A CONTRIBUTION TO THE STUDIES OF CONVERGENCE OF FOURIER SERIES

As usual, let

$$S_n = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx),$$

$$\sigma_n = \frac{S_0 + S_1 + \cdots + S_{n-1}}{n}.$$ 

I. Theorem. If a sequence of integers $n_m \ (m = 1, 2, \cdots)$ satisfies

$$n_{m+1}/n_m > \lambda > 1,$$

then for the Fourier series of any function the sequence $S_{nm}$ converges to a given function almost everywhere.

Proof. It is known that the sequence $\{\sigma_{n_m}\}$ converges almost everywhere to a given function, therefore, it is enough to prove that the sequence $\{S_{n_m} - \sigma_{n_m}\}$ converges almost everywhere to zero. However, it is easy to see that this is a resul of convergence of the following series:

$$\sum_{m=1}^{\infty} \int_0^{2\pi} (S_{n_m} - \sigma_{n_m})^2 \, dx. \quad (1)$$

Consider the $p^{th}$ partial sum of the series (1). We have

$$\frac{1}{\pi} \sum_{m=1}^{p} \int_0^{2\pi} (S_{n_m} - \sigma_{n_m})^2 \, dx = \sum_{m=1}^{p} \frac{1}{n_m^2} \sum_{k=1}^{k=n_m} k^2 (a_k^2 + b_k^2) =$$

$$= \sum_{k=1}^{n_p} k^2 (a_k^2 + b_k^2) \left[ \frac{1}{n_{m_k}^2} + \frac{1}{n_{m_k+1}^2} + \cdots + \frac{1}{n_{p}^2} \right], \quad (2)$$

where $n_{m_k}$ is defined by the inequality

$$n_{m_k-1} < k \leq n_{m_k}.$$ 


\(^2\) Translation from French to Russian by I.A. Vinogradova.
Clearly,
\[
\frac{1}{n_{m_k}^2} + \frac{1}{n_{m_{k+1}}^2} + \cdots + \frac{1}{n_p^2} \leq \frac{1}{n_{m_k}^2} \left( 1 + \frac{1}{\lambda^2} + \cdots + \frac{1}{\lambda^{2p}} \right) < \\
\frac{\lambda^2}{\lambda^2 - 1} \sum_{k=1}^{k=n_p} (a_k^2 + b_k^2).
\]

and, therefore, the sum (2) is no greater then

\[
\frac{\lambda^2}{\lambda^2 - 1} \sum_{k=1}^{k=n_p} (a_k^2 + b_k^2).
\]

This leads to convergence of the series (1) and the proof is complete.

II. Theorem. If the only non-zero terms of a Fourier–Lebesgue have indices \( n_m \) (the sequence \( n_m \) satisfies the condition of theorem I), then the series converges almost everywhere.

Proof. If we only consider functions integrable with a square, then theorem II follows immediately from theorem I; however the statement is true for all integrable functions. The sequence \( \sigma_{n_m} \) converges almost everywhere, therefore we only need to consider

\[
|S_{n_m - 1} - \sigma_{n_m - 1}| \leq \sum_{k=1}^{m-1} \frac{n_k}{n_m} (|a_{n_k}| + |b_{n_k}|).
\]  

(3)

Since \(|a_{n_k}| + |b_{n_k}|\) approaches 0 as \( k \to \infty \) and, on the other hand,

\[
\sum_{k=1}^{m-1} \frac{n_k}{n_m} < \sum_{k=1}^{m-1} < \frac{1}{\lambda - 1}
\]

it can be seen that the difference (3) approaches 0 as \( m \to \infty \).

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