'18].

First attempt: Sierpiński carpet



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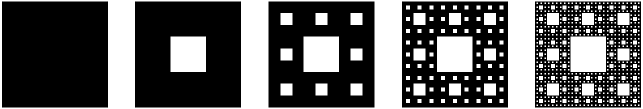

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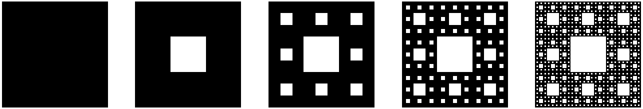

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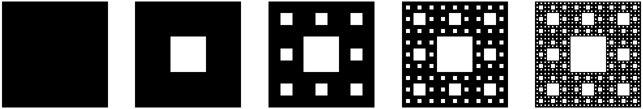

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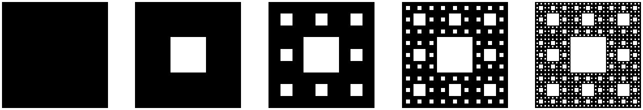

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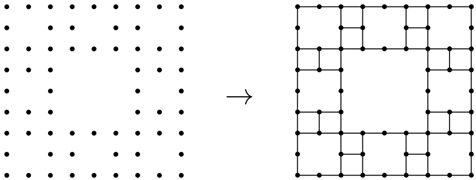

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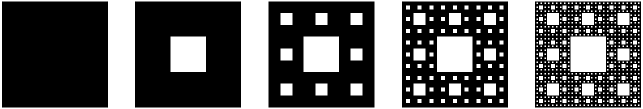

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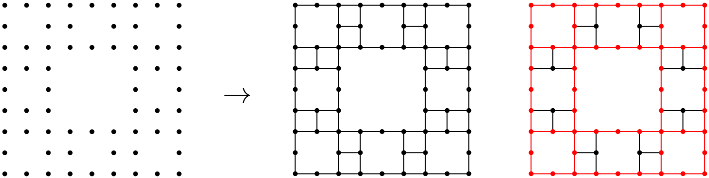
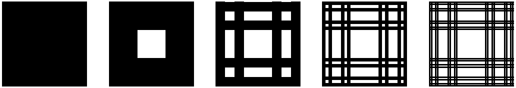


A treewidth extremal fractal: Cantor crossbar

The Cantor crossbar in $\mathbb{R}^2$:



$$= R_1(CS \times [0,1]) \cup R_2(CS \times [0,1])$$

The Cantor set (CS):

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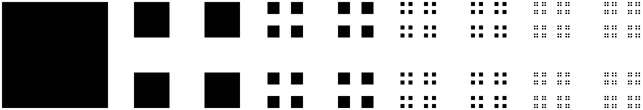
The Cantor dust (CD_d):



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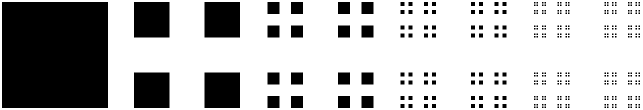

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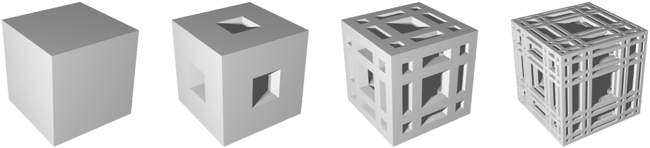

$$R_1(CD_{d-1} \times [0,1]) \cup \ldots \cup R_d(CD_{d-1} \times [0,1])$$

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- We can generalize the above construction so that δ takes any desired value in the range $(1, d)$.
- In the definition of Cantor dust, we start with a Cantor set of smaller dimension. This can be done by removing the central interval of length $\alpha \in (0, 1)$, instead of $\frac{1}{3}$, and recursing on the remaining two intervals of length $\frac{1-\alpha}{2}$.

From treewidth to running time lower bounds

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Efficient NP-Hardness reductions, using gadgets, as in the NP-Hardness proof of TSP in $\mathbb{R}^2$ by Papadimitriou.

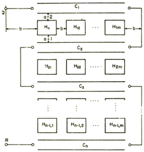

Fig. 4. The skeleton, schematically.

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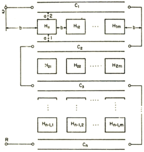

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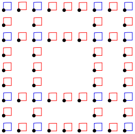
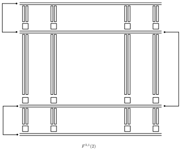
Arrange gadgets along a Cantor crossbar.





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Thank You.