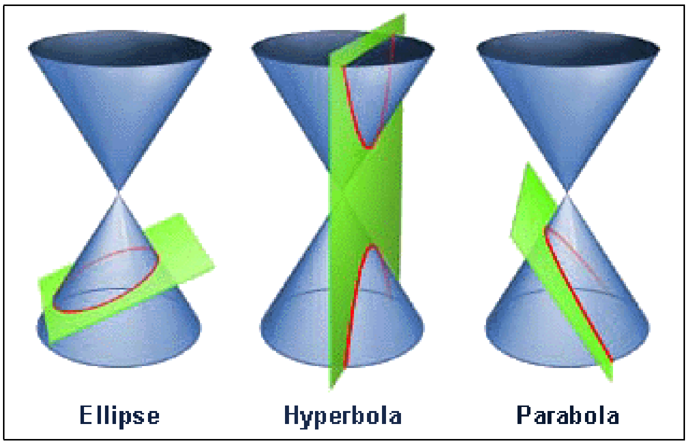
Optical properties of conic sections

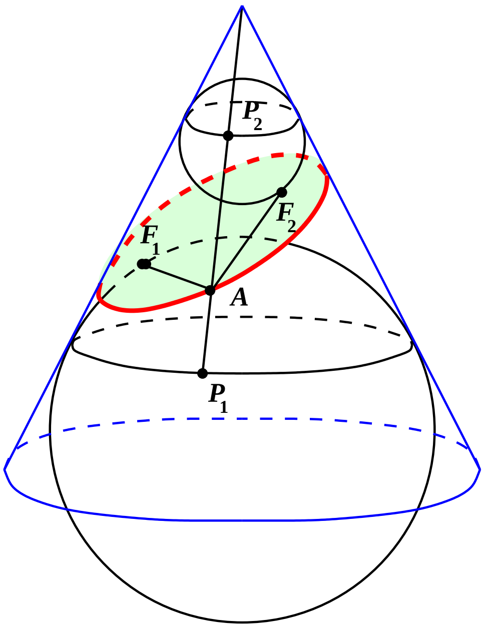
Michael Chmutov



Archimedes of Syracuse

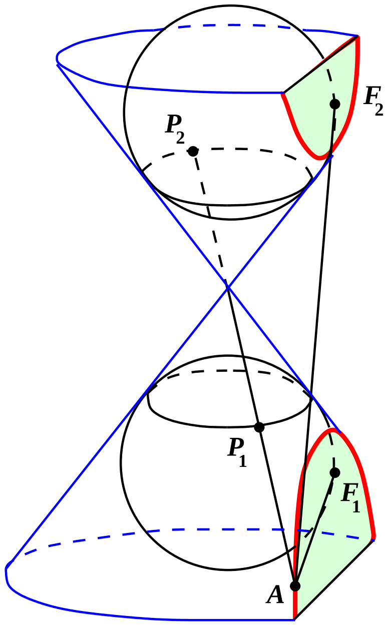
The pictures are borrowed from the web sites: `http://ccins.camosun.bc.ca/~jbritton/jbconics.htm` and `http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/Archimedes.html`

FOCI



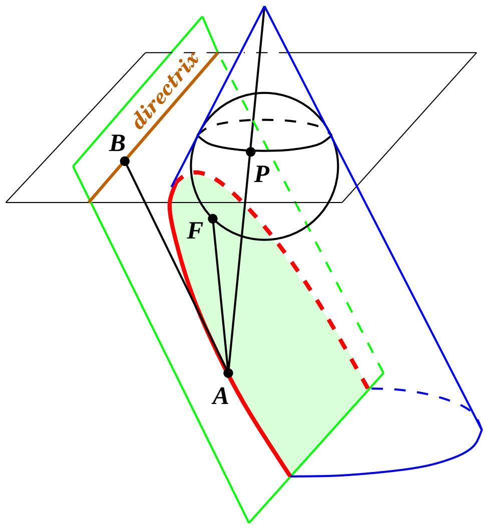
$$|AF_1| = |AP_1|,\ \ |AF_2| = |AP_2|,$$

$$|AF_1| + |AF_2| = |AP_1| + |AP_2| = |P_1P_2| = const$$

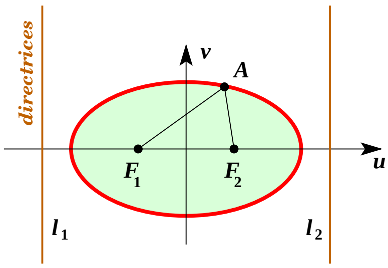


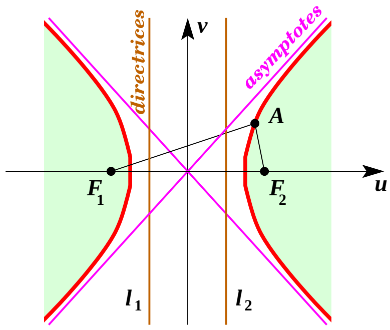
$$|AF_1| = |AP_1|,\ \ |AF_2| = |AP_2|,$$

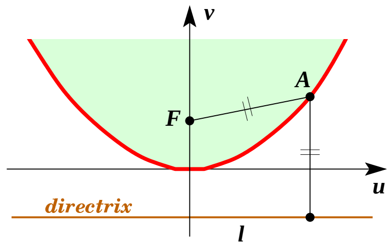
$$|AF_2| - |AF_1| = |AP_2| - |AP_1| = |P_1P_2| = const$$



$$|AF| = |AP| = |AB|$$

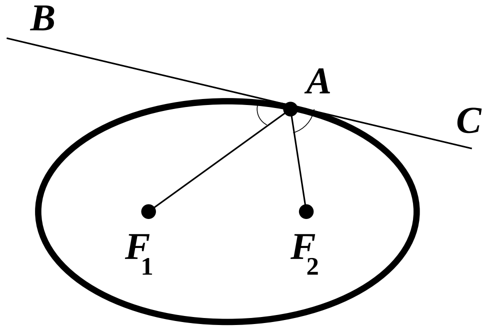




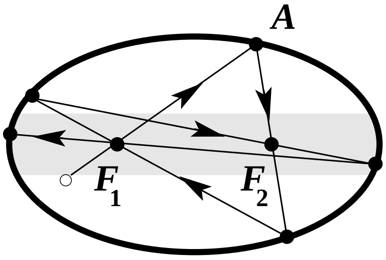


	Ellipse	Hyperbola	Parabola
Definitions	**1.** The set of points A with a constant $(2a)$ sum of distances to two given points F_1 and F_2 (*foci*). $\lvert AF_1\rvert + \lvert AF_2\rvert = 2a$	**1.** The set of points A with a constant $(2a)$ difference of distances to two given points F_1 and F_2 (*foci*). $\Big\lvert \lvert AF_1\rvert - \lvert AF_2\rvert \Big\rvert = 2a$	The set of points A on the same distance from a given point F (*focus*) and a given line l (*directrix*). $\lvert AF\rvert = \lvert Al\rvert$
	2. The set of points A such that the ratio of distances to the given point F_1 (*focus*) and to the given line l_1 (*directrix*) is a constant e (*eccentricity*). $\dfrac{\lvert AF_1\rvert}{\lvert Al_1\rvert} = \dfrac{\lvert AF_2\rvert}{\lvert Al_2\rvert} = e$		
	$0 < e < 1$	$e > 1$	$e = 1$
Equation	$\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} = 1$ $\boxed{0 < b \leq a}$	$\dfrac{u^2}{a^2} - \dfrac{v^2}{b^2} = 1$	$u^2 = 4cv$
Eccentricity	$e = \dfrac{c}{a}$, where		$e = 1$
	$c = \sqrt{a^2 - b^2}$	$c = \sqrt{a^2 + b^2}$	
Foci	$F_1 = (-c, 0); \quad F_2 = (c, 0)$		$F = (0, c)$
Directrices	$x = -\dfrac{a}{e}; \quad x = \dfrac{a}{e}$		$y = -c$
Asymptotes	No	$v = -\dfrac{b}{a}\,u; \quad v = \dfrac{b}{a}\,u$	No
Vertices	$(-a, 0); \quad (a, 0)$		$(0, 0)$

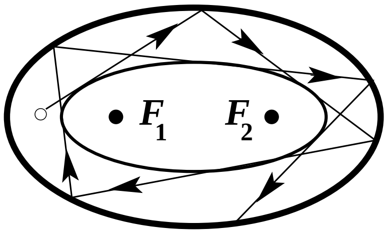
OPTICAL PROPERTIES



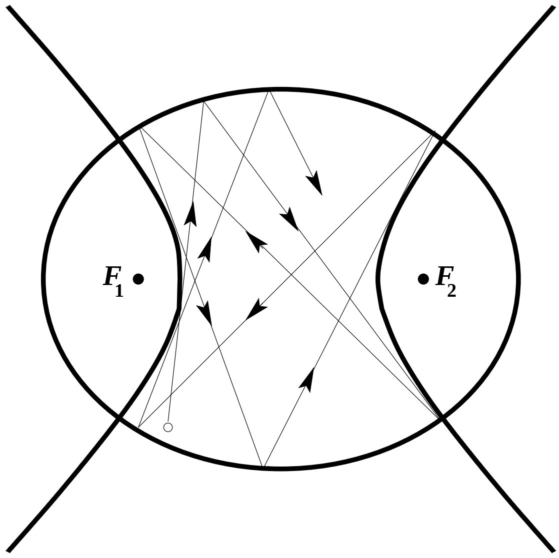
$$\angle F_1AB = \angle F_2AC$$



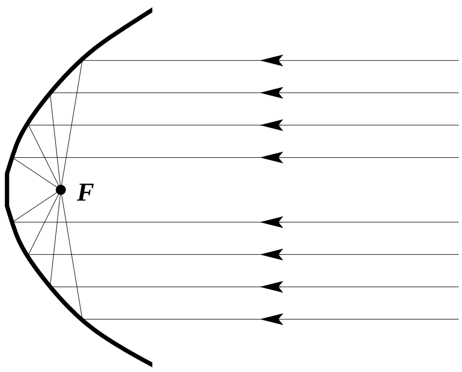
The trajectory of a light ray passing through the foci asymptotically approaches the major axis



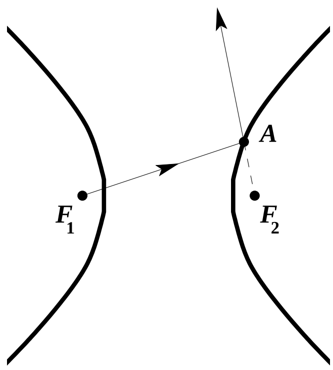
The trajectory of a light ray passing outside the segment between the foci is tangent to an ellipse confocal with the original one



The trajectory of a light ray intersecting the segment between the foci is tangent to a hyperbola confocal with the original ellipse



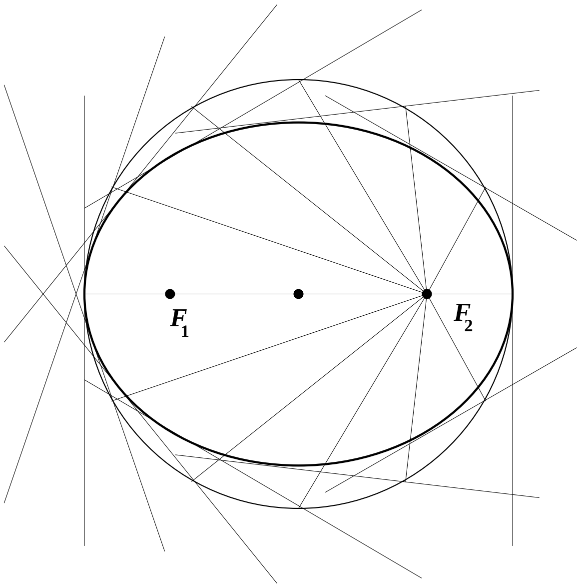
Trajectories of all light rays parallel to the axis reflect off the parabola and pass through the focus

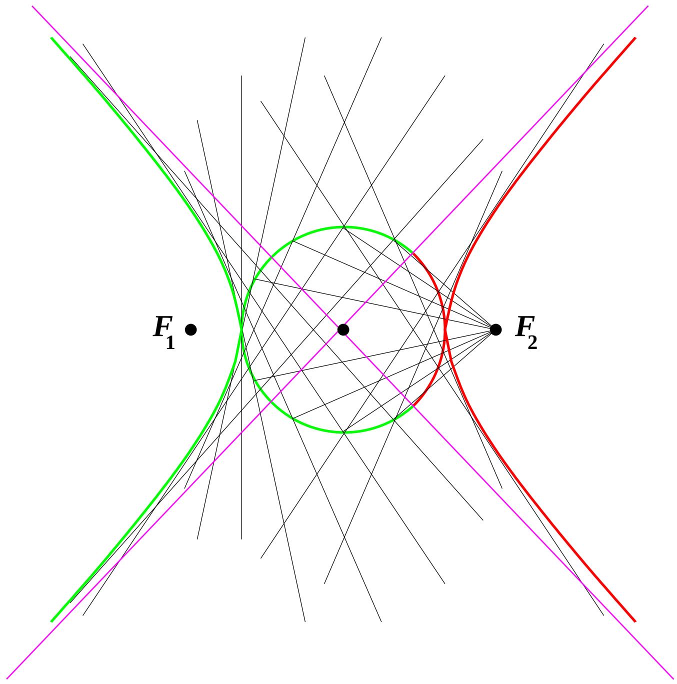


The trajectory of a light ray coming out of one focus will reflect of the opposite branch of the hyperbola as if it were coming from the other focus

PEDAL CURVES

Definition. Given a curve Γ and a point p, the *pedal curve* Γ_p of Γ with respect to p is a curve formed by the bases of perpendiculars from p onto all tangent lines of Γ.





Proposition. *The pedal curve of an ellipse or a hyperbola with respect to one of its foci is a circle.*

Homework. Find the pedal curve of a parabola with respect to its focus.