Fundamental Theorem of Algebra
Lecture notes from the Reading Classics (Euler) Working Group, Autumn 2003

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Abstract

These notes are based on a lecture given by Dan File Lecturer on Wednesday, November 12, 2003, and were LaTeX-ed in real time by Steven J. Miller and additions were made by Dan File.

1 History of the Fundamental Theorem of Algebra

D'Alembert (1746) observed that if $p(x)$ is a polynomial with real coefficients and z is a solution, then $\overline{z}$ is also a solution. His intent was to integrate rational functions by means of partial fractions. His observation permitted him to separate any real polynomial into linear and quadratic terms and hence find antiderivatives.

Euler became interested in this problem: Euler worked on the quartic and quintic. For the quartic, Euler showed that there was an x-intercept. He was relying on the fact that if you have roots β_i ($i \in \{1, 2, 3, 4\}$), then $-(\beta_1 \cdots \beta_4)^2$ is negative. This is fine if the βs are real or in complex conjugate pairs, but had some trouble with the quntic.

Nicolas Bernouli claimed that $x^4 - 4x^3 + 2x^2 + 4x + 4$ is irreducible over $\mathbb{R}$, but Euler factored it.

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$$x^4 - 4x^3 + 2x^2 + 4x + 4 \;=\; (x^2 - \sqrt{2 \pm \sqrt{4 + 2\sqrt{7}}})x + (1 \pm \sqrt{4 + 2\sqrt{7}} + \sqrt{7})), \tag{1}$$

where above the two factors come from taking the $+$ sign each time, or the $-$ sign each time. Note factoring a quartic into two real quadratics is different than trying to find four complex roots.

Definition: A function f is *analytic* on an open subset $R \subset \mathbb{C}$ if f is complex differentiable everywhere on R; f is *entire* if it is analytic on all of $\mathbb{C}$.

2 Proof of the Fundamental Theorem via Liouville

Theorem 2.1 (Liouville). *If $f(z)$ is analytic and bounded in the complex plane, then $f(z)$ is constant.*

We now prove

Theorem 2.2 (Fundamental Theorem of Algebra). *Let $p(z)$ be a polynomial with complex coefficients of degree n. Then $p(z)$ has n roots.*

Proof. It is sufficient to show any $p(z)$ has one root, for by division we can then write $p(z) = (z - z_0)g(z)$, with g of lower degree.

Note that if

$$p(z) \;=\; a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0, \tag{2}$$

then as $|z| \to \infty$, $|p(z)| \to \infty$. This follows as

$$p(z) \;=\; z^n \cdot \left| a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n} \right|. \tag{3}$$

Assume $p(z)$ is non-zero everywhere. Then $\frac{1}{p(z)}$ is bounded when $|z| \geq R$. Also, $p(z) \neq 0$, so $\frac{1}{p(z)}$ is bounded for $|z| \leq R$ by continuity. Thus, $\frac{1}{p(z)}$ is a bounded, entire function, which must be constant. Thus, $p(z)$ is constant, a contradiction which implies $p(z)$ must have a zero (our assumption). □

[Lev]

3 Proof of the Fundamental Theorem via Rouche

Theorem 3.1 (Rouche). *If f and h are each analytic functions inside and on a domain C with bounding curve ∂C, and $|h(z)| < |f(z)|$ on ∂C, then f and $f + h$ have the same number of zeros in C.*

We now prove the Fundamental Theorem of Algebra:

Proof. Let

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \\ f(z) &= a_n z^n \\ h(z) &= p(z) - f(z) = a_{n-1} z^{n-1} + \cdots + a_0. \end{aligned} \tag{4}$$

On the circle $|z| = R$, $|f(z)| = |a_n| R^n$, and

$$|h(z)| \leq |a_{n-1}| R^{n-1} + ... + |a_1| R + |a_0|.$$

Make R large enough so that

$$\frac{|a_{n-1}| + ... + |a_1| + |a_0|}{|a_n|} < R.$$

Then $|h(z)| < |f(z)|$ holds on the boundary of the circle centered at the origin of radius R. Since f clearly has n zeros, we are done. □

[Saff]

4 Proof of the Fundamental Theorem via Picard's Theorem

Theorem 4.1. *If there are two distinct points that are not in the image of an entire function $p(z)$ (ie, $\exists z_1 \neq z_2$ such that for all $z \in \mathbb{C}$, $p(z) \neq z_1$ or z_2), then $p(z)$ is constant.*

We now prove the Fundamental Theorem of Algebra:
Let $p(z)$ be a non-constant polynomial, and assume $p(z)$ is never 0.

Claim 4.2. *If $p(z)$ is as above, $p(z)$ does not take on one of the values $\frac{1}{k}$ for $k \in \mathbb{N}$.*

Proof. Assume not. Thus, $\exists z_k \in \mathbb{C}$ such that $p(z_k) = \frac{1}{k}$. If we take a circle D centered at the origin with sufficiently large radius, then $|p(z)| > 1$ for all z outside D. Thus, each $z_i \in D$. By Bolzano-Weierstrasss, as all the points $z_k \in D$, we have a convergent subsequence. Thus, we have $z_{n_i} \to z'$. But

$$p(z') = \lim_{n_i \to \infty} p(z_{n_i}) = 0. \tag{5}$$

Thus, there must be some k such that $p(z) \neq \frac{1}{k}$. Since there are two distinct values not in the image of p, by Picard's Theorem it is now constant. This contradicts our assumption that $p(z)$ is non-constant. Therefore, $p(z_0) = 0$ for some z_0. □

Remark 4.3. *One can use a finite or countable version of Picard. Rather than missing just two points, we can modify the above to work if Picard instead stated that if we miss finitely many (or even countably many) points, we are constant. Just look at the method above, gives $\frac{1}{k_1}$. We can then find another larger one, say $\frac{1}{k_2}$. And so on. We can even get uncountably many such points by looking at numbers such as $\frac{\pi}{k}$ (using now the transcendence of $\mathbb{C}$ is 1).*

[Boas, 1935]

5 Proof of the Fundamental Theorem via Cauchy's Integral Theorem

Theorem 5.1 (Cauchy Integral Theorem). *Let $f(z)$ be analytic inside on on the boundary of some region C. Then*

$$\int_{\partial C} f(z)dz = 0. \tag{6}$$

We now prove the Fundamental Theorem of Algebra:

Proof. Without loss of generality let $p(z)$ be a non-constant polynomial and assume $p(z) = 0$. For $z \in \mathbb{R}$, assume $p(z) \in \mathbb{R}$. (Otherwise, consider $p(z)\overline{p}(z)$).

Therefore, $p(z)$ doesn't change signs for $z \in \mathbb{R}$, or by the Intermediate Value Theorem it would have a zero.

$$\int_0^{2\pi} \frac{d\theta}{p(2\cos\theta)} \neq 0. \tag{7}$$

This follows from our assumption that $p(z)$ is of constant sign for real arguments, bounded above from 0. This integral equals the contour integral

$$\frac{1}{i}\int_{|z|=1}\frac{dz}{zp(z+z^{-1})} \;=\; \frac{1}{i}\int_{|z|=1}\frac{z^{n-1}}{Q(z)}, \tag{8}$$

where

$$Q(z) \;=\; z^n p(z+z^{-1}). \tag{9}$$

If $z \neq 0$, $Q(z) \neq 0$.
If $z = 0$, then $Q(z) \neq 0$ since

$$\begin{aligned} p(z+z^{-1}) &= a_n(z+z^{-1})^n + \cdots \\ z^n p(z+z^{-1}) &= z^n\left(\cdots a_n z^{-n}\right) + \cdots \\ &= a_n + z(\cdots). \end{aligned} \tag{10}$$

Thus, $Q(z) = a_n$, which is non-zero. Hence, $Q(z) \neq 0$, and consequently $\frac{z^{n-1}}{Qz}$ is analytic. By the Cauchy Integral Formula $\frac{1}{i}\int_{|z|=1}\frac{z^{n-1}}{Q(z)} \neq 0$. Thus, a contradiction!

□

[Boas 1964]

6 Proof of the Fundamental Theorem via Maximum Modulus Principle

Theorem 6.1 (Maximum (Minimum) Modulus Principle). *No entire function attains its maximum in the interior.*

We now prove the Fundamental Theorem of Algebra:

Proof. Assume $p(z)$ is non-constant and never zero. $\exists M$ such that $|p(z)| \geq |a_0| \neq 0$ if $|z| > M$. Since $|p(z)|$ is continuous, it achieves its minimum on a closed interval. Let z_0 be the value in the circle of radius M where $p(z)$ takes its minimum value.

So $|p(z_0)| \leq |p(z)|$ for all $z \in \mathbb{C}$, and in particular $|p(z_0)| \leq |p(0)| = |a_0|$.

Translate the polynomial. Let $p(z) = p((z - z_0) + z_0)$; let $p(z) = Q(z - z_0)$. Note the minimum of Q occurs at $z = 0$: $|Q(0)| \leq |Q(z)|$ for all $z \in \mathbb{C}$.

$$Q(z) \;=\; c_0 + c_j z^j + \cdots + c_n z^n, \tag{11}$$

where j is such that c_j is the first coefficient (after c_0) that is non-zero. I must show $Q(0) = 0$. Note if $c_0 = 0$, we are done.

We may rewrite such that

$$Q(z) \;=\; c_0 + c_j z^j + z^{j+1} R(z). \tag{12}$$

We will extract roots. Let

$$re^{i\theta} \;=\; -\frac{c_0}{c_j}. \tag{13}$$

Further, let

$$z_1 \;=\; r^{\frac{1}{j}} e^{\frac{i\theta}{j}}. \tag{14}$$

So,

$$c_j z^j = -c_0. \tag{15}$$

Let $\epsilon > 0$ be a small real number. Then

$$\begin{aligned} Q(\epsilon z_1) &= c_0 + c_j \epsilon^j z_1^j + \epsilon^{j+1} z_1^{j+1} R(\epsilon z_1) \\ |Q(\epsilon z_1)| &\leq |c_0 + c_j \epsilon^j z_j^j| + \epsilon^{j+1} |z_1|^{j+1} |R(\epsilon z_1)| \\ &\quad |c_0| - \epsilon^j |c_0| + \epsilon^{j+1} |z_1|^{j+1} N, \end{aligned} \tag{16}$$

where N is chosen such that $N > |R(\epsilon z_1)|$, and ϵ is chosen so that

$$\epsilon^{j+1} |z_1|^{j+1} < \epsilon^j |c_0|.$$

Thus,

$$|Q(\epsilon z_1)| \;<\; |c_0|, \tag{17}$$

but this was supposed to be our minimum. Thus, a contradiction! □

[Fefferman]

7 Proof of the Fundamental Theorem via Radius of Convergence

We now prove the Fundamental Theorem of Algebra: As always, $p(z)$ is a non-constant polynomial. Consider

$$f(z) \;=\; \frac{1}{p(z)} \;=\; b_0 + b_1 z + \cdots\,, \tag{18}$$

and

$$p(z) \;=\; a_n z^n + \cdots + a_0, \quad a_0 \neq 0. \tag{19}$$

Lemma 7.1. *$\exists c, r \in \mathbb{C}$ such that $|b_k| > cr^k$ for infinitely many k.*

Now, $1 = p(z)f(z)$. Thus, $a_0 b_0 = 1$. This is our basis step. Assume we have some coefficient such that $|b_k| > cr^k$. We claim we can always find another. Suppose there are no more. Then the coefficient of z^{k+n} in $p(z)f(z)$ is

$$a_0 b_{k+n} + a_1 b_{k+n-1} + \cdots + a_n b_k \;=\; 0. \tag{20}$$

Thus, as we have $|b_j| > cr^j$ in this range, we have the coefficient satisfies

$$|a_0| r^n + |a_1| r^{n-1} + \cdots + |a_{n-1}| r \;\leq\; |a_n| \tag{21}$$

if

$$r \;\leq\; \min\{1, \frac{|a_n|}{|a_0| + \cdots + |a_{n-1}|}. \tag{22}$$

This will give that

$$\begin{aligned} |b_k| \quad &= \quad \frac{|a_0 b_{k+n} + \cdots + a_{n-1} b_{k+1}|}{|a_n|} \\ &\leq \quad \frac{|a_0 b_{k+n}| + \cdots + |a_{n-1} b_{k+1}|}{|a_n|} \;\leq\; cr^k \end{aligned} \tag{23}$$

for sufficiently small.
Let $z = \frac{1}{r}$. Then

$$\left| b_k z^k \right| \;=\; \frac{|b_k|}{r^k} \;>\; c. \tag{24}$$

This is true for infinitely many k, hence the power series diverges, contradicting the assumption that the function is analytic and its power series converges everywhere.

[Velleman]

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