# The Conjecture of Birch and Swinnerton-Dyer

(Overheads for a talk at Ohio State, 11/10/2005; they weren't used due to technical difficulties)

Warren Sinnott

# Diophantine problems

$x^2 + y^2 = z^2$ (the "Pythagorean" equation)

$x^2 - Ny^2 = 1$ ("Pell's" Equation)
$N$ is a given integer, not a square.

$x^3 = y^2 + N$ (One of Fermat's challenges to English mathematicians was to show that when $N = 2$ the only positive integer solution is $x = 3, y = 5$.)

$x^N + y^N = z^N$ (Fermat's Last Theorem)
$N$ is an integer $> 2$.

(from Diophantus:) If a rational number is the difference of two positive rational cubes then it is the sum of two positive rational cubes.

(from a 10th century Arabic mss.) Given a natural number $N$, does there exist a right triangle with rational sides and area $N$? ("congruence number problem")

# Hilbert's 10th problem

*Find an algorithm to decide whether a polynomial equation* $f(x, y, z, \dots) = 0$ *(with integer coefficients) has any integer solutions.*

Matijasevič (following work of J. Robinson, M. Davis, and others) 1970: *There is no such algorithm.*

Still open: the "rational" form of Hilbert's 10th problem: *Find an algorithm to decide whether* $f(x, y, z, \dots) = 0$ *has any* rational *solutions.*

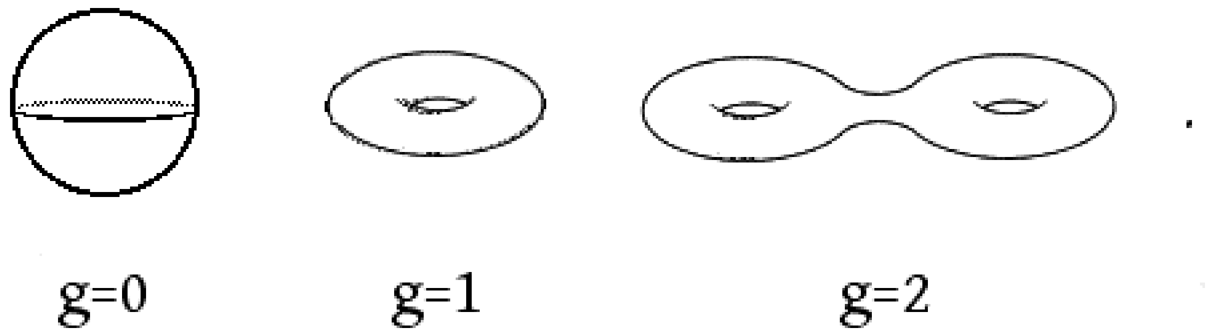
Even the following is still open:

*Find an algorithm to decide, given integers* $a, b$*, whether the equation* $y^2 = x^3 + ax + b$ *has a solution in the rational numbers.*

Consider the problem of finding the *rational* zeroes to (absolutely irreducible) polynomial equations in two variables (with integer or rational coefficients): $f(x, y) = 0$. Roughly, the problem gets harder as the degree of the $f$ increases.

But the correct measure of the "difficulty" of solving $f(x, y) = 0$ is the *genus* of the equation.

Consider the set $X(\mathbb{C}) = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ of complex solutions to $f(x, y) = 0$. If we complete $X$ (add several points at $\infty$) and desingularize it, we get a compact Riemann surface $\hat{X}$; topologically, $\hat{X}$ is a compact oriented surface, and we let $g$ be its genus (the number of holes...)


g=0
g=1
g=2
. . .

A brief and incomplete outline of what is known about rational solutions to $f(x, y) = 0$:

$$g = 0$$

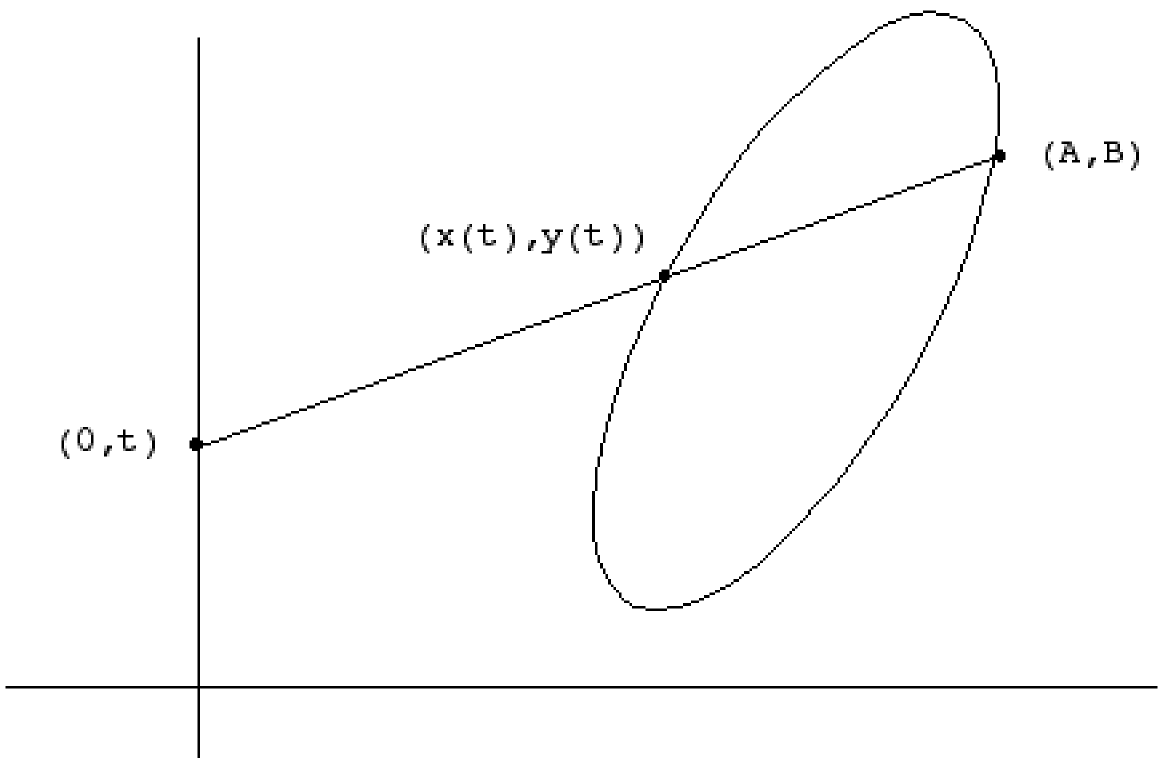
This is the case for example if $\deg f = 1$ or 2. The set of $X(\mathbb{Q})$ of rational solutions to $f(x, y) = 0$ is either empty or infinite. If $\deg f = 1$, then there are infinitely many points (which form a "1-parameter family.") When $\deg f = 2$, it is a problem to decide whether $X(\mathbb{Q})$ is empty, and the problem is solved by the "Hasse principle:" $X(\mathbb{Q})$ is non-empty if and only if there are real solutions and $p$-adic solutions for each prime $p$, i.e.

$$X(\mathbb{Q}) \neq \emptyset$$

$$\Longleftrightarrow$$

$$X(\mathbb{R}) \neq \emptyset \text{ and } X(\mathbb{Q}_p) \neq \emptyset \text{ for all primes } p$$

Moreover, as soon as we have one solution $(A, B) \in X(\mathbb{Q})$ we get infinitely many, and we can parametrize them by the rational points on a line:


(A,B)
(x(t),y(t))
(0,t)

$$g = 1$$

This case occurs, for example, if $f(x, y) = y^2 - x^3 - ax - b$ and $x^3 + ax + b$ has distinct roots. The set $X(\mathbb{Q})$ of rational solutions to $f(x, y) = 0$ can be finite (including possibly empty) or infinite. There are no algorithms at present to decide which. But if we allow "points at infinity" the set $X(\mathbb{Q})$, when nonempty, can be made into an abelian group. (E.g. for $f(x, y) = y^2 - x^3 - ax - b$, there is one point at infinity, which serves as the identity for the group.)

$$g > 1$$

Remarkably little is known in general beyond one spectacular result, due to Faltings: $X(\mathbb{Q})$ is *finite* (including possibly empty). But we have no effective procedure for deciding whether $X(\mathbb{Q})$ is empty or for enumerating its elements if it is non-empty.

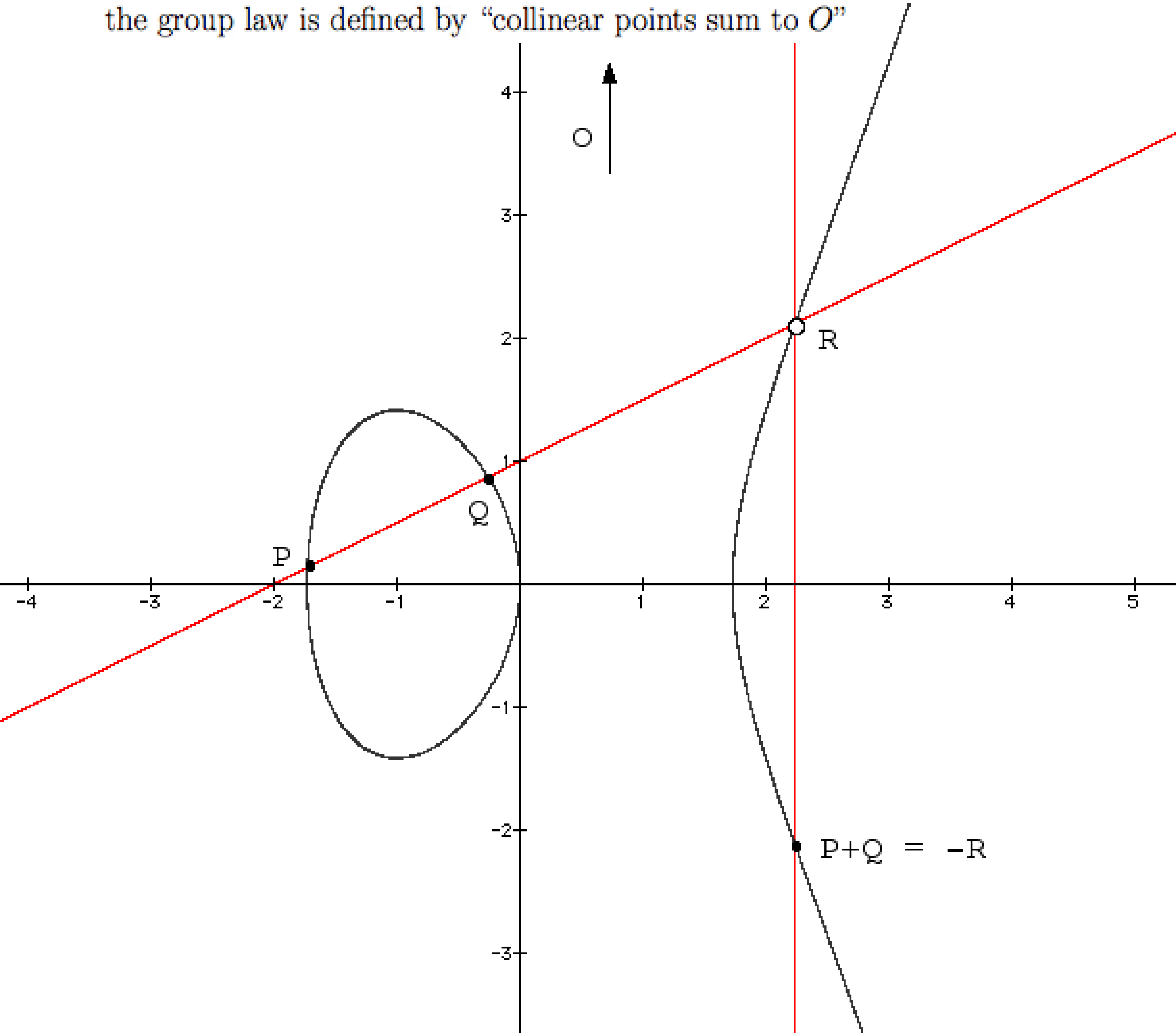
We consider the case $g = 1$ in more detail.

A curve $E : y^2 = x^3 + ax + b$ is called an *elliptic curve* when $-4a^3 - 27b^2 \neq 0$ (which guarantees that the roots of $x^3+ax+b$ are distinct). One can make the points of $E$ with values in any field into a group: given points $P$ and $Q$ on $E$, we can construct the line through $P$ and $Q$; this line will intersect $E$ in a third point $R$, and a group law on $E$ is then determined by the condition $P+Q+R = O$ (where $O$ is the point at infinity and the identity of the group).

(When $P = Q$, we use the tangent line to the curve at $P$. There are other special cases to consider as well.)

A key feature of the situation: if $P$ and $Q$ have coordinates in a field $F$, so does $P + Q$. So $E(\mathbb{Q})$, $E(\mathbb{R})$, and $E(\mathbb{C})$ all become groups under this construction.

the point at infinity ($O$) is the identity
the lines through $O$ are the *vertical* lines
reflection across the $x$-axis is negation
the group law is defined by "collinear points sum to $O$"



This picture yields the following formulas:

If $P = (x_1, y_1)$, $Q = (x_2, y_2)$ are distinct points and $x_1 \neq x_2$, then the coordinates $(x_3, y_3)$ of $P + Q$ are

$$\left(\left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - x_1 - x_2, \left(\frac{y_2 - y_1}{x_2 - x_1}\right) x_3 - \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}\right)$$

If $P$ and $Q$ are distinct points with $x_1 = x_2$, then $P + Q = O$, the point at infinity.

If $P = Q$, but $y_1 = y_2 \neq 0$, then the coordinates of $P + Q = 2P$ are

$$\left(\left(\frac{3x_1^2 + a}{2y_1}\right)^2 - 2x_1, \left(\frac{3x_1^2 + a}{2y_1}\right) x_3 + y_1 - \left(\frac{3x_1^2 + a}{2y_1}\right) x_1\right)$$

Finally, if $P = Q$ and $y_1 = y_2 = 0$, then $P + Q = 2P = O$.

The picture shows $E(\mathbb{R})$. $E(\mathbb{C})$ is a torus: we have

$$\mathbb{C}/L \simeq E(\mathbb{C})$$

for some lattice $L \subseteq \mathbb{C}$, the isomorphism being given by the Weierstrass $\wp$-function for $L$.

We want to understand $E(\mathbb{Q})$.

Theorem (Mordell, 1922) *$E(\mathbb{Q})$ is a finitely generated abelian group.*

So $E(\mathbb{Q}) \simeq T \oplus \mathbb{Z}^r$, where $T$ is finite abelian. $T$ is easy to compute:

Theorem (Lutz, Nagell c. 1935) *If $(x, y)$ is a torsion point, then $x$ and $y$ are integers and either $y = 0$ or $y^2 \mid 4a^3 + 27b^2$.*

What about $r$? ($r$ called the *rank* of $E$) We can compute an upper bound for $r$ but there's no known bound for the *heights* of the generators of $E(\mathbb{Q})$. (So unless the upper bound *is* the rank, we don't know when to stop looking.)

If $P = (x, y) \in E(\mathbb{Q})$, the height of $P$, denoted $H(P)$, is the maximum size of the numerator and denominator of $x$ and $y$.

How can we determine $r$??

Digression (?):

Consider $E(\mathbb{F}_p) = \{(x, y) \in \mathbb{F}_p^2 : y^2 \equiv x^3 + ax + b \bmod p\} \cup \{O\}$. If $p \neq 2$ and $p \nmid 4a^3 + 27b^2$, then the formulas above make $E(\mathbb{F}_p)$ into a group. What is $N_p = \#E(\mathbb{F}_p)$? For each $x = 0, 1, \ldots, p-1$ we get

- no points if $x^3 + ax + b$ is not a square mod$p$
- one point if $x^3 + ax + b \equiv 0 \bmod p$
- two points if $x^3 + ax + b$ is a nonzero square mod$p$

plus one for the point at infinity.

Since a randomly chosen nonzero element of $\mathbb{F}_p$ is equally as likely to be a square as a non-square, the first and third possibilities might tend to be equally likely, which suggests that $N_p = \#E(\mathbb{F}_p)$ should be about $p + 1$. In fact,

Theorem (Hasse, 1934) $|p + 1 - N_p| \leq 2\sqrt{p}$. (For $p > 2, p \nmid 4a^3 + 27b^2$.)

In the late 1950s, Birch and Swinnerton-Dyer had the happy thought (suggested by work of Siegel on quadratic forms in the 1930s) that if $r = \text{rank} E(\mathbb{Q})$ is large ($> 0$) then we should get more points in $E(\mathbb{F}_p)$ than expected.

(There is a "reduction map" $E(\mathbb{Q}) \to E(\mathbb{F}_p)$).

Or maybe, if there are more points in lots of $E(\mathbb{F}_p)$'s than there should be, we have a better chance of being able to "piece them together" into a rational point on $E$.

In any case, they tried calculating

$$\pi_E(x) = \prod_{p \leq x} \frac{N_p}{p}.$$

for various elliptic curves $E$, on the idea (hope?) that this would grow more rapidly when $r = r_E$ is positive. Here are the results for some curves of the form $E_d : y^2 = x^3 - d^2 x$:

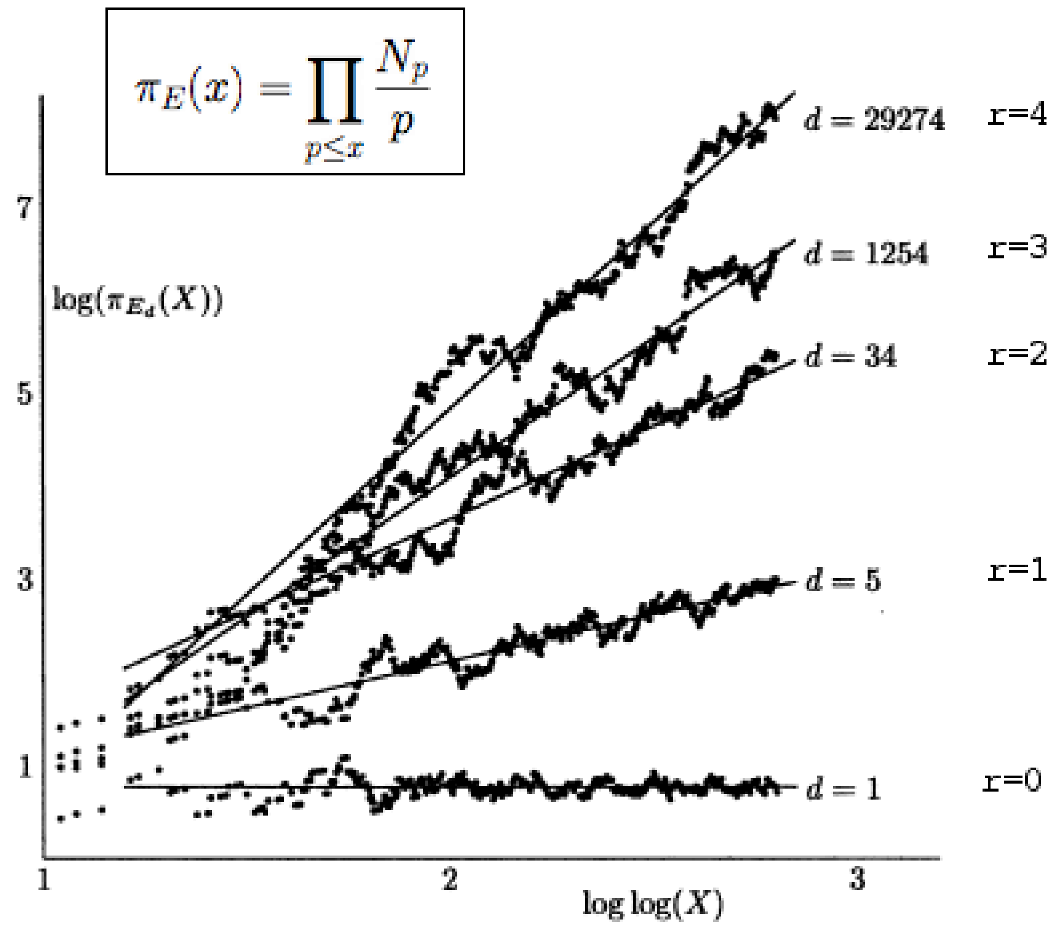



FIGURE 2. Birch and Swinnerton-Dyer data for $y^2 = x^3 - d^2x$

(Source: Rubin, Silverberg: *Ranks of elliptic curves*, BAMS 39 4 2002)

This leads to the conjecture that $\log \pi_E(x)$ grows like $r_E \log\log x$:

Birch Swinnerton-Dyer Conjecture (First form):

For any elliptic curve defined over $\mathbb{Q}$,

$$\pi_E(x) \sim C_E(\log x)^{r_E},$$

for some constant $C_E$, with $r_E$ the rank of $E(\mathbb{Q})$.

Another digression (?): zeta and $L$-functions:

The Riemann zeta function $\zeta(s)$ has a number of striking properties — an expression as a product ("Euler product") over the primes, analytic continuation to $\mathbb{C}$ (except for a simple pole at $s = 1$), a functional equation relating $\zeta(s)$ and $\zeta(1-s)$.

The Euler product has the form

$$\begin{aligned}\zeta(s) &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \prod_{\text{maximal ideals } P \text{ of } \mathbb{Z}} \left(1 - \frac{1}{[\mathbb{Z} : P]^s}\right)^{-1}\end{aligned}$$

If $k$ is an number field, then we can define analogously

$$\zeta_k(s) = \prod_{\text{maximal ideals } \mathfrak{p} \text{ of } \mathfrak{o}} \left(1 - \frac{1}{[\mathfrak{o} : \mathfrak{p}]^s}\right)^{-1}$$

—which has the same striking properties.

Quite generally, if $A$ is any ring of finite type over $\mathbb{Z}$ (i.e. $A = \bar{\mathbb{Z}}[a_1, \ldots, a_n]$ for some $a_i \in A$ and with $\bar{\mathbb{Z}}$= the image of $\mathbb{Z}$ in $A$), then $A/P$ is a finite field for any maximal ideal of $A$, so that we could define

$$\zeta(A, s) = \prod_{\text{maximal ideals } P \text{ of } A} \left(1 - \frac{1}{[A : P]^s}\right)^{-1}$$

and ask about its properties.

E.g. take $A = \mathbb{F}_p[x]$:

$$\begin{aligned}
\zeta(\mathbb{F}_p[x], s) &= \prod_{\text{monic irreducibles } \pi(x)} \left(1 - \frac{1}{p^{(\deg \pi)s}}\right)^{-1} \\
&= \sum_{\text{monic polynomials } m(x)} \frac{1}{p^{(\deg m)s}} \\
&= \sum_{n=0}^{\infty} p^n \frac{1}{p^{ns}} \\
&= \frac{1}{1 - \frac{1}{p^{s-1}}}.
\end{aligned}$$

and therefore (now taking $A = \mathbb{Z}[x]$)

$$\zeta(\mathbb{Z}[x], s) = (!) \prod_p \zeta(\mathbb{F}_p[x], s) = \zeta(s - 1)$$

If we take $A = \mathbb{F}_p[x,y]/(y^2 - x^3 - ax - b)$, we get a "zeta function" attached to the elliptic curve $E$ mod $p$. In the 1930s, Hasse showed that

$$\zeta(E/\mathbb{F}_p, s) = \frac{1 - a_p x + p x^2}{(1-x)(1-px)}$$

where $x = p^{-s}$, and $a_p = p + 1 - N_p$. (This is not exactly $\zeta_A(s)$, but takes into account the point at $\infty$.)

(Note that the zeroes of $\zeta(E/\mathbb{F}_p, s)$ occur where $p^{-s}$ is a root of $1 - a_p x + p x^2$. If you use Hasse's estimate $|a_p| \le 2\sqrt{p}$, you find that the zeroes of $\zeta(E/\mathbb{F}_p, s)$ occur on the line $\Re(s) = 1/2$.)

Hasse suggested multiplying these $\zeta(E/\mathbb{F}_p, s)$ together to get

$$\begin{aligned}\zeta(E/\mathbb{Q}, s) &= \prod_p^{*} \zeta(E/\mathbb{F}_p, s) \\ &= \zeta(s)\zeta(s-1) \prod_p^{*} 1 - a_p p^{-s} + p^{1-2s}\end{aligned}$$

(the "$*$" means that things need to be adjusted at the finite number of primes $p$ where $p = 2$ or $p \mid 4a^3 + 27b^2$)

(Note that this function is essentially $\zeta(A, s)$, where now $A = \mathbb{Z}[x, y]/(y^2 - x^3 - ax - b)$.)

The function

$$L(E/\mathbb{Q}, s) = \prod_p^* (1 - a_p p^{-s} + p^{1-2s})^{-1}$$

is called the Hasse-Weil $L$-function of $E$. It only converges for $\Re(s) > 3/2$, but if we formally set $s = 1$ we find

$$L(E/\mathbb{Q}, 1) = \prod_p^* \frac{p}{N_p},$$

since $N_p = p + 1 - a_p$. This suggests that $L(E/\mathbb{Q}, 1)$ should vanish if $r_E > 0$ and perhaps should vanish to order $r_E$. This is the second form of the Birch Swinnerton-Dyer Conjecture:

Birch Swinnerton-Dyer Conjecture (Second Form):

For any elliptic curve defined over $\mathbb{Q}$,

$$\mathrm{ord}_{s=1} L(E/\mathbb{Q}, s) = r_E$$

with $r_E$ the rank of $E(\mathbb{Q})$.

Note that this presumes that $L(E/\mathbb{Q}, s)$ can be analytically continued at least to $s = 1$; it is now known that $L(E/\mathbb{Q}, s)$ can be analytically continued to the entire complex plane, for all elliptic curves defined over $\mathbb{Q}$, by work of Wiles, Taylor, Breuil, Conrad, and Diamond.

Here's a heuristic argument that relates the two forms, and "explains" the growth rate $(\log x)^r$: the usual zeta function has a simple pole at $s = 1$; and standard arguments allow one to deduce from this that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} \approx \log x$$

and therefore

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{r} \approx \frac{1}{(\log x)^r},$$

which arises from $1/\zeta(s)^r$, which has a zero of order $r$ at $s = 1$. By analogy one might expect

$$\prod_{p \le x}^{*} \frac{p}{N_p} \approx \frac{1}{(\log x)^r},$$

if $L(E, s)$ has a zero of order $r$ at $s = 1$.

(the "$\approx$" above means the ratio tends to a nonzero constant.)

What's known?

- If $\text{ord}_{s=1} L(E/\mathbb{Q}, s) = 0$ or 1, then the second form of the conjecture is valid. (Gross, Zagier, and Kolyvagin)

  So, for example, if $L(E/\mathbb{Q}, 1) \neq 0$, then the only rational solutions to the equation $y^2 = x^3 + ax + b$ correspond to torsion points and can therefore be determined by the Lutz/Nagell theorem.

  And if $L(E/\mathbb{Q}, 1) = 0$ but $L'(E/\mathbb{Q}, 1) \neq 0$, then there is a rational solution $P = (x_0, y_0)$ to $y^2 = x^3 + ax + b$ such that every solution is a multiple of $P$ plus a torsion point ("multiple" and "plus" in the sense of the group law on $E$).
- The first form implies the second. (Dorian Golfeld)