COMPLEX VARIABLES AND EULER

Abstract. Euler contributed significantly in the development of the theory of complex functions. The now well known as Euler's identity, the identification of the roots of unity, the standardization of the symbol i are just few examples in this direction. Euler defined the logarithm and trigonometric functions of complex numbers, and via that he solved such equations as $\sin z = 2$.

1. History of pre-Euler era

The existence of imaginary numbers arose from solving cubic equations.

- (1515) Scipione del Ferro of Bologna: a solution of the "depressed cubic" $x^3 = mx + n$ is given by

$$x = \sqrt[3]{\frac{n}{2} + \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}} + \sqrt[3]{\frac{n}{2} - \sqrt{\frac{n^2}{4} - \frac{m^3}{27}}}$$

- (1545) Girolamo Cardano in *Ars Magna*: $x^3 + ax^2 + bx + c = 0$ can be reduced to a reduced cubic by $x = z - a/3$ ("Cardano's formula")

  Remark the graph of $x^3 = 6x + 4$ shows there exist three real solutions, the above formula gives the solution $x = \sqrt[3]{2 + 2\sqrt{-1}} + \sqrt[3]{2 - 2\sqrt{-1}}$! Two possibilities: either Cardano's formula is wrong or the "imaginary" somehow cancels out.
- (1570) Rafael Bombelli in *Algebra*: imaginary numbers are a temporary annoyance. Notice that $2 + 2\sqrt{-1} = (-1 + \sqrt{-1})$ and $2 - 2\sqrt{-1} = (-1 - \sqrt{-1})$ which implies that a solution is $x = -2$.

  Remark What about the other solutions? Bombelli had no answer, and one and a half century passed with great mathematicians treating these imaginary quantities with no rigor. For example, Leibniz called $\sqrt{-1}$ *the amphibian between being and non-being.* Descartes in *Geometrie* (1637) addressed the question of square roots of negative numbers by saying "neither the true nor the false roots are always real; sometimes they are imaginary".

2. Euler's input

- (1749) on the controversy between G. W. Leibniz and Johann Bernoulli: both are mistaken!
- (1749) on complex numbers:
  - definition of logarithm of complex numbers, existence of infinite branches
  - definition of trigonometric functions of complex numbers
  - transcendental quantities can be reduced to complex numbers

3. Post-Euler era

- Dirichlet, Riemann
- Gauss, Cauchy, Weierstrass

4. References

(1) Leonhardi Euleri, Opera Omnia, Ser.1, Vol 6
Recherches sur les racines imaginaires des équations, p.78-147
(2) Leonhardi Euleri, Opera Omnia, Ser.1, Vol 19
Sur les logarithmes des nombres négatifs et imaginaires, p.417-438
(3) L. Euler, Introduction to analysis of the Infinite
(4) W. Dunham, Euler: The Master of Us All
(5) A. Shenitzer and J. Stillwell, Mathematical evolutions