On Archimedian Spirals

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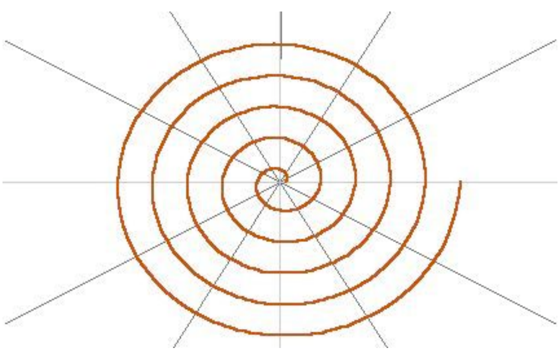
If a straight line drawn in a plane revolve at a uniform rate about one extremity which remains fixed and return to the position from which it started, and if, at the same time as the line revolves, a point move at a uniform rate along the straight line beginning from the extremity which remains fixed, the point will describe a spiral in the plane. Let the length which the point that moves along the straight line describes the first distance, and similarly for the nth revolution. Let the circle drawn with the origin at the center and radius of the first distance be called the first circle and so on.



Archemedes found that the area bounded by the spiral in one revolution is one third the area of the first circle. He was also able to use this to trisect angles and for circle squaring.

The proof of this requires us to prove some formula to get nice expressions for the sum of approximating sectors.

In proposition 10 On spirals Archemedes proves that

$$(n+1)A_n^2+A_1(A_1+A_2+A_3+\ldots+A_n)=3(A_1^2+A_2^2+A_3^2+\ldots+A_n^2).$$

In proposition 11 he proves that

$$(n+1)A_n^2 : (A_2^2+A_3^2+\ldots+A_n^2) < A_n^2 : (A_n * A_1+1/3(A_n-A_1)^2)$$

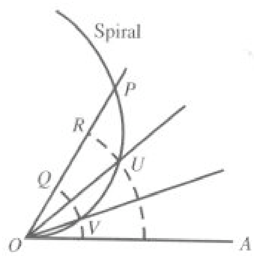
but

$$(n-1)A_n^2 : (A_1^2+A_2^2+A_3^2+\ldots+A_{n-1}^2) < A_n^2 : (A_n * A_1+1/3(A_n-A_1)^2)$$

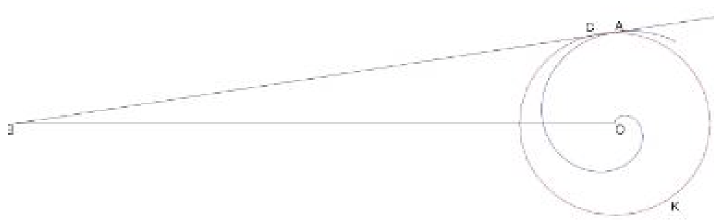
Proposition 24 The area bounded by the first turn of the spiral and the initial line is equal to one-third the area of the first circle.

Proposition 26 If BC be any arc measured in the forward direction on any turn of the spiral, not being greater than the complete turn, and if a circle be drawn with O as its center and OC as its radius meeting OB at B' then

$$(\text{area of spiral between } OB, OC) : (\text{sector } OB'C) = (OC * OB + 1/3(OC - OB)^2) : OC^2$$



To trisect an angle one needs to orient the angle so that one edge lies in the initial position of the spiral. We can find the point P where the angle intersects the spiral. Now that we are given this length OP we can take a third of it OQ and two thirds of it OR and form circles with radius OQ and Or centered at O. These circles will intersect the spiral so that the angles form an arithmetic progression thus angle AOV is the trisection of POA. This can be generalized to divide any angle into n equal angles.



If we take the tangent at the end of the first revolution and then intersect it with the perpendicular of the first line at the origin then the distance from the origin to the intersection is the circumference of the first circle.