Reflections on Mathematics

Math 1118: Spring 2013

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Preface

These notes are designed with architecture and design students in mind. It is our hope that the reader will find these notes both interesting and challenging. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of geometry and algebra. We hope to show the reader that geometry and algebra are deeply connected.
- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.
- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered, other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

?

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then I encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at: `snapp@math.osu.edu`

Thanks and Acknowledgments

A brief history of this document: In 2009, Greg Williams, a Master of Arts in Teaching student at Coastal Carolina University, worked with Bart Snapp to produce an early draft of the chapter on isometries. In the summer and fall of 2011, Bart Snapp wrote the remaining chapters of this set of notes.

Much thanks goes to Herb Clemens, Vic Ferdinand, Betsy McNeal, and Daniel Shapiro who introduced me to ideas which have been used in this course.

Contents

Chapter 1

Isometries

> And since you know you cannot see yourself, so well as by reflection, I, your glass, will modestly discover to yourself, that of yourself which you yet know not of.
>
> —William Shakespeare

1.1 Matrices as Functions

We're going to discuss some basic functions in geometry. Specifically, we will talk about translations, reflections, and rotations. To start us off, we need a little background on matrices.

Question What is a *matrix*?

You might think of a matrix as just a jumble of large brackets and numbers. However, we are going to think of matrices as *functions*. Just as we write $f(x)$ for a function f acting on a number x, we'll write:

$$\mathsf{M}\mathbf{p} = \mathbf{q}$$

to represent a matrix M mapping point $\mathbf{p}$ to point $\mathbf{q}$. A point $\mathbf{p}$ is often represented as an ordered pair of coordinates, $\mathbf{p} = (x, y)$. However, to make things work out nicely, we need to write our points all straight and narrow, with a little buddy at the end:

$$(x, y) \rightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Throughout this chapter, we will abuse notation slightly, freely interchanging several notations for a point:

$$\mathbf{p} \leftrightsquigarrow (x, y) \leftrightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

With this in mind, our work will be done via matrices and points that look like this:

$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Now recall the nitty gritty details of *matrix multiplication*:

$$\begin{aligned} \mathsf{M}\mathbf{p} &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \cdot 1 \\ dx + ey + f \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix} \end{aligned}$$

Question Fine, but what does this have to do with geometry?

In this chapter we are going to study a special type of functions, called *isometries*. These are function that preserves distances. Let's see what we mean by this:

Definition An **isometry** is a function M that maps points in the plane to other points in the plane such that

$$d(\mathbf{p}, \mathbf{q}) = d(\mathsf{M}\mathbf{p}, \mathsf{M}\mathbf{q}),$$

where d is the distance function.

Question How do you compute the distance between two points again?

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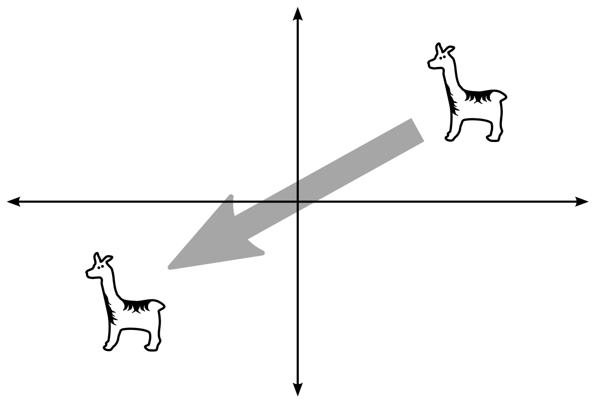
We're going to see that several ideas in geometry, specifically translations, reflections, and rotations which all seem very different, are actually all isometries. Hence, we will be thinking of these concepts as matrices.

1.1.1 Translations

Of all the isometries, *translations* are probably the easiest. With a translation, all we do is move our object in a straight line, that is, every point in the plane is moved the same distance and the same direction. Let's see what happens to

Louie Llama when he is translated:

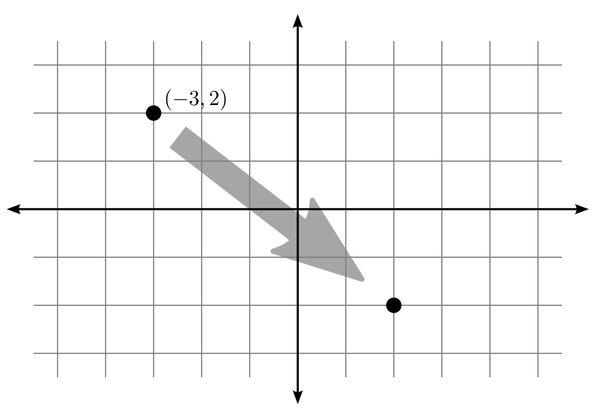


Pretty simple eh? We can give a more "mathematical" definition of a translation involving our newly-found knowledge of matrices! Check it:

Definition A **translation**, denoted by $\mathsf{T}_{(u,v)}$, is a function that moves every point a given distance u in the x-direction and a given distance v in the y-direction. We will use the following matrix to represent translations:

$$\mathsf{T}_{(u,v)} = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (-3, 2)$. *Use a matrix to translate* $\mathbf{p}$ 5 *units right and* 4 *units down.*



Here is how you do it:

$$\begin{aligned}
\mathsf{T}_{(5,-4)}\mathbf{p} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -3+0+5 \\ 0+2-4 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence, we end up with the point $(2, -2)$. But you knew that already, didn't you?

Question Can you demonstrate with algebra why translations are isometries?

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Question We know how to translate individual points. How do we move entire figures and other funky shapes?

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1.1.2 Reflections

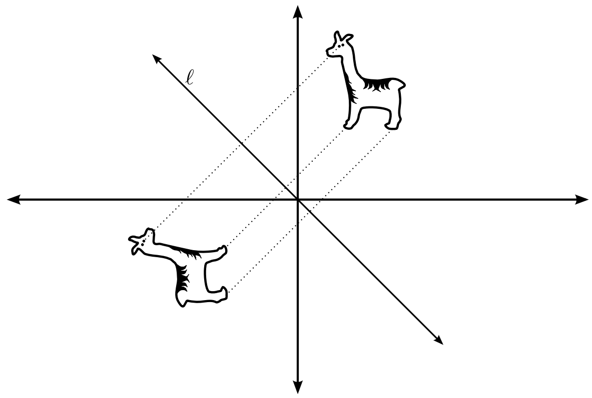
The act of reflection has fascinated humanity for millennia. It has a strong effect on our perception of beauty and has a defined place in art—not to mention how useful it is for the application of make-up. Here is our definition of a reflection:

Definition The **reflection** across a line ℓ, denoted by F_ℓ, is the function that maps a point $\mathbf{p}$ to a point $\mathsf{F}_\ell\mathbf{p}$ such that:

(1) If $\mathbf{p}$ is on ℓ, then $\mathsf{F}_\ell\mathbf{p} = \mathbf{p}$.

(2) If $\mathbf{p}$ is not on ℓ, then ℓ is the perpendicular bisector of the segment connecting $\mathbf{p}$ and $\mathsf{F}_\ell\mathbf{p}$.

You might be saying, "Huh?" It's not as hard as it looks. Check out this picture of the situation, again Louie Llama will help us out:



A Collection of Reflections

We are going to begin with a trio of reflections. We'll start with a **horizontal reflection** across the y-axis. Using our matrix notation, we write:

$$\mathsf{F}_{x=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

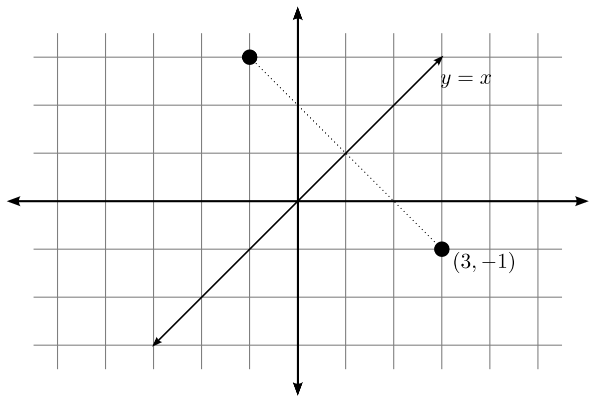
The next reflection in our collection is a **vertical reflection** across the x-axis. Using our matrix notation, we write:

$$\mathsf{F}_{y=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final reflection to add to our collection is a **diagonal reflection** across the line $y = x$. Using our matrix notation, we write:

$$\mathsf{F}_{y=x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (3, -1)$*. Use a matrix to reflect* $\mathbf{p}$ *across the* *line $y = x$.*



Here is how you do it:

$$\begin{aligned}
\mathsf{F}_{y=x}\mathbf{p} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 - 1 + 0 \\ 3 + 0 + 0 \\ 0 + 0 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence we end up with the point $(-1, 3)$.

Question Let $\mathbf{p}$ be some point in Quadrant I of the (x, y)-plane. What reflection will map this point to Quadrant II? What about Quadrant IV? What about Quadrant III?

?

Question Can you demonstrate with algebra why each of our reflections above are isometries?

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1.1.3 Rotations

Imagine that you are on a swing set, going higher and higher until you are actually able to make a full circle[1]. At the point where you are directly above

[1] Face it, I think we all dreamed of doing that when we were little—or in my case, last week.

where you would be if the swing were at rest, where is your head, comparatively? Your feet? Your hands?

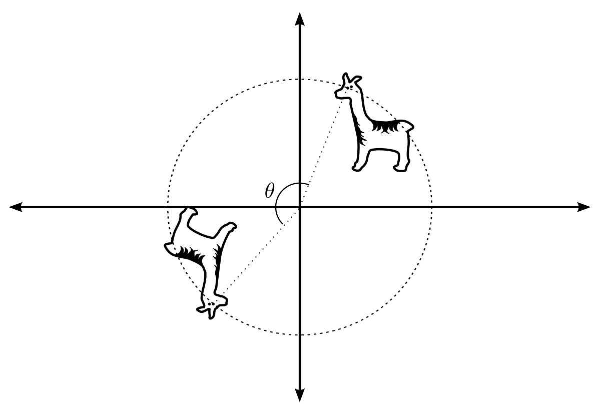
Rotations should bring circles to mind. This is not a coincidence. Check out our definition of a *rotation*:

Definition A **rotation** of θ degrees about the origin, denoted by R_θ, is a function that maps a point $\mathbf{p}$ to a point $\mathsf{R}_\theta\mathbf{p}$ such that:

(1) The points $\mathbf{p}$ and $\mathsf{R}_\theta\mathbf{p}$ are equidistant from the origin.

(2) An angle of θ degrees is formed by $\mathbf{p}$, the origin, and $\mathsf{R}_\theta\mathbf{p}$.

Louie Llama, can you do the honors?

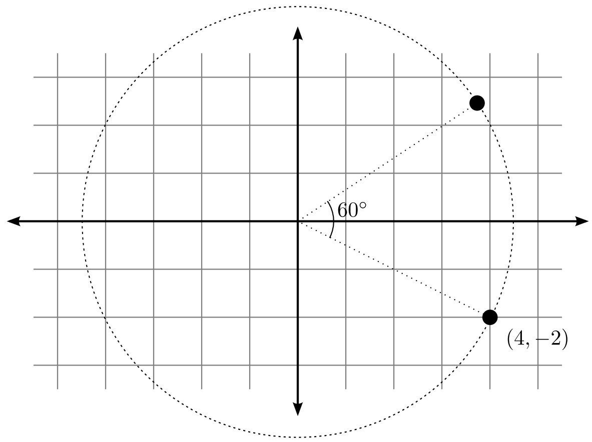


WARNING *Positive angles denote a counterclockwise rotation. Negative angles denote a clockwise rotation.*

Looking back on trigonometry, there were some angles that kept on coming up. Some of these were 90°, 60°, and 45°. We'll focus on these angles too.

$$\mathsf{R}_{90} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{R}_{60} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{R}_{45} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (4, -2)$. *Use a matrix to rotate* $\mathbf{p}$ $60°$ *about* *the origin.*



Here is how you do it:

$$\begin{aligned}
\mathsf{R}_{60}\mathbf{p} &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2+\sqrt{3}+0 \\ 2\sqrt{3}-1+0 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 2+\sqrt{3} \\ 2\sqrt{3}-1 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence, we end up with the point $(2+\sqrt{3}, 2\sqrt{3}-1)$.

Question Do the numbers in the matrices above look familiar? If so, why?

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Question How do you rotate a point 180 degrees?

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Question Can you demonstrate with algebra why our rotations above are isometries?

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Problems for Section 1.1

(1) How do you compute the distance between two points $\mathbf{p}$ and $\mathbf{q}$ in the plane?

(2) Use algebra to explain why:

$$d(\mathbf{p},\mathbf{q}) = d(\mathbf{p}-\mathbf{q},\mathbf{o}) = d(\mathbf{o},\mathbf{p}-\mathbf{q})$$

where $\mathbf{o} = (0,0)$.

(3) What is an isometry?

(4) What is a translation?

(5) What is a rotation?

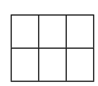
(6) What is a reflection?

(7) Reflecting back on this chapter, suppose I translate a point $\mathbf{p}$ to $\mathbf{p}'$. Does it make any difference if I move the point $\mathbf{p}$ along a wiggly path

or a straight path? Explain your reasoning.

(8) Reflecting back on this chapter, is a rotation the continuous *act* of *moving* a point through an angle around some fixed point, or is it just a final picture compared to the initial one? Explain your reasoning.

(9) In the vector illustrator *Inkscape* there is an option to transform an image via a "matrix." If you select this tool, you are presented with 6 boxes to fill in with numbers:



Use what you've learned in this chapter to make a guess as to how this tool works.

(10) In what direction does a positive rotation occur?

(11) Is a $270°$ rotation the same as a $-90°$ rotation? Explain your reasoning.

(12) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(13) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 8 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(14) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(15) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(16) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(17) Use a matrix to translate the point $(-1, 6)$ three units right and two units up. Sketch this situation and explain your reasoning.

(18) The matrix $\mathsf{T}_{(-2,6)}$ was used to translate the point $\mathbf{p}$ to $(-1, -3)$. What is $\mathbf{p}$? Sketch this situation and explain your reasoning.

(19) Use a matrix to reflect the point $(5, 2)$ across the x-axis. Sketch this situation and explain your reasoning.

(20) Use a matrix to reflect the point $(-3, 4)$ across the y-axis. Sketch this situation and explain your reasoning.

(21) Use a matrix to reflect the point $(-1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(22) Use a matrix to reflect the point $(1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(23) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point $\mathbf{p}$ to $(4, 3)$. What is $\mathbf{p}$? Explain your reasoning.

(24) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point $\mathbf{p}$ to $(0, -8)$. What is $\mathbf{p}$? Explain your reasoning.

(25) The matrix $\mathsf{F}_{x=0}$ was used to reflect the point $\mathbf{p}$ to $(-5, -1)$. What is $\mathbf{p}$? Explain your reasoning.

(26) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point $\mathbf{p}$ to $(9, -2)$. What is $\mathbf{p}$? Explain your reasoning.

(27) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point $\mathbf{p}$ to $(-3, -3)$. What is $\mathbf{p}$? Explain your reasoning.

(28) Considering the point $(3, 2)$, use a matrix to rotate this point $60°$ about the origin. Sketch this situation and explain your reasoning.

(29) Considering the point $(\sqrt{2}, -\sqrt{2})$, use a matrix to rotate this point $45°$ about the origin. Sketch this situation and explain your reasoning.

(30) Considering the point $(-7, 6)$, use a matrix to rotate this point $90°$ about the origin. Sketch this situation and explain your reasoning.

(31) Considering the point $(-1, 3)$, use a matrix to rotate this point $0°$ about the origin. Sketch this situation and explain your reasoning.

(32) Considering the point $(0, 0)$, use a matrix to rotate this point $120°$ about the origin. Sketch this situation and explain your reasoning.

(33) Considering the point $(1, 1)$, use a matrix to rotate this point $-90°$ about the origin. Sketch this situation and explain your reasoning.

(34) The matrix R_{90} was used to rotate the point $\mathbf{p}$ to $(2, -5)$. What is $\mathbf{p}$? Explain your reasoning.

(35) The matrix R_{60} was used to rotate the point $\mathbf{p}$ to $(0, 2)$. What is $\mathbf{p}$? Explain your reasoning.

(36) The matrix R_{45} was used to rotate the point $\mathbf{p}$ to $(-\frac{1}{2}, \frac{5}{2})$. What is $\mathbf{p}$? Explain your reasoning.

(37) The matrix R_{-90} was used to rotate the point $\mathbf{p}$ to $(4, 3)$. What is $\mathbf{p}$? Explain your reasoning.

(38) If someone wanted to plot the graph of $y = x^2$, they might start by filling in the following table:

x	x^2
0	
1	
-1	
2	
-2	
3	
-3	

Reflect each point you obtain from the table above about the line $y = x$. Give a plot of this situation. What curve do you obtain? What is this new curve's relationship to $y = x^2$? Explain your reasoning.

(39) Some translation T was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (1, 2)$ and $\mathbf{q} = (3, 4)$, find T and explain your reasoning.

(40) Some translation T was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (-2, 3)$ and $\mathbf{q} = (2, 3)$, find T and explain your reasoning.

(41) Some reflection F was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (1, 4)$ and $\mathbf{q} = (1, -4)$, find F and explain your reasoning.

(42) Some reflection F was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (5, 0)$ and $\mathbf{q} = (0, 5)$, find F and explain your reasoning.

(43) Some rotation R was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (3, 0)$ and $\mathbf{q} = (0, 3)$, find R and explain your reasoning.

(44) Some rotation R was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (\sqrt{2}, \sqrt{2})$ and $\mathbf{q} = (0, 2)$, find R and explain your reasoning.

(45) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (0,0), \\ (1,0) &\mapsto (3,0), \\ (0,1) &\mapsto (0,5). \end{aligned}$$

Find M and explain your reasoning.

(46) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (-1,1), \\ (1,0) &\mapsto (3,0), \\ (0,1) &\mapsto (0,5). \end{aligned}$$

Find M and explain your reasoning.

(47) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (1,1), \\ (1,0) &\mapsto (2,1), \\ (0,1) &\mapsto (1,2). \end{aligned}$$

Find M and explain your reasoning.

(48) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (2,2), \\ (1,1) &\mapsto (3,3), \\ (-1,1) &\mapsto (1,3). \end{aligned}$$

Find M and explain your reasoning.

(49) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (0,0), \\ (1,1) &\mapsto (0,3), \\ (-1,1) &\mapsto (5,0). \end{aligned}$$

Find M and explain your reasoning.

(50) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (1,2), \\ (1,1) &\mapsto (-3,1), \\ (-1,1) &\mapsto (2,-3). \end{aligned}$$

Find M and explain your reasoning.

1.2 The Algebra of Matrices

1.2.1 Matrix Multiplication

We know how to multiply a matrix and a point. Multiplying two matrices is a similar procedure:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj+bm+cp & ak+bn+cq & al+bo+cr \\ dj+em+fp & dk+en+fq & dl+eo+fr \\ gj+hm+ip & gk+hn+iq & gl+ho+ir \end{bmatrix}$$

Variables are all good and well, but let's do this with actual numbers. Consider the following two matrices:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathsf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's multiply them together and see what we get:

$$\begin{aligned} \mathsf{MI} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot1+2\cdot0+3\cdot0 & 1\cdot0+2\cdot1+3\cdot0 & 1\cdot0+2\cdot0+3\cdot1 \\ 4\cdot1+5\cdot0+6\cdot0 & 4\cdot0+5\cdot1+6\cdot0 & 4\cdot0+5\cdot0+6\cdot1 \\ 7\cdot1+8\cdot0+9\cdot0 & 7\cdot0+8\cdot1+9\cdot0 & 7\cdot0+8\cdot0+9\cdot1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \mathsf{M} \end{aligned}$$

Question What is IM equal to?

?

It turns out that we have a special name for I. We call it the **identity matrix**.

WARNING *Matrix multiplication is **not** generally commutative. Check it out:*

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textit{and} \qquad \mathsf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When we multiply these matrices, we get:

$$\begin{aligned}
\mathsf{FR} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1\cdot 0 + 0\cdot 1 + 0\cdot 0 & 1\cdot(-1) + 0\cdot 0 + 0\cdot 0 & 1\cdot 0 + 0\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + (-1)\cdot 1 + 0\cdot 0 & 0\cdot(-1) + (-1)\cdot 0 + 0\cdot 0 & 0\cdot 0 + (-1)\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 0\cdot 1 + 1\cdot 0 & 0\cdot(-1) + 0\cdot 0 + 1\cdot 0 & 0\cdot 0 + 0\cdot 0 + 1\cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

On the other hand, we get:

$$\mathsf{RF} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question Can you draw some nice pictures showing geometrically that matrix multiplication is not commutative?

?

Question Is it always the case that $(\mathsf{LM})\mathsf{N} = \mathsf{L}(\mathsf{MN})$?

?

1.2.2 Compositions of Matrices

It is often the case that we wish to apply several isometries successively to a point. Consider the following:

$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{N} = \begin{bmatrix} g & h & i \\ j & k & l \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Now let's compute

$$\begin{aligned}
\mathsf{M}(\mathsf{N}\mathbf{p}) &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} g & h & i \\ j & k & l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \right) \\
&= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} gx + hy + i \\ jx + ky + l \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} agx + ahy + ai + bjx + bky + bl + c \\ dgx + dhy + di + ejx + eky + el + f \\ 1 \end{bmatrix}
\end{aligned}$$

Now *you* compute $(\mathsf{MN})\mathbf{p}$ and compare what *you* get to what we got above.

Compositions of Translations

A composition of translations occurs when two or more successive translations are applied to the same point. Check it out:

$$\begin{aligned}
\mathsf{T}_{(5,-4)}\mathsf{T}_{(-3,2)} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \mathsf{T}_{(5+(-3),(-4)+2)} \\
&= \mathsf{T}_{(2,-2)}
\end{aligned}$$

Theorem 1 *The composition of two translations $\mathsf{T}_{(u,v)}$ and $\mathsf{T}_{(s,t)}$ is equal to the translation $\mathsf{T}_{(u+s,v+t)}$.*

Question How do you prove the theorem above?

?

Question Can you give a single translation that is equal to the following composition?

$$\mathsf{T}_{(-7,5)}\mathsf{T}_{(0,-6)}\mathsf{T}_{(2,8)}\mathsf{T}_{(5,-4)}$$

?

Question Are compositions of translations commutative? Are they associative?

?

Compositions of Reflections

A composition of reflections occurs when two or more successive reflections are applied to the same point. Check it out:

$$\begin{aligned}
\mathsf{F}_{y=0}\mathsf{F}_{y=x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Question Is the composition $\mathsf{F}_{y=0}\mathsf{F}_{y=x}$ still a reflection?

?

Question Are compositions of reflections commutative? Are they associative?

?

Compositions of Rotations

A composition of rotations occurs when two or more successive rotations are applied to the same point. Check it out:

$$\begin{aligned}
\mathsf{R}_{60}\mathsf{R}_{60} &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

Theorem 2 *The product of two rotations R_θ and R_φ is equal to the rotation $\mathsf{R}_{\theta+\varphi}$.*

From this we see that:

$$\mathsf{R}_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question What is the rotation matrix for a 360° rotation? What about a 405° rotation?

?

Question Are compositions of rotations commutative? Are they associative?

?

1.2.3 Mixing and Matching

Life gets interesting when we start composing translations, reflections, and rotations together. First we'll compose a reflection with a rotation:

$$
\begin{aligned}
\mathsf{F}_{y=0}\mathsf{R}_{60} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Question Does this result look familiar?

?

Now how about a rotation composed with a translation:

$$
\begin{aligned}
\mathsf{R}_{90}\mathsf{T}_{(3,-4)} &= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 4 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Question Does $\mathsf{R}_{90}\mathsf{T}_{(3,-4)} = \mathsf{T}_{(3,-4)}\mathsf{R}_{90}$?

?

Question Find a matrix that represents the reflection $\mathsf{F}_{y=-x}$.

I'll take this one. Note that

$$
\begin{aligned}
\mathsf{F}_{y=-x} &= \mathsf{R}_{180}\mathsf{F}_{y=x} \\
&= \mathsf{R}_{90}\mathsf{R}_{90}\mathsf{F}_{y=x} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

OK looks good, but you, the reader, are going to have to check the above computation yourself.

Question How do we deal with reflections that are not across the lines $y = 0$, $x = 0$, or $y = x$? How would you reflect points across the line $y = 1$?

?

Problems for Section 1.2

(1) Give a single translation that is equal to $\mathsf{T}_{(-3,2)}\mathsf{T}_{(5,-1)}$. Explain your reasoning.

(2) Consider the two translations $\mathsf{T}_{(-4,8)}$ and $\mathsf{T}_{(4,-8)}$. Do these commute? Explain your reasoning.

(3) Give a single reflection that is equal to $\mathsf{F}_{x=0}\mathsf{F}_{y=0}$. Sketch this situation and explain your reasoning.

(4) Given any point $\mathbf{p} = (x, y)$, express $\mathsf{T}_{(4,2)}\mathsf{T}_{(6,-5)}\mathbf{p}$ as $\mathsf{T}_{(u,v)}\mathbf{p}$ for some values of u and v. Sketch this situation and explain your reasoning.

(5) Give a matrix for R_{-45}. Explain your reasoning.

(6) Give a matrix for R_{-60}. Explain your reasoning.

(7) Sam suggests that:

$$\mathsf{R}_{-90} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why does he suggest this? Is it even true? Explain your reasoning.

(8) Give a matrix for $\mathsf{F}_{y=-x}$. Explain your reasoning.

(9) Given the point $\mathbf{p} = (-4, 2)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{F}_{y=x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(10) Given the point $\mathbf{p} = (5, 0)$, use matrices to compute $\mathsf{F}_{y=x}\mathsf{F}_{y=-x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(11) Give a single rotation that is equal to $\mathsf{R}_{45}\mathsf{R}_{60}$. Explain your reasoning.

(12) Given the point $\mathbf{p} = (1, 3)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{90}\mathbf{p}$. Sketch this situation and explain your reasoning.

(13) Given the point $\mathbf{p} = (-7, 2)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{-45}\mathbf{p}$. Sketch this situation and explain your reasoning.

(14) Given the point $\mathbf{p} = (-2, 5)$, use matrices to compute $\mathsf{R}_{90}\mathsf{R}_{-90}\mathsf{R}_{360}\mathbf{p}$. Sketch this situation and explain your reasoning.

(15) Given the point $\mathbf{p} = (5, 4)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{T}_{(2,-4)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(16) Given the point $\mathbf{p} = (-1, 6)$, use matrices to compute $\mathsf{R}_{45}\mathsf{T}_{(0,0)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(17) Given the point $\mathbf{p} = (11, 13)$, use matrices to compute $\mathsf{T}_{(-6,-3)}\mathsf{R}_{135}\mathbf{p}$. Sketch this situation and explain your reasoning.

(18) Given the point $\mathbf{p} = (-7, -5)$, use matrices to compute $\mathsf{R}_{540}\mathsf{F}_{x=0}\mathbf{p}$. Sketch this situation and explain your reasoning.

(19) Give a composition of matrices that will take a point and reflect it across the x-axis and then rotate the result $90°$ around the origin. Sketch this situation and explain your reasoning.

(20) Give a composition of matrices that will take a point and translate it three units up and 2 units left and then rotate it $90°$ clockwise around the origin. Sketch this situation and explain your reasoning.

(21) Give a composition of matrices that will take a point and rotate it $270°$ around the origin, reflect it across the line $y = x$, and then translate the result down 5 units and 3 units to the right. Sketch this situation and explain your reasoning.

(22) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $x = 1$. Sketch this situation and explain your reasoning.

(23) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = -4$. Sketch this situation and explain your reasoning.

(24) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = x + 5$. Sketch this situation and explain your reasoning.

(25) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it $45°$ around the point $(2, 3)$. Sketch this situation and explain your reasoning.

(26) Give a composition of translations and any of the following matrices

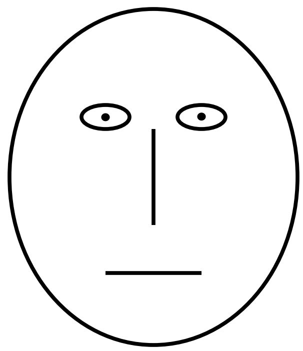
$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it $90°$ clockwise around the point $(-3, 4)$ Sketch this situation and explain your reasoning.

1.3 Symmetry

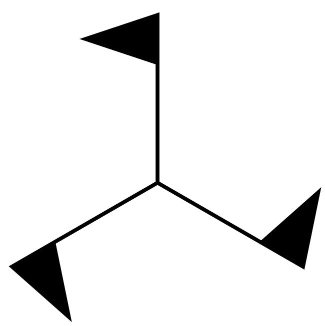
Question What is *symmetry*?

You should think about the question above **before** reading on—though, uncharistically, we will give an answer eventually. Let's start by giving some examples of symmetry. An object can have symmetry *through reflections*:



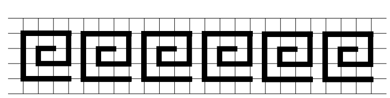
Check it out—if the nose is along line line $x = 0$, then the reflection $\mathsf{F}_{x=0}$ does not change the picture.

An object can have symmetry *through rotations*:



Check it out—if this object is centered at the origin, then the rotation R_{120} does not change the picture.

An object can have symmetry *through translations* if it extends infinitely in some direction, in this case imagine the pattern below extending infinitely both to the left and the right:

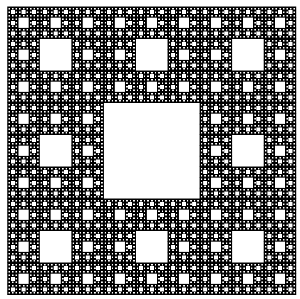


Now the translation $\mathsf{T}_{(4,0)}$ does not change the picture—note, we assumed that each square above is 1 unit.

Question What other translation don't change the picture?

?

An object can have symmetry *through dilations*, here we need to imagine the entire plane filled with tiles as below:



This picture (assuming it extended infinitely) would have symmetry through D_s for some scale factor s, see Activity A.3.

Trying to combine all these different ideas of symmetry is not easy. In [7], it is said

Symmetry is immunity to a possible change.

Question Can you explain what the definition of symmetry above is saying?

?

1.3.1 Symmetry Groups

One of the most fundamental notions in all of modern mathematics is that of a *group*. Sadly, many students never see a group in their education.

Definition A **group** is a set of elements (in our case matrices) which we will call $\mathcal{G}$ such that:

(1) There is an associative operation (in our case matrix multiplication).

(2) The set is closed under this operation (the product of any two matrices in the set is also in the set).

(3) There exists an identity $\mathsf{I} \in \mathcal{G}$ such that for all $\mathsf{M} \in \mathcal{G}$,

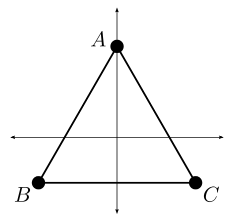
$$\mathsf{IM} = \mathsf{MI} = \mathsf{M}.$$

(4) For all $M \in \mathcal{G}$ there is an inverse $\mathsf{M}^{-1} \in \mathcal{G}$ such that

$$\mathsf{MM}^{-1} = \mathsf{M}^{-1}\mathsf{M} = \mathsf{I}.$$

1.3.2 Groups of Rotations

Let's see a group. Here we have an equilateral triangle centered at the origin of the (x, y)-plane:



Question The matrix R_{360} will rotate this triangle completely around the origin. What matrix will rotate this triangle one-third of a complete rotation?

As a gesture of friendship, I'll take this one. One-third of 360 is 120. So we see that R_{120} will rotate the triangle one-third of a full rotation. Do you remember this matrix? Here it is:

$$\mathsf{R}_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In spite of the fact that this matrix is messy and that matrix multiplication is somewhat tedious, you should realize that

$$\mathsf{R}_{120}^2 = \mathsf{R}_{240} \qquad \text{and} \qquad \mathsf{R}_{120}^3 = \mathsf{R}_{360}.$$

Let's put these facts (and a few more) together in what is called a *group table*. Remember multiplication tables from elementary school? Well, we're going to make something like a "multiplication table" of rotations. We'll start by listing the identity and powers of a one-third rotation along the top and left-hand sides. Setting $\mathsf{R} = \mathsf{R}_{120}$ we have:

$\circ$	I	R	R^2	R^3	$\cdots$
I					
R					
R^2					
R^3					
$\vdots$					

Since $\mathsf{R}^3 = \mathsf{I}$, we need only take our table to R^2. At this point we can write out the complete table:

$\circ$	I	R	R^2
I	I	R	R^2
R	R	R^2	I
R^2	R^2	I	R

Since matrix multiplication is associative, and we see from the table that every element has an inverse, we see that

$$\{\mathsf{I}, \mathsf{R}, \mathsf{R}^2\}$$

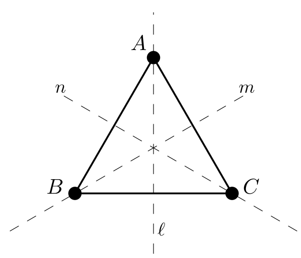
is a group.

Question What rotation matrices would we use when working with a square? A pentagon? A hexagon?

?

1.3.3 Groups of Reflections

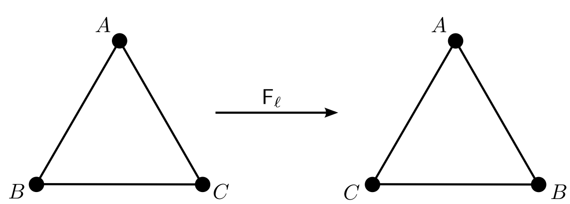
Let's see another group. Again consider an equilateral triangle. This time we are interested in the three lines of reflection that preserve this triangle:



Question Suppose that the triangle above is centered at the origin of the (x, y)-plane. What are equations for ℓ, m, and n?

?

The easiest of the reflections above is the reflection over F_ℓ.



We'll start our group table off with just two elements: I and $\mathsf{F} = \mathsf{F}_\ell$.

$\circ$	I	F
I	I	F
F	F	I

Notice that when we apply F twice we're right back where we started. Hence, $\mathsf{FF} = \mathsf{I}$. Since matrix multiplication is associative, we see that

$$\{\mathsf{I}, \mathsf{F}\}$$

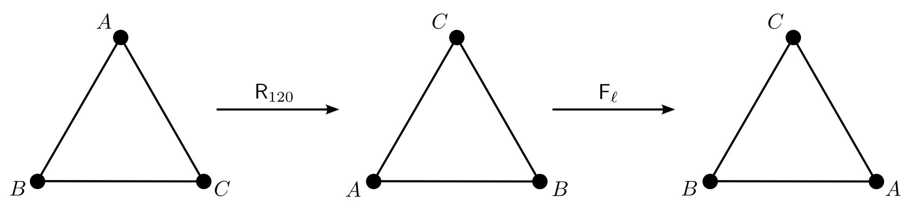
forms a group. Specifically this is a group of reflections of the triangle.

Question Above we used F_ℓ. What would happen if we used F_m or F_n? Also, what are the equations for the lines of symmetry of the square centered at the origin?

?

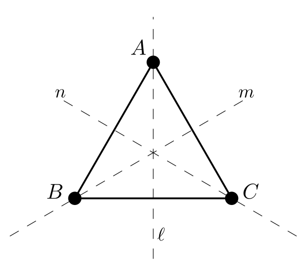
1.3.4 Symmetry Groups

Now let's mix our rotations and reflections. Consider our original triangle and apply $\mathsf{F}_\ell \mathsf{R}_{120}$:



What you may not immediately notice is that we obtain the same transformation by taking the original triangle and applying F_m.

As it turns out, every possible symmetry of the equilateral triangle can be represented using reflections and rotations. Each of these reflections and rotations can be expressed as a composition of a single reflection and a single rotation. The collection of all symmetries forms a group called the *symmetry group* of the equilateral triangle.



Let's see the group table, note we'll let $\mathsf{R} = \mathsf{R}_{120}$ and $\mathsf{F} = \mathsf{F}_\ell$:

$\circ$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2	F	FR
R^2	R^2	I	R	FR	FR^2	F
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2	I	R
FR^2	FR^2	F	FR	R	R^2	I

This table shows every symmetry of the triangle, including the identify I. By comparing the rows and columns of the group table, you can see that every element has an inverse. This combined with the fact that the matrix multiplication is associative shows that the symmetries of the triangle,

$$\{\mathsf{I}, \mathsf{R}, \mathsf{R}^2, \mathsf{F}, \mathsf{FR}, \mathsf{FR}^2\}$$

form a group.

Question Can you express the symmetries of squares in terms of reflections and rotations? What does the group table look like for the symmetry group of the square?

?

Problems for Section 1.3

(1) Explain what *symmetry* is.

(2) State the definition of a group of matrices.

(3) How many lines of reflectional symmetry exist for a square? Provide a drawing to justify your answer.

(4) What are the equations for the lines of reflectional symmetry that exist for the square? Explain your answers.

(5) How many lines of reflectional symmetry exist for a regular hexagon? Provide a drawing to justify your answer.

(6) What are the equations for the lines of reflectional symmetry for a regular hexagon? Explain your answers.

(7) How many consecutive rotations are needed to return the vertexes of a square to their original position? Provide a drawing to justify your answer, labeling the vertexes.

(8) How many degrees are in one-fourth of a complete rotation of the square? Explain your answer.

(9) How many degrees are in one-sixth of a complete rotation of the regular hexagon? Explain your answer.

(10) In this section, we've focused on a 3-sided figure, a 4-sided figure, and a 6-sided figure. Why do we not include the rotation group for the pentagon in this section? If we did, how many degrees would be in one-fifth of a complete rotation?

(11) With notation used in this section, draw pictures representing the action of the following isometries F_ℓ, R, RF_ℓ and $\mathsf{F}_\ell\mathsf{R}$ on the equilateral triangle.

(12) Consider a square centered at the origin. Draw pictures representing the action of $\mathsf{F}_{y=0}$, R_{90}, $\mathsf{R}_{90}\mathsf{F}_{y=0}$, and $\mathsf{F}_{y=0}\mathsf{R}_{90}$ on this square.

(13) Consider a hexagon centered at the origin. Draw pictures representing the action of $\mathsf{F}_{x=0}$, R_{60}, $\mathsf{R}_{60}\mathsf{F}_{x=0}$, and $\mathsf{F}_{x=0}\mathsf{R}_{60}$ on this hexagon.

(14) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}. Explain and illustrate your answer.

(15) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is $\mathsf{F}_{x=0}$. Explain and illustrate your answer.

(16) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}^2. Explain and illustrate your answer.

(17) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{180}. Explain and illustrate your answer.

(18) Find two symmetries of the square, neither of which is the identity, such that their composition is F_ℓ. Explain and illustrate your answer.

(19) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{270}. Explain and illustrate your answer.

(20) Use a group table to help you write out the symmetries of the equilateral triangle. List all elements that commute with every other element in the table. Explain your reasoning.

(21) Use a group table to help you write out the symmetries of the square. List all elements that commute with every other element in the table. Explain your reasoning.

(22) Use a group table to help you write out the symmetries of the regular hexagon. List all elements that commute with every other element in the table. Explain your reasoning.

(23) Let M be a symmetry of the equilateral triangle. Define

$$C(\mathsf{M}) = \{\text{all symmetries that commute with } \mathsf{M}\}.$$

Write out $C(\mathsf{M})$ for every symmetry M of the equilateral triangle. Make some observations.

Chapter 2

Folding and Tracing Constructions

> We don't even know if Foldspace introduces us to one universe or many...
>
> —Frank Herbert

2.1 Constructions

While origami as an art form is quite ancient, folding and tracing constructions in mathematics are relatively new. The earliest mathematical discussion of folding and tracing constructions that I know of appears in T. Sundara Row's book *Geometric Exercises in Paper Folding*, [8], first published near the end of the Nineteenth Century. In the Twentieth Century it was shown that every construction that is possible with a compass and straightedge can be done with folding and tracing. Moreover, there are constructions that are possible via folding and tracing that are *impossible* with compass and straightedge alone. This may seem strange as you can draw a circle with a compass, yet this seems impossible to do via paper-folding. We will address this issue in due time. Let's get down to business—here are the rules of folding and tracing constructions:

Rules for Folding and Tracing Constructions

(1) You may only use folds, a marker, and semi-transparent paper.

(2) Points can only be placed in two ways:

   (a) As the intersection of two lines.

   (b) By marking "through" folded paper onto a previously placed point. Think of this as when the ink from a permanent marker "bleeds" through the paper.

(3) Lines can only be obtained in three ways:

  (a) By joining two points—either with a drawn line or a fold.
  (b) As a crease created by a fold.
  (c) By marking "through" folded paper onto a previously placed line.

(4) One can only fold the paper when:

  (a) Matching up points with points.
  (b) Matching up a line with a line.
  (c) Matching up two points with two intersecting lines.

Now we are going to present several basic constructions. We will proceed by the order of difficulty of the construction.

Construction (Transferring a Segment) *Given a segment, we wish to move it so that it starts on a given point, on a given line.*

Construction (Copying an Angle) *Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.*

Transferring segments and copying angles using folding and tracing without a "bleeding marker" can be tedious. Here is an easy way to do it:

Use 2 sheets of paper and a pen that will mark through multiple sheets.

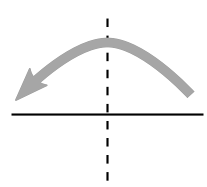
Question Can you find a way to do the above constructions without using a marker whose ink will pass through paper?

?

Construction (Bisecting a Segment) *Given a segment, we wish to cut it in half.*

(1) *Fold the paper so that the endpoints of the segment meet.*

(2) *The crease will bisect the given segment.*



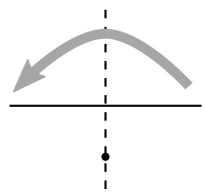
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Perpendicular through a Point) *Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.*

(1) *Fold the given line onto itself so that the crease passes though the given point.*

(2) *The crease will be the perpendicular line.*



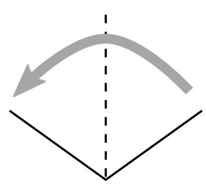
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Bisecting an Angle) *We wish to divide an angle in half.*

(1) *Fold a point on one leg of the angle to the other leg so that the crease passes though the vertex of the angle.*

(2) *The crease will bisect the angle.*



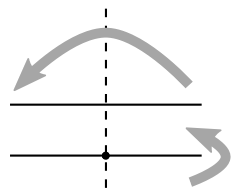
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Parallel through a Point) *Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.*

(1) *Fold a perpendicular line through the given point.*

(2) *Fold a line perpendicular to this new line through the given point.*



Now there may be a pressing question in your head:

Question How the heck are we going to fold a circle?

First of all, remember the definition of a circle:

Definition A **circle** is the set of points that are a fixed distance from a given point.

Question Is the center of a circle part of the circle?

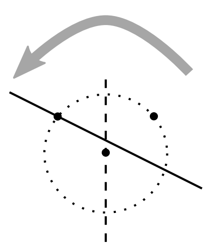
?

Secondly, remember that when doing compass and straightedge constructions we can **only** mark points that are intersections of lines and lines, lines and circles, and circles and circles. Thus while we technically draw circles, we can only actually mark certain points on circles. When it comes to folding and tracing constructions, drawing a circle amounts to marking points a given distance away from a given point—that is exactly what we can do with compass and straightedge constructions.

Construction (Intersection of a Line and a Circle) *We wish to construct the points where a given line meets a given circle. Note: A circle is given by a point on the circle and the central point.*

(1) *Fold the point on the circle onto the given line so that the crease passes through the center of the circle.*

(2) *Mark this point though both sheets of paper onto the line.*



Question Which rule for folding and tracing constructions are we using above?

?

Question How could you check that your folding and tracing construction is correct?

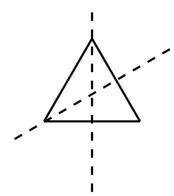
?

Construction (Equilateral Triangle) *We wish to construct an equilateral triangle given the length of one side.*

(1) *Bisect the segment.*

(2) *Fold one end of the segment onto the bisector so that the crease passes through the other end of the segment. Mark this point onto the bisector.*

(3) *Connect the points.*



Question Which rules for folding and tracing constructions are we using above?

?

Construction (Intersection of Two Circles) *We with to intersect two circles, each given by a center point and a point on the circle.*

(1) *Use four sheets of tracing paper. On the first sheet, mark the centers of both circles. On the next two sheets, mark the center and point on each of the circle—one circle per sheet.*

(2) *Simply move the two sheets with the centers and points on the circles, so that the centers are over the centers from the first sheet, and the points on the circles coincide. Now on the fourth sheet, mark all points.*

?

Question How do you use folding and tracing to construct a regular hexagon? What other regular polygons can you construct?

?

Problems for Section 2.1

(1) What are the rules for folding and tracing constructions?

(2) Use folding and tracing to bisect a given line segment. Explain the steps in your construction.

(3) Given a line segment with a point on it, use folding and tracing to construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction.

(4) Use folding and tracing to bisect a given angle. Explain the steps in your construction.

(5) Given a point and line, use folding and tracing to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(6) Given a point and line, use folding and tracing to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a circle (a center and a point on the circle) and line, use folding and tracing to construct the intersection. Explain the steps in your construction.

(8) Given a line segment, use folding and tracing to construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction.

(9) Explain how to use folding and tracing to transfer a segment.

(10) Given an angle and some point, use folding and tracing to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction.

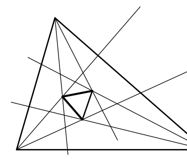
(11) Explain how to use folding and tracing to construct envelope of tangents for a parabola.

(12) Use folding and tracing to construct a square. Explain the steps in your construction.

(13) Use folding and tracing to construct a regular hexagon. Explain the steps in your construction.

(14) Morley's Theorem states: If you trisect the angles of any triangle with lines, then those lines form a new equilateral triangle inside the original

triangle.



Give a folding and tracing construction illustrating Morley's Theorem. Explain the steps in your construction.

(15) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction.

(16) Construct a 30-60-90 right triangle. Explain the steps in your construction.

(17) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction.

2.2 Anatomy of Figures

Remember, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for folding and tracing constructions.

Question In regards to folding and tracing constructions, what is a *point*?

?

Question In regards to folding and tracing constructions, what is a *line*?

?

Question In regards to folding and tracing constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them (I know we've already gone over this, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

2.2.1 Lines Related to Triangles

Believe it or not, in mathematics we often try to study the simplest objects as deeply as possible. After the objects listed above, triangles are among the most basic of geometric figures, yet there is much to know about them. There are several lines that are commonly associated to triangles. Here they are:

- Perpendicular bisectors of the sides.
- Bisectors of the angles.
- Altitudes of the triangle.
- Medians of the triangle.

The first two lines above are self-explanatory. The next two need definitions.

Definition An **altitude** of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.

Definition A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

Question The intersection of any two lines containing the altitudes of a triangle is called an **orthocenter**. How many orthocenters does a given triangle have?

?

Question The intersection of any two medians of a triangle is called a **centroid**. How many centroids does a given triangle have?

?

Question What is the physical meaning of a centroid?

?

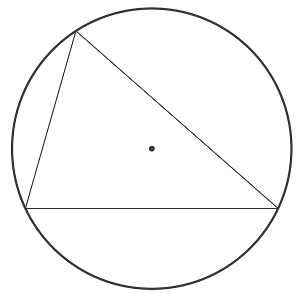
2.2.2 Circles Related to Triangles

There are also two circles that are commonly associated to triangles. Here they are:

- The circumcircle.
- The incircle.

These aren't too bad. Check out the definitions.

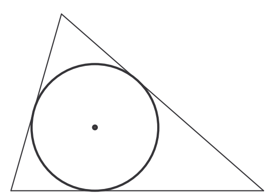
Definition The **circumcircle** of a triangle is the circle that contains all three vertexes of the triangle. Its center is called the **circumcenter** of the triangle.



Question Does every triangle have a circumcircle?

?

Definition The **incircle** of a triangle is the largest circle that will fit inside the triangle. Its center is called the **incenter** of the triangle.



Question Does every triangle have an incircle?

?

Question Are any of the lines described above related to these circles and/or centers? Clearly articulate your thoughts.

?

Problems for Section 2.2

(1) In regards to folding and tracing constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle.

(2) Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.

(3) Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.

(4) Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.

(5) Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.

(6) What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.

(7) What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.

(8) Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(9) Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(10) Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(11) Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(12) Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(13) Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:

  (a) More than one.
  (b) How many triangles are in the picture you drew?

(14) Given a triangle, construct the circumcenter. Explain the steps in your construction.

(15) Given a triangle, construct the orthocenter. Explain the steps in your construction.

(16) Given a triangle, construct the incenter. Explain the steps in your construction.

(17) Given a triangle, construct the centroid. Explain the steps in your construction.

(18) Given a triangle, construct the incircle. Explain the steps in your construction.

(19) Given a triangle, construct the circumcircle. Explain the steps in your construction.

(20) Where is the circumcenter of a right triangle? Explain your reasoning.

(21) Where is the orthocenter of a right triangle? Explain your reasoning.

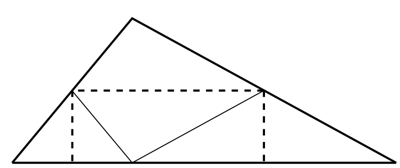
(22) Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(23) True or False: Explain your conclusions.

   (a) An altitude of a triangle is always perpendicular to a line containing some side of the triangle.
   (b) An altitude of a triangle always bisects some side of the triangle.
   (c) The incenter is always inside the triangle.
   (d) The circumcenter can be outside the triangle.
   (e) The orthocenter is always inside the triangle.
   (f) The centroid is always inside the incircle.

(24) Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.

(25) The following picture shows a triangle that has been folded along the dotted lines:



Explain how the picture "proves" the following statements:

   (a) The interior angles of a triangle sum to 180°.
   (b) The area of a triangle is given by $bh/2$.

(26) Use folding and tracing to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(27) Use folding and tracing to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(28) Use folding and tracing to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(29) Use folding and tracing to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

2.3 Similar Triangles

In geometry, we may have several different segments all with the same length. We don't want to say that the segments are *equal* because that would mean that they are *exactly* the same—position included. Hence we need a new concept:

Definition When different segments have the same length, we say they are **congruent segments**.

Definition In a similar fashion, if there is an isometry that maps one angle to the another angle, then we say they are **congruent angles**.

Put these together and we have the definition for triangles:

Definition Two triangles are said to be **congruent triangles** if there is an isometry that maps one triangle to the other triangle.

After working with triangles for short time, one quickly sees that the notion of congruence is not the only "equivalence" one wants to make between triangles. In particular, we want the notion of *similarity.*

One day when aloof old Professor Rufus was trying to explain similar triangles to his class, he merely wrote

$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{array}{c} \angle A \simeq \angle A' \\ \angle B \simeq \angle B' \\ \angle C \simeq \angle C' \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of similar triangles?

?

Question Devise a way to use folding and tracing constructions to help explore this notion of similar triangles.

?

Another day when aloof old Professor Rufus was trying to explain similar triangles to his class, he merely wrote

$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{array}{c} AB = k \cdot A'B' \\ BC = k \cdot B'C' \\ CA = k \cdot C'A' \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of similar triangles?

?

Question Devise a way to use folding and tracing constructions to help explore this notion of similar triangles.

?

Question What's going on in aloof old Professor Rufus' head[1]—why are his explanations so different?

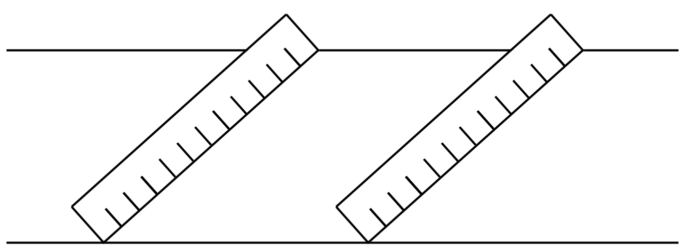
Well in fact, **both** definitions of similar triangles given above are **equivalent**.

Definition Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be **similar** if either equivalent condition holds:

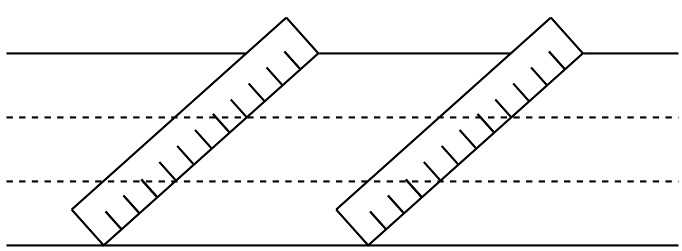
$$\begin{array}{lcl} \angle A \simeq \angle A' & & AB = k \cdot A'B' \\ \angle B \simeq \angle B' & \text{or} & BC = k \cdot B'C' \\ \angle C \simeq \angle C' & & CA = k \cdot C'A' \end{array}$$

Let's see if we can put similar triangles to work. Here is an old carpenter's trick. Suppose you have two parallel lines.

If you want to make two more lines that divide the space into three equal regions, you can place a ruler diagonally across the parallel lines, and then mark of one-third of the diagonal distance.



From this you can get your three regions.

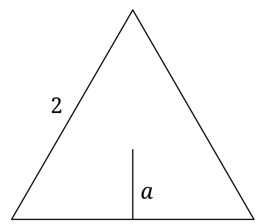


[1] I realize that this is a dangerous question!

Question Can you explain why this works using similar triangles?

?

Now for a more challenging situation. Suppose we have an equilateral triangle of side length 2. Use similar triangles and the Pythagorean Theorem to find the length of the segment a (the segment that goes from the **center** of the triangle to a side at a right angle) in the picture below.



Question How do you do this?

?

Problems for Section 2.3

(1) Compare and contrast the ideas of *equal triangles*, *congruent triangles*, and *similar triangles*.

(2) Explain why all equilateral triangles are similar to each other.

(3) Explain why all isosceles right triangles are similar to each other.

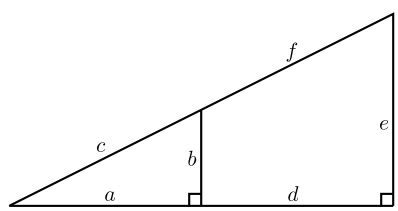
(4) Explain why when given a right triangle, the altitude of the right angle divides the triangle into two smaller triangles each similar to the original right triangle.

(5) The following sets contain lengths of sides of similar triangles. Solve for all unknowns—give all solutions. In each case explain your reasoning.

(a) $\{3, 4, 5\}$, $\{6, 8, x\}$
(b) $\{3, 3, 5\}$, $\{9, 9, x\}$
(c) $\{5, 5, x\}$, $\{10, 4, y\}$
(d) $\{5, 5, x\}$, $\{10, 8, y\}$
(e) $\{3, 4, x\}$, $\{4, 5, y\}$

(6) A *Pythagorean Triple* is a set of three positive integers $\{a, b, c\}$ such that $a^2 + b^2 = c^2$. Write down an infinite list of Pythagorean Triples. Explain your reasoning and justify all claims.

(7) Here is a right triangle, note it is **not** drawn to scale:



Solve for all unknowns in the following cases.

(a) $a = 3$, $b = ?$, $c = ?$, $d = 12$, $e = 5$, $f = ?$
(b) $a = ?$, $b = 3$, $c = ?$, $d = 8$, $e = 13$, $f = ?$
(c) $a = 7$, $b = 4$, $c = ?$, $d = ?$, $e = 11$, $f = ?$
(d) $a = 5$, $b = 2$, $c = ?$, $d = 6$, $e = ?$, $f = ?$

In each case explain your reasoning.

(8) Suppose you have two similar triangles. What can you say about the area of one in terms of the area of the other? Be specific and explain your reasoning.

(9) During a solar eclipse we see that the apparent diameter of the Sun and Moon are nearly equal. If the Moon is around 24000 miles from Earth, the Moon's diameter is about 2000 miles, and the Sun's diameter is about 865000 miles how far is the Sun from the Earth?

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(10) When jets fly above 8000 meters in the air they form a vapor trail. Cruising altitude for a commercial airliner is around 10000 meters. One day I reached my arm into the sky and measured the length of the vapor trail with my hand—my hand could just span the entire trail. If my hand spans 9 inches and my arm extends 25 inches from my eye, how long is the vapor trail? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(11) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Dave sits comfortably with his dog Fritz at a distance of 10 feet. How far must Michael stand from his TV to have the "same" viewing experience? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(12) You love IMAX movies. While the typical IMAX screen is 72 feet by 53 feet, your TV is only a 32 inch screen—it has a 32 inch diagonal. How close do you have to sit to your screen to simulate the IMAX format? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(13) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Michael stands and watches his TV at a distance of 2 feet. Dave sits comfortably with his dog Fritz at a distance of 10 feet. Whose TV appears bigger to the respective viewer? Explain your reasoning.

(a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(14) Here is a personal problem: Suppose you are out somewhere and you see that when you stretch out your arm, the width of your thumb is the same apparent size as a distant object. How far away is the object if you know the object is:

(a) 6' long (as tall as a person).

(b) 16' long (as long as a car).

(c) 40' long (as long as a school bus).

(d) 220' long (as long as a large passenger airplane).

(e) 340' long (as long as an aircraft carrier).

Explain your reasoning.

(15) I was walking down Woody Hayes Drive, standing in front of St. John Arena when a car pulled up and the driver asked, "Where is Ohio Stadium?" At this point I was a bit perplexed, but nevertheless I answered, "Do you see the enormous concrete building on the other side of the street that looks like the Roman Colosseum? That's it."

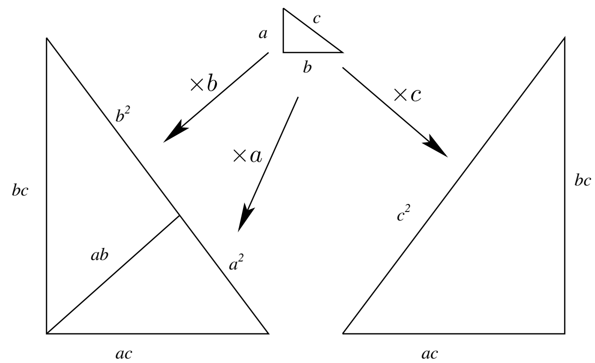
The person in the car then asked, "Where are the Twin-Towers then?" Looking up, I realized that the towers were in fact just covered by top of Ohio Stadium. I told the driver to just drive around the stadium until they found two enormous identical towers—that would be them. They thanked me and I suppose they met their destiny.

I am about 2 meters tall, I was standing about 100 meters from the Ohio Stadium and Ohio Stadium is about 40 meters tall. If the Towers are around 500 meters from the rotunda (the front entrance of the stadium), how tall could they be and still be obscured by the stadium? Explain your reasoning—for the record, the towers are about 80 meters tall.

(16) Consider the following combinations of S's and A's. Which of them produce a *Congruence Theorem*? Which of them produce a *Similarity Theorem*? Explain your reasoning.

SSS, SSA, SAS, SAA, ASA, AAA

(17) Explain how the following picture "proves" the Pythagorean Theorem.



Chapter 3

Another Dimension

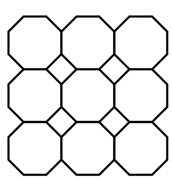
> Another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension, another dimension!
>
> —The Beastie Boys

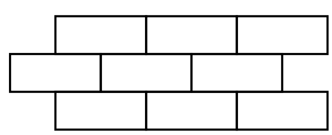
3.1 Tessellations

Go to the internet and look up M.C. Escher. He was an artist. Look at some of his work. When you do your search be sure to include the word "tessellation" OK? Back already? Very good. Sometimes Escher worked with tessellations. What's a tessellation? I'm glad you asked:

Definition A **tessellation** is a pattern of polygons fitted together to cover the entire plane without overlapping.

While it is impossible to actually cover the entire plane with shapes, if we give you enough of a tessellation, you should be able to continue it's pattern indefinitely. Here are pieces of tessellations:

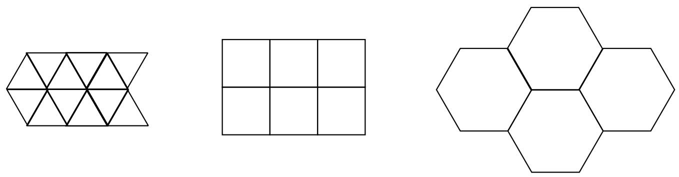




On the left we have a tessellation of a square and an octagon. On the right we have a "brick-like" tessellation.

Definition A tessellation is called a **regular tessellation** if it is composed of copies of a single regular polygon and these polygons meet vertex to vertex.

Example *Here are some examples of regular tessellations:*



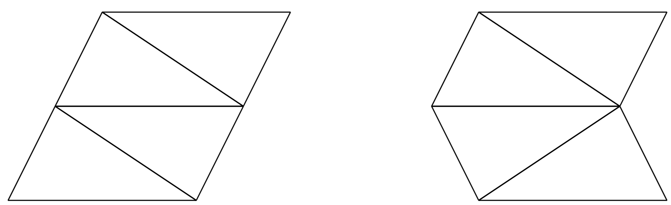
Johannes Kepler, who lived from 1571–1630, was one of the first people to study tessellations. He certainly knew the next theorem:

Theorem 3 *There are only* 3 *regular tessellations.*

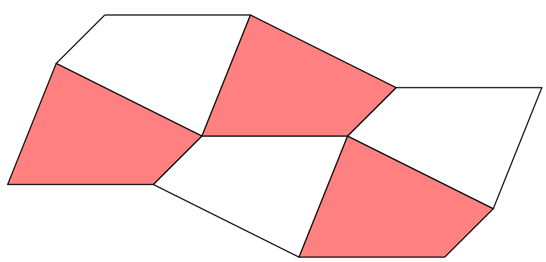
Question Why is the theorem above true?

?

Since one can prove that there are only three regular tessellations, and we have shown three above, then that is all of them. On the other hand there are lots of nonregular tessellations. Here are two different ways to tessellate the plane with a triangle:

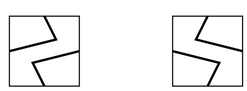


Here is a way that you can tessellate the plane with any old quadrilateral:

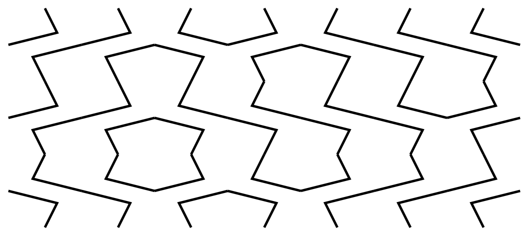


3.1.1 Tessellations and Art

How does one make art with tessellations? To start, a little decoration goes a long way. Check this out: Decorate two squares as such:



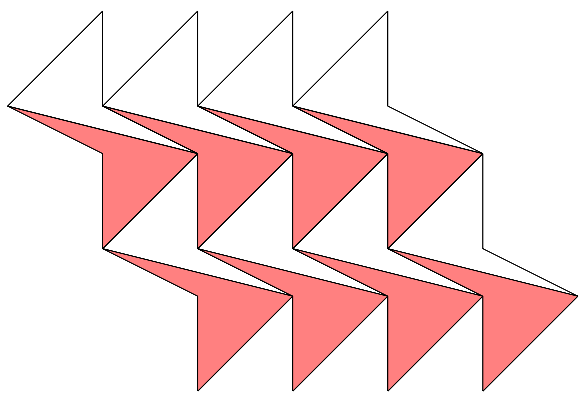
Tessellate them randomly in the plane to get this lightning-like picture:



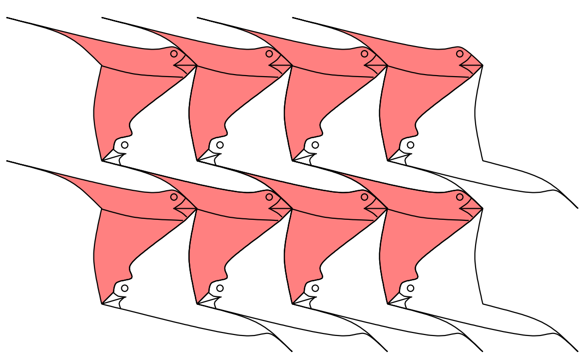
Question What sort of picture do you get if you tessellate these decorated squares randomly in a plane?

?

Another way to go is to start with your favorite tessellation:



Then you modify it a bunch to get something different:



Question What kind of art can you make with tessellations?

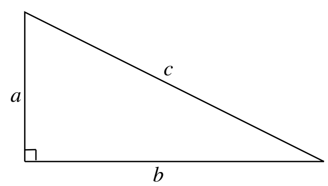
?

I'm not a very good artist, but I am a mathematician. So let's use a tessellation to give a proof! Let me ask you something:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

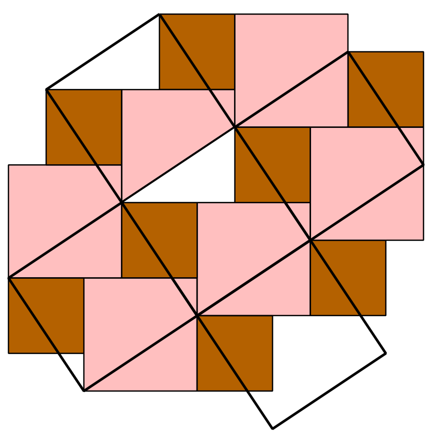
Theorem 4 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

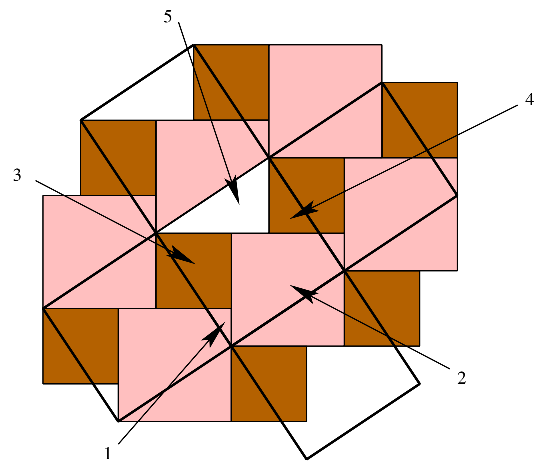
$$a^2 + b^2 = c^2.$$

Let's give a proof! Check out this tessellation involving 2 squares:



Question How does the picture above "prove" the Pythagorean Theorem?

Solution The white triangle is our right triangle. The area of the middle overlaid square is c^2, the area of the small dark squares is a^2, and the area of the medium lighter square is b^2. Now label all the "parts" of the large overlaid square:



From the picture we see that

$$a^2 = \{3 \text{ and } 4\}$$
$$b^2 = \{1,\ 2, \text{ and } 5\}$$
$$c^2 = \{1,\ 2,\ 3,\ 4, \text{ and } 5\}$$

Hence

$$c^2 = a^2 + b^2$$

Since we can always put two squares together in this pattern, this proof will work for any right triangle. ■

Question Can you use the above tessellation to give a dissection proof of the Pythagorean Theorem?

?

Problems for Section 3.1

(1) Show two different ways of tessellating the plane with a given scalene triangle. Label your picture as necessary.

(2) Show how to tessellate the plane with a given quadrilateral. Label your picture.

(3) Show how to tessellate the plane with a nonregular hexagon. Label your picture.

(4) Give an example of a polygon with 9 sides that tessellates the plane.

(5) Give examples of polygons that tessellate and polygons that do not tessellate.

(6) Give an example of a triangle that tessellates the plane where both 4 and 8 angles fit around each vertex.

(7) True or False: Explain your conclusions.

- (a) There are exactly 5 regular tessellations.
- (b) Any quadrilateral tessellates the plane.
- (c) Any triangle will tessellate the plane.
- (d) If a triangle is used to tessellate the plane, then it is always the case that exactly 6 angles will fit around each vertex.
- (e) If a polygon has more than 6 sides, then it cannot tessellate the plane.

(8) Given a regular tessellation, what is the sum of the angles around a given vertex?

(9) Given that the regular octagon has 135 degree angles, explain why you cannot give a regular tessellation of the plane with a regular octagon.

(10) Fill in the following table:

Regular n-gon	Does it tessellate?	Measure of an angle	If it tessellates, how many surround each vertex?
3-gon			
4-gon			
5-gon			
6-gon			
7-gon			
8-gon			
9-gon			
10-gon			

Hint: A regular n-gon has interior angles of $180(n-2)/n$ degrees.

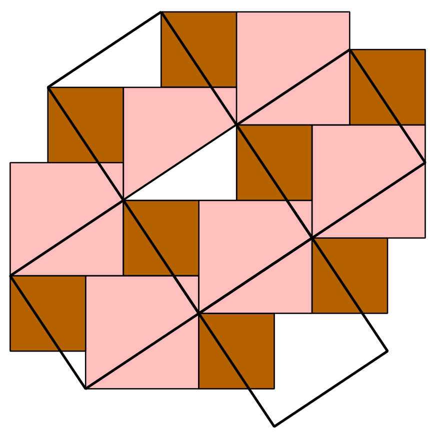
(a) What do the shapes that tessellate have in common?

(b) Make a graph with the number of sides of an n-gon on the horizontal axis and the measure of a single angle on the vertical axis. Briefly describe the relationship between the number of sides of a regular n-gon and the measure of one of its angles.

(c) What regular polygons *could* a bee use for building hives? Give some reasons that bees seem to use hexagons.

(11) Considering that the regular n-gon has interior angles of $180(n-2)/n$ degrees, and Problem (10) above, prove that there are only 3 regular tessellations of the plane.

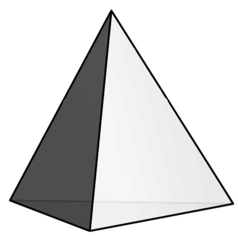
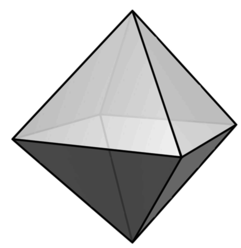
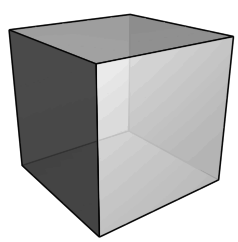
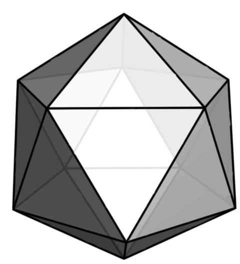
(12) Explain how the following picture "proves" the Pythagorean Theorem.



3.2 Solids

3.2.1 Platonic Solids

Around 2400 years ago, there was a group of mystics—today we might call it a cult—who called themselves the *Pythagoreans*. Being mystics, the Pythagoreans had some strange ideas, but on the other hand they were an enlightened group of people, because they believed that they could better understand the universe around them by studying mathematics. As part of the Pythagoreans' numerological religion, they thought that some polyhedra had special powers. The Pythagoreans associated the following polyhedra to elements of nature as follows:

Fire: Tetrahedron	Air: Octahedron
	
Earth: Cube	Water: Icosahedron
	

While the idea that these polyhedra are somehow connected to "arcane elements of nature" is complete nonsense, there is actually something special about the solids above. They are *regular convex polyhedra*. Let's dissect those last words:

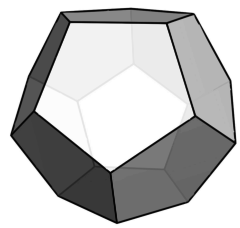
Definition A **polyhedron** is a three dimensional solid that is bounded by a finite number of polygons.

Definition An object is **convex** if given any two points inside the object, the segment connecting those two points is also contained inside the object.

Definition A polyhedron is **regular** if all its faces are the same regular polygon and if the same number of polygons meet at every vertex.

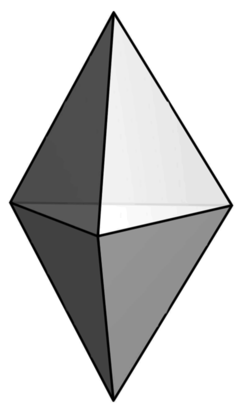
Apparently the Pythagoreans discovered the above four regular convex polyhedra first. Only later did they discover a fifth, the dodecahedron:

Æther: Dodecahedron



Since the first four regular convex solids had elements associated to them, the Pythagoreans reasoned that this fifth solid must also be associated to an element. However, all the elements were accounted for. So the Pythagoreans associated the dodecahedron to what we might call the æther, a mysterious non-earthly substance.

Question Consider the **triangular dipyramid**, the solid where two tetrahedrons are joined at a face:



Is this a regular convex polyhedron? Explain your reasoning.

?

Question Are there other regular convex polygons other than the ones shown above?

We'll give you the answer to this one:

Theorem 5 *There are at most* 5 *regular convex polyhedra.*

Proof To start, a corner of a three-dimensional object made of polygons must have at least 3 faces. Now start with the simplest regular polygon, an equilateral triangle. You can make a corner by placing:

(1) 3 triangles together.

(2) 4 triangles together.

(3) 5 triangles together.

Thus each of the above configuration of triangles could give rise to a regular convex polyhedra. However, you cannot make a corner out of 6 or more equilateral triangles, as 6 triangles all connected lie flat on the plane.

Now we can make a corner with 3 square faces, but we cannot make a corner with 4 or more square faces as 4 or more squares lie flat on the plane.

Finally we can make a corner with 3 faces each shaped like a regular pentagon, but we cannot make a corner with 4 or more regular pentagonal faces as they will be forced to overlap.

Could we make a corner with 3 faces each shaped like a regular hexagon? No, because any number of hexagons will lie flat on the plane. A similar argument works to show that we cannot make a corner with 3 faces shaped like a regular n-gon where $n > 6$, except now instead of the shapes lying flat on the plane, they overlap.

Thus we could at most have 5 regular convex solids. ■

Question Above we prove that there are at most 5 regular convex polyhedra. How do we know that these actually exist?

?

The five regular convex polyhedra came to be known as the **Platonic Solids**, over 2000 years ago when Plato discussed them in his work *Timaeus*. Here is a empty table of facts about the Platonic Solids—could you be so kind and fill it in?

Solid	Vertexes	Edges	Faces
tetrahedron			
octahedron			
cube			
icosahedron			
dodecahedron			

These solids have haunted men for centuries. People who were trying to understand the universe wondered what was the reason that there were only 5 regular convex polyhedra. Some even thought there was something *special* about certain numbers.

Question In what ways does the number 5 come up in life that might lead a person into believing it is a special number? Does this mean that the number 5 is more special than any other number?

?

Problems for Section 3.2

(1) Explain what a *polyhedron* is.

(2) Explain what it means for an object to be *convex*.

(3) Explain what it means for a polyhedron to be *regular*.

(4) State which of the following are convex sets:

(a) A hollow sphere.
(b) A half-space.
(c) The intersection of two spherical solids.
(d) A solid cube.
(e) A solid cone.
(f) The union of two spherical solids.

In each case explain your reasoning.

(5) Use words and pictures to describe the following objects:

(a) A tetrahedron.
(b) An octahedron.
(c) A cube.
(d) An icosahedron.
(e) A dodecahedron.
(f) A triangular dipyramid.

(6) How is a regular convex polyhedron different from any old convex polyhedron?

(7) True or False:

(a) There are only 5 convex polyhedra.
(b) The icosahedron has exactly 12 faces.
(c) Every vertex of the tetrahedron touches exactly 3 faces.
(d) The octahedron has exactly 6 vertexes.
(e) Every vertex of the dodecahedron touches exactly 3 faces.

In each case explain your reasoning.

(8) While there are only 5 regular convex polyhedra, there are also nonconvex regular polyhedra. Draw some examples.

(9) Draw picture illustrating the steps of the proof of Theorem 5.

(10) Where does the proof of Theorem 5 use the fact that the solids are convex?

(11) A **dual polyhedron** is the polyhedron obtained when one connects the centers of all the pairs of adjacent faces of a given polyhedron. What are the dual polyhedra of the Platonic Solids? Explain your reasoning.

(12) How many rotational symmetries does the tetrahedron have? Explain your reasoning.

(13) How many rotational symmetries does the cube have? Explain your reasoning.

(14) How many rotational symmetries does the octahedron have? Explain your reasoning.

(15) How many rotational symmetries does the dodecahedron have? Explain your reasoning.

(16) How many rotational symmetries does the icosahedron have? Explain your reasoning.

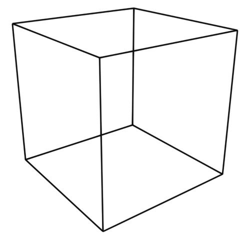
(17) Consider a cube whose side has a length of 1 unit. Imagine an octahedron inside this cube whose vertexes are the centers of the faces of the cube. How long are the edges of this new octahedron?

(18) Consider a tetrahedron whose side has a length of 1 unit. Imagine another tetrahedron inside whose vertexes are the centers of the faces of the original tetrahedron. How long are the edges of this new tetrahedron?

(19) Consider a cube whose side has a length of 1 unit. Imagine an octahedron around this cube so that the centers of the faces of the octahedron are on the vertexes of the cube. How long are the edges of this new octahedron?

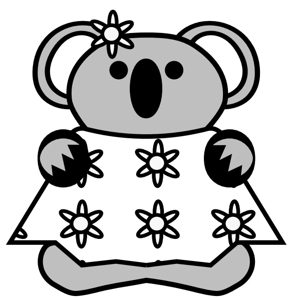
3.3 Higher Dimensions

Consider this picture:



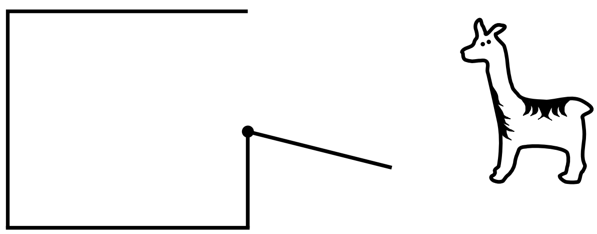
Ceci n'est pas un cube.

Here, the French got it right. That is not a cube—at best it is what a *shadow* of a real cube might look like on this sheet of paper. Being that paper is effectively 2-dimensional, it is seemingly impossible to have an actual cube embedded in this text. The miracle is that when we look at this shadow of a cube, we know that it represents a real cube. You may not think this is much of a miracle, let's see if I can convince you otherwise. Meet Sugarbear:

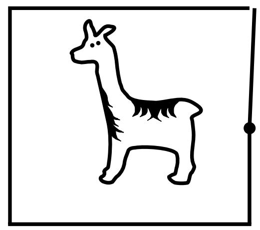


Just as the image at the beginning of this section was a *picture* of a three-dimensional cube, this is **not** actually Sugarbear—rather this is a *picture* of Sugarbear who is also actually 3-dimensional.

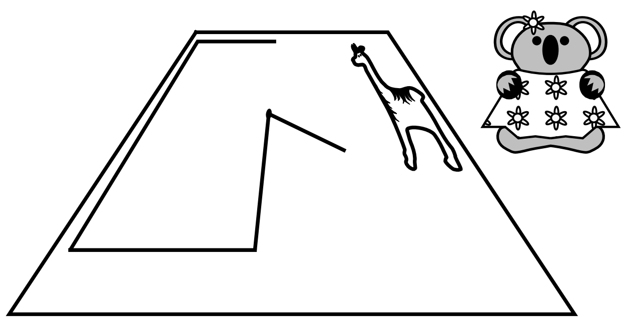
Sugarbear has decided to take a vacation to *Flatland*. The curious reader should see [1] for a history of Flatland. In Flatland there are only two directions *left/right* and *up/down*. Flatlanders (as they're called) have no concept of *inward/outward*. As it so happens, you already know someone from Flatland, Louie Llama! Here's Louie Llama chillin in front of his house:



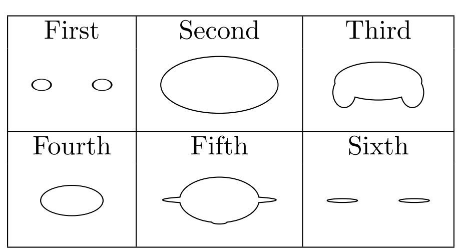
Notice, that even when Louie Llama is standing comfortably inside his house with the door closed, you can still see inside his house! This is a privilege we receive for living in the third-dimension.



On the other hand, Louie Llama cannot see you. From his perspective, he is surrounded by four walls, nothing could possibly get inside or out of his house. So our favorite Llama was quite confused when Sugarbear greeted him with a jolly "Hello!" as she arrived in Flatland.



Remember, Louie Llama can only see stuff in the plane that is Flatland. From Louie Llama's perspective, Sugarbear's "Hello!" came from all around him and even from *inside* himself! So to help out the puzzled llama, Sugarbear decided to take the plunge into Flatland, actually jumping through the plane that Louie Llama lives in. When she does this, Louie Llama sees something quite strange. Our llama sees a series of cross-sections of Sugarbear:



However, each cross-section is very confusing and seems to morph into the next. This wasn't very helpful for our friend Louie Llama.

Question Can you make sense of what Louie Llama saw?

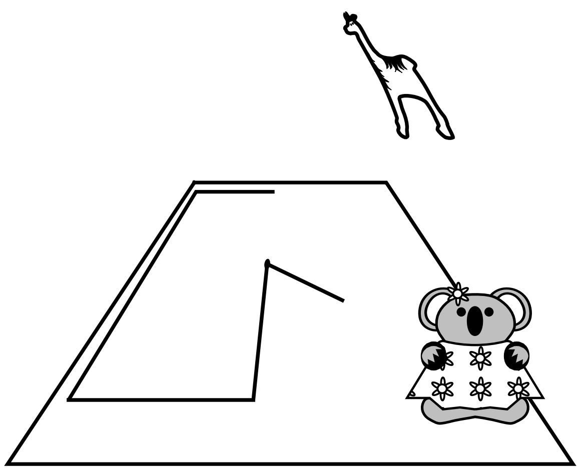
?

Sugarbear now says "Greetings from the third-dimension." Louie Llama is quite perplexed—he only knows two dimensions. To help him out Sugarbear tries to tell him what a cube is.

Question Can you help Sugarbear describe what a cube is to a two-dimensional Flatlander? How might Louie Llama react to a real cube?

?

At this point, Sugarbear gets an idea. Why not "pop" Louie Llama out of the plane so that he can see Flatland the way we do? Being rather rambunctious for a bear, Sugarbear decides to do this sending Louie Llama off into the air floating above Flatland. This is a very strange experience for Louie Llama.



As he floats above Flatland, he can see his world in a way that he had never seen before. He can see inside closed building and even inside his friends! Finally, he drifts back down into Flatland. From his friends' perspective he just "appeared" out of nowhere. When his friends ask him where he came from, he can only reply, "up" but cannot show them where this is. Louie Llama's worldview has changed forever.

At the end of her vacation in Flatland, Sugarbear turns to you and asks you the following question:

> If Flatlanders had no idea that there could be a third dimension, who's to say that we are not as blind as they are. Could there be a fourth dimension?

Question What would *you* say to Sugarbear? Be careful not to upset her, her claws are quite sharp.

?

While I honestly don't know if there is a *real* fourth dimension, we can work by analogy to figure out what it would be like should it actually exist. We'll start with dimension 0 and see how high we can work up to.

Zero Dimensions A 0-dimensional object is just a point. Draw a point in the box below:

One Dimension A 1-dimensional object is a segment, what you get from connecting two points. Draw two points connected by a segment below:

Two Dimensions There are many different 2-dimensional objects, we'll make a square. To do this, simply copy the segment you drew above and connect corresponding end-points. Let's see it in the box below:

Three Dimensions If you're playing along at home, and I sincerely hope you are, you may realize that we are going to have a bit of trouble at this point. No matter, instead of drawing a real cube, we'll draw a "shadow" of a cube. To do this, simply copy the square you drew before, place it next to the original but offset a tad. Then connect corresponding vertexes. Let's see your shadow of the cube in the box below:

Four Dimensions Wow, if we had trouble with three-dimensions, what do we do now? The answer is simple. We do the same thing we've done before. Simply copy your shadow of the cube and connect corresponding vertexes. Let's see it in the box below:

Pretty strange eh? This is called a **hypercube**.

Higher Dimensions We need not stop at dimension four, but as a gesture of friendship—we will. If we had continued on, we would make a shadow of a hyperhypercube! That would be pretty mind-blowing.

Some Observations and Thoughts Living in our three-dimensional world as we do, it truly seems utterly impossible that there could be other dimensions. Yet, it is starting to seem that perhaps our common sense isn't so sensible. Equipped with the tool of mathematics, physicist are discovering that our universe is filled with symmetry. The symmetries we are finding cannot be explained as the symmetry of a 2-dimensional or 3-dimensional objects—to explain the observed phenomenon, we need higher dimensions.

Problems for Section 3.3

(1) For each of the activities below, explain if they are essentially 1-dimensional, 2-dimensional, or 3-dimensional.

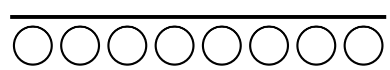
(a) Driving a car down the freeway.
(b) Riding on a roller coaster.
(c) Driving a car in a parking-lot.
(d) Driving a snow-machine.
(e) Flying an airplane.
(f) Swimming underwater.

In each case explain your reasoning.

(2) Suppose you stuck your fingers into Flatland. Describe what a Flatlander would see.

(3) Suppose a 4-dimensional creature stuck its "fingers" into our three dimensional space. What do you think we would see?

(4) Do you know what *bubble tea* is? If not, look it up. It is drunk with a large straw that can be used to suck up tapioca balls. Some people love bubble tea, others hate it. If you were a tapioca ball in a bubble tea straw, you could only move forward and backward, like this:



How many dimensions would you view your world in? How many dimensions would your world actually have? Explain your reasoning.

(5) Fill in the following table:

	Vertexes	Edges	Faces	Solids
Point				
Segment				
Square				
Cube				
Hypercube				
Hyperhypercube				

(6) Someone once told me that a simple way to draw a hypercube was to draw a regular octagon and then simply make squares on the inside of each edge of the octagon. Does this work? Explain your reasoning.

(7) We've seen how to draw n-dimensional cubes. Explain how to draw n-dimensional triangles. Draw a triangle, a tetrahedron, and a hypertetrahedron. Explain your reasoning.

(8) An n-sphere is the set of points in n-dimensions that are all equidistant from a given point. Using this definition, explain why:

(a) A 2-sphere is a circle.

(b) A 3-sphere is a sphere.

In addition, explain what a 1-sphere would be and do your best to describe a 4-sphere.

(9) Consider a square whose side has a length of 2 units.

(a) Compute its perimeter and area.

(b) Increase the side-length of this square by a factor of 3. What is the new perimeter and area?

Explain your reasoning.

(10) Consider a cube whose side has a length of 2 units.

(a) Compute its perimeter, area, and volume.

(b) Increase the side-length of this square by a factor of 3. What is the new perimeter, area, and volume?

Explain your reasoning.

(11) Consider a hypercube whose side has a length of 2 units.

(a) Compute its perimeter, area, volume, and hypervolume.

(b) Increase the side-length of this square by a factor of 3. What is the new perimeter, area, volume, and hypervolume?

Explain your reasoning.

(12) If one can only travel along edges, what is the maximum distance between two vertexes of:

(a) A square who side has length 1 unit.

(b) A cube who side has length 1 unit.

(c) A hypercube who side has length 1 unit.

(d) A hyperhypercube who side has length 1 unit.

(e) An n-dimensional cube who side has length 1 unit.

Explain your reasoning.

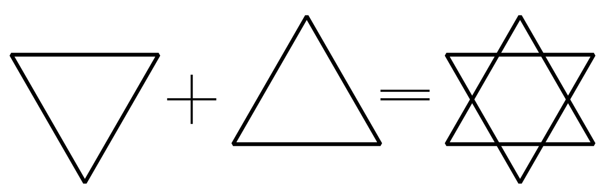
Chapter 4

String Art

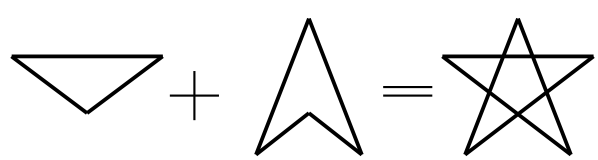
> The artist is a receptacle for the emotions that come from all over the place: from the sky, from the earth, from a scrap of paper, from a passing shape, from a spider's web.
>
> —Pablo Picasso

4.1 I'm Seeing Stars

Can you remember when you first learned to draw a star? I can! I was first taught to draw a star like this:

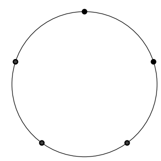


But I wanted to know how to draw 5-pointed stars like the other kids. So one day I taught myself to draw a star like this:

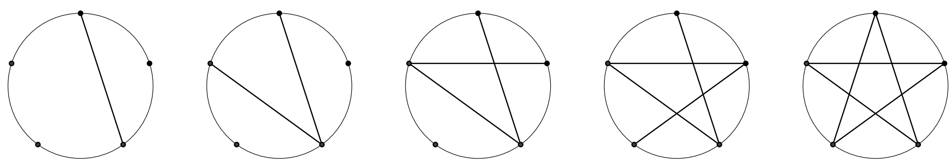


This may seem like a silly way to draw this star—it certainly makes me chuckle now. Let's see if we can give a theory for drawing stars that will connect the two different stars above, teach you how to draw some new stars, and learn a little more mathematics along the way.

We'll start with 5 points equally spaced on a circle:

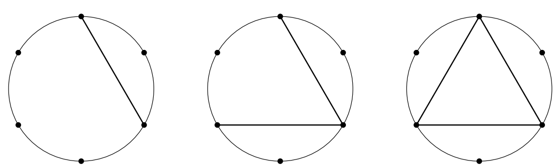


Think of these points as *pins*, next we'll start at the top and draw lines that we'll think of as *strings*, moving clockwise to the point two steps away each time:

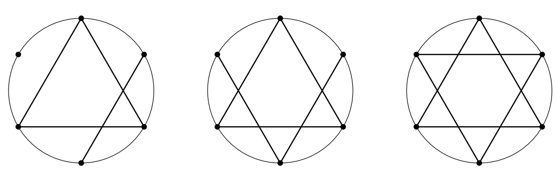


Check it out, we got a star! If we had used pins and string to make this star, we could have done it with one piece of string.

Now start with 6 points equally spaced on a circle. Again, we'll move clockwise two steps each time:



Oops, we've run out of points! No worries, just start again at one of the points that hasn't been touched by a line yet, drawing lines and moving two steps each time:



Ah! The other star! In this case, if we used pins and strings, we'd need to use *two* pieces of string. Hmmm—at this point I have a question:

Question How do we actually draw n equally spaced points on a circle?

?

You can figure that one out—or just ask someone who is standing up at the front of some room talking to themselves. Oh! And I have another question:

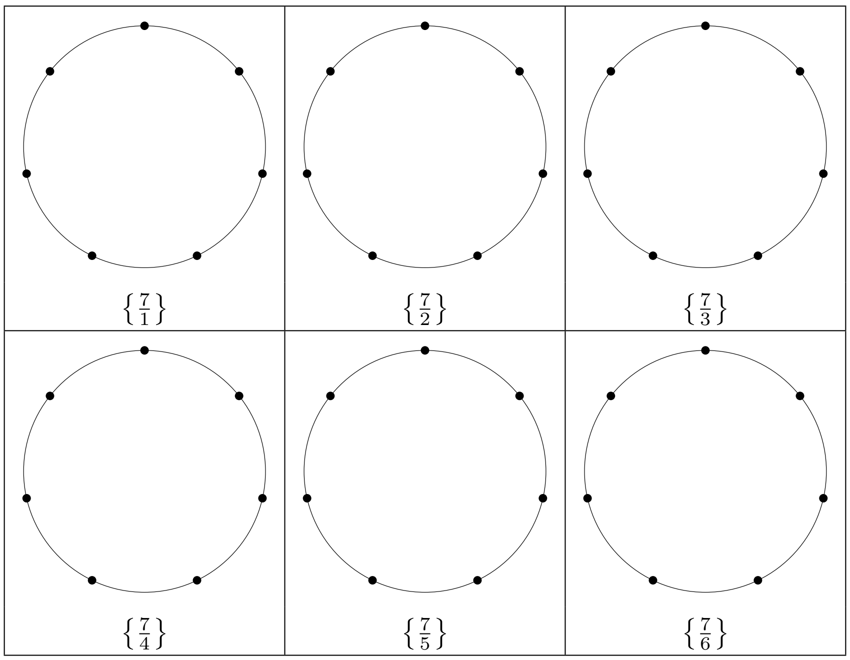
Question What happens if you put n points, and connect them by moving s steps each time?

I'm feeling generous, so I'll help out with this one. Let's do a bunch of examples together. I'm hoping that the following notation will help out:

$$\left\{\frac{n}{s}\right\} = \text{the star with } n \text{ points where we move } s \text{ steps}$$

WARNING *Remember,* $\left\{\frac{n}{s}\right\}$ *is a star* ***not*** *a fraction!*

Armed with our new notation, let's draw the following stars!



Question Do you notice any patterns? Why not share your observations with a friend—or an enemy?

?

Question Looking back, we see that some stars are one piece like $\left\{\frac{7}{2}\right\}$. Other stars are more than one piece, like $\left\{\frac{6}{2}\right\}$. What's the general rule?

?

Problems for Section 4.1

(1) Explain what is meant by the symbol $\left\{\frac{n}{s}\right\}$.

(2) Explain how to draw n equally spaced points on a circle.

(3) Use a compass and protractor to draw $\left\{\frac{3}{s}\right\}$ for $s = 1, 2$. Which of these look the same? How many pieces are there in each case?

(4) Use a compass and protractor to draw $\left\{\frac{4}{s}\right\}$ for $s = 1, 2, 3$. Which of these look the same? How many pieces are there in each case?

(5) Use a compass and protractor to draw $\left\{\frac{5}{s}\right\}$ for $s = 1, 2, 3, 4$. Which of these look the same? How many pieces are there in each case?

(6) Use a compass and protractor to draw $\left\{\frac{6}{s}\right\}$ for $s = 1, 2, 3, 4, 5$. Which of these look the same? How many pieces are there in each case?

(7) Use a compass and protractor to draw $\left\{\frac{7}{s}\right\}$ for $s = 1, \ldots, 6$. Which of these look the same? How many pieces are there in each case?

(8) Use a compass and protractor to draw $\left\{\frac{8}{s}\right\}$ for $s = 1, \ldots, 7$. Which of these look the same? How many pieces are there in each case?

(9) Use a compass and protractor to draw $\left\{\frac{9}{s}\right\}$ for $s = 1, \ldots, 8$. Which of these look the same? How many pieces are there in each case?

(10) Fill in the following table with stars:

$n\backslash s$	1	2	3	4	5	6	7	8
3								
4								
5								
6								
7								
8								

Here you may simply sketch the stars, though the sketch must be clear and show the distinguishing features of the stars.

(11) Give three different stars each made of 2 connected pieces. Explain your reasoning.

(12) Give three different stars each made of 3 connected pieces. Explain your reasoning.

(13) Give three different stars each made of 4 connected pieces. Explain your reasoning.

(14) Give three different stars each made of 5 connected pieces. Explain your reasoning.

(15) Draw $\left\{\frac{7}{9}\right\}$, what happens? Explain why we never need to discuss the case $s \geqslant n$ when drawing $\left\{\frac{n}{s}\right\}$.

(16) Perceptive Persephone claims that $\left\{\frac{5}{2}\right\}$ is *not* equal to $\left\{\frac{5}{3}\right\}$. They look the same to me! Please explain what is she going on about.

(17) Give a precise description of when two stars $\left\{\frac{n}{s}\right\}$ and $\left\{\frac{n}{t}\right\}$ will look the same. Explain why your description holds.

(18) Give a precise description based on n of how many different looking stars $\left\{\frac{n}{s}\right\}$ can be formed. Explain why your description holds.

(19) Give a precise description based on n of when a star $\left\{\frac{n}{s}\right\}$ is one piece for all values of s. Explain why your description holds.

(20) Give a precise description based on n and s of when a star $\left\{\frac{n}{s}\right\}$ is one piece. Explain why your description holds.

(21) Explain how to figure out how many pieces a star $\left\{\frac{n}{s}\right\}$ will have *without* drawing the star.

(22) Give a precise description of what happens when we "reduce to lowest terms." As an example, think about how the stars $\left\{\frac{6}{2}\right\}$ and $\left\{\frac{3}{1}\right\}$ are related. Explain why your description holds.

(23) Draw $\left\{\frac{4}{2}\right\}$, $\left\{\frac{6}{3}\right\}$, $\left\{\frac{8}{4}\right\}$, what do you notice? Let's call these stars *asterisk stars*. Give a precise description of when you can draw asterisk stars—and when you cannot draw asterisk stars. Explain why your description holds.

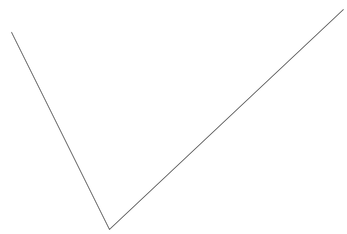
(24) Interested Isabel finds $\left\{\frac{7}{2}\right\}$ and $\left\{\frac{7}{3}\right\}$ interesting. Why?

(25) Interested Isabel finds $\left\{\frac{10}{2}\right\}$ and $\left\{\frac{10}{4}\right\}$ interesting. Why?

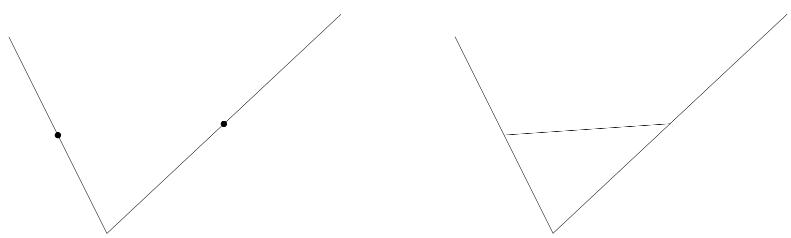
4.2 Drawing Curves

4.2.1 Envelope of Tangents

Get out a piece of paper, a ruler, and a writing utensil. Really do it! Seriously! Anyway here is what we're going to do, draw two lines that come to a point:

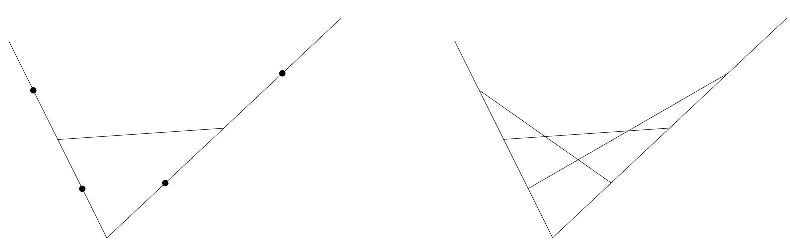


Now take both of those lines and mark the midpoints. Next connect the midpoints with a line:



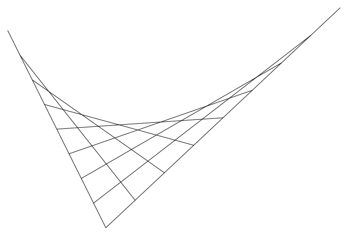
1-line envelope of tangents

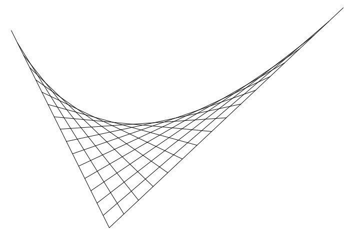
Each original line is now divided into two segments. Mark the midpoints of those segments and cross-connect them:



3-line envelope of tangents

If you continue this cross-connecting process, you'll get groovy pictures like these:


7-line envelope of tangents


15-line envelope of tangents

Over 2000 year ago, Apollonius of Perga explained why the curve we just produced is a parabola. Today we call what we just drew an *envelope of tangents*. If we cross-connect n pairs of lines we'll call this an n*-line envelope of tangents*.

Question What does it mean for a line to be tangent to a curve?

?

4.2.2 Bézier Curves

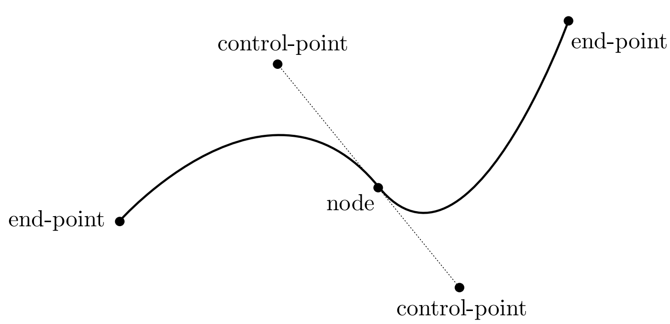
While some of us like pointy drawings, others like smooth drawings. To make smooth drawings, we'll need the help of *Pierre Bézier*, pronounced "beh-zeeay." Here is the deal, Pierre Bézier was an engineer with the Renault car company in the 1960's. He needed a convenient way to draw curves, a way that would allow curves to be:

(1) Easy to render—meaning easy for computer (or person!) to reproduce the curve described by the artist.

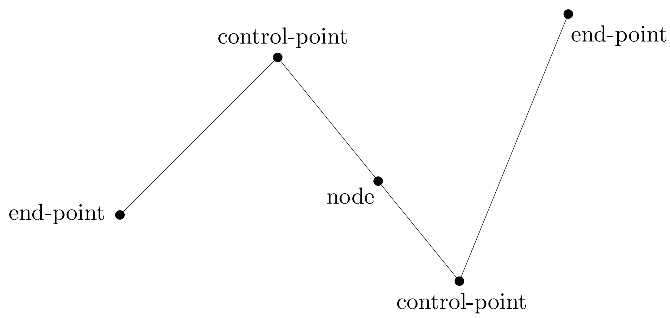
(2) Easy to transform—meaning easy to manipulate the curve with geometric transformations.

(3) Easy to draft—meaning easy for an artist to shape a desired curve.

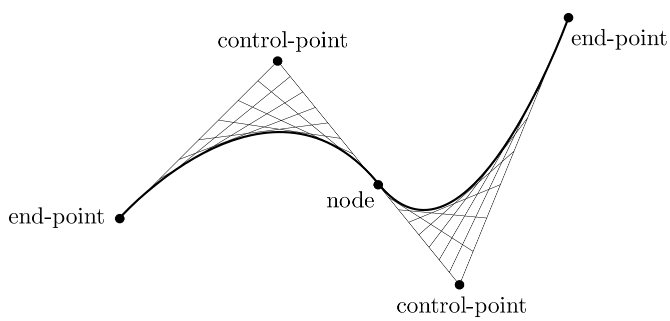
Given two end-points, Bézier devised a way to work with curves via a *node* and two *control-points*:



With the end-points remaining fixed, the artist can move the node and control-points to change the way the curve looks. But how do those so-called control-points actually control the way the curve looks? Well the end-points, node, and control-points define a polygonal-path:



This polygonal-path in turn gives an envelop of tangents, which then gives the desired curve:



While this may seem all fancy and highfalutin, this is the essence of Bézier's method for drawing curves. It has become so popular that this is basically how curves are currently drafted using vector graphics programs such as *Adobe Illustrator* and *Inkscape*. Why are Bézier curves so popular? We'll see that they satisfy the requirements above quite well.

Easy to Render A computer does not draw the parabola via an envelope of tangents—however, the process we use above can be done by hand. The key point is that one can obtain an arbitrary amount of smoothness to the curve by simply adding more lines to the envelope of tangents.

Easy to Transform Each curve is completely determined by the polygonal-path from the first end-point to first control-point to the node to the other control-point and the final end-point. Hence if I simply handed you five points:

$$(1,4) \to (4,2) \to (6,6) \to (7,8) \to (9,4)$$

you can then draw a curve, and tell someone else how to draw the exact same curve. Moreover, if you take a geometric transformation like one of the matrices we've studied before, you could transform the entire curve merely by transforming those five points.

Easy to Draft Of course, as you are aware, most graphic artists don't think much about coordinates and yet are still able to draw Bézier curves with impunity. This is really the genius of Bézier's method. By fixing the end-points, one only need to know the position of the node and the control points. So by using a mouse, an artist can move the control-points and node to get a plethora of interesting curves.

Question What interesting pictures can *you* make with Bézier curves?

?

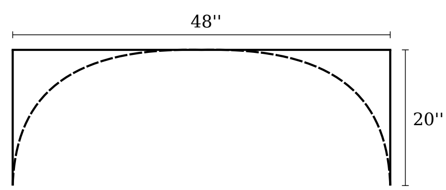
Problems for Section 4.2

(1) Sketch the graph of $y = x^2$. On the same plot, draw a 7-line envelope of tangents defined by the line segments that go from $(2, 4)$ to $(0, -4)$ and $(-2, 4)$ to $(0, -4)$.

(2) Sketch the graph of $y = x^2/8$. On the same plot, draw a 7-line envelope of tangents defined by the line segments that go from $(8, 8)$ to $(0, -8)$ and $(-8, 8)$ to $(0, -8)$.

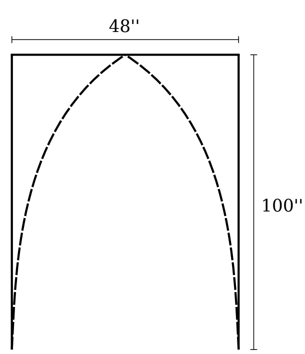
(3) Use a 7-line envelope of tangents to approximate one-quarter of a circle. Compare your drawing with an actual quarter of a circle. What do you notice?

(4) Your client gives you the following rough sketch of an arched ceiling in a hallway:



Give an accurately scaled drawing of this arch, and explain how it can be enlarged to actual-size.

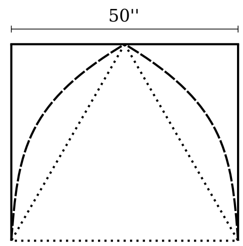
(5) Your client gives you the following rough sketch of a lancet arch:



Use an envelope of tangents to give an accurately scaled drawing of this arch, and explain how it can be enlarged to actual-size.

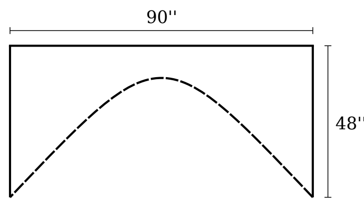
(6) Your client gives you the following rough sketch of a pointed arch where

they want the dotted triangle to be an equilateral triangle:



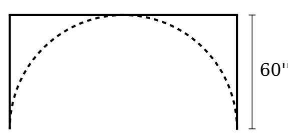
Use an envelope of tangents to give an accurately scaled drawing of this arch, and explain how it can be enlarged to actual-size.

(7) Your client gives you the following rough sketch of a parabolic arch:



Use an envelope of tangents to give an accurately scaled drawing of this arch, and explain how it can be enlarged to actual-size.

(8) Your client gives you the following rough sketch of a round arch:



Give *two* accurately scaled drawings of this arch: One given by envelope of tangents, and one that is actually semi-circular. In both cases, explain how your drawings can be enlarged to actual-size.

(9) Suppose you draw an envelope of tangents by continually taking midpoints as we did above. This will form a 1-line envelope of tangents, then a 3-line envelope of tangents, then a 7-line envelope of tangents, then a 15-line envelope of tangents, and so on. What is the pattern? Explain why the pattern holds.

(10) Concerning Bézier curves, why are the control points and node always on a straight line? What would happen if they weren't?

(11) The following five points determine a polygonal-path:

$$(1,4) \to (4,2) \to (6,6) \to (7,8) \to (9,4)$$

Using a 7-line envelope of tangents, draw the correspond Bézier curve.

(12) The following five points determine a polygonal-path:

$$(1, 2) \to (2, 7) \to (4, 6) \to (8, 4) \to (9, 7)$$

Using a 7-line envelope of tangents, draw the correspond Bézier curve.

(13) The following five points determine a polygonal-path:

$$(4, 2) \to (1, 6) \to (4, 6) \to (6, 6) \to (8, 9)$$

Using a 7-line envelope of tangents, draw the correspond Bézier curve.

(14) The following five points determine a polygonal-path:

$$(1, 4) \to (4, 2) \to (6, 6) \to (7, 8) \to (9, 4)$$

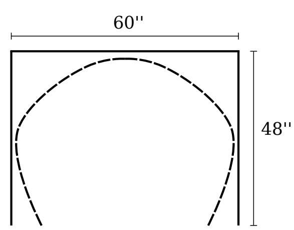
Using a 7-line envelope of tangents, draw the correspond Bézier curve.

(15) The following five points determine a polygonal-path:

$$(3, 1) \to (1, 6) \to (5, 7) \to (9, 8) \to (8, 3)$$

Using a 7-line envelope of tangents, draw the correspond Bézier curve.

(16) Your client gives you the following rough sketch of a horseshoe arch:



Use an envelope of tangents to give an accurately scaled drawing of this arch, and explain how it can be enlarged to actual-size.

(17) Compare the two polygonal-paths:

$$(1, 2) \to (2, 7) \to (4, 6) \to (8, 4) \to (9, 7)$$
$$(1, 2) \to (8, 4) \to (4, 6) \to (2, 7) \to (9, 7)$$

Using a 7-line envelope of tangents, draw the correspond Bézier curve for each. What does this tell you about the order of the points in the polygonal-path?

(18) In the examples of polygonal-paths we've given you so far

$$\mathbf{a} \to \mathbf{b} \to \mathbf{c} \to \mathbf{d} \to \mathbf{e}$$

the points **b**, **c**, and **d** have all been on a line. What will happen if they are not on a line? Explain your reasoning.

(19) The following points determine a polygonal-path:

$$(8, 0) \to (0, 0) \to (0, 8)$$

(a) Draw the corresponding 7-line envelope of tangents.
(b) Apply the matrices

$$\mathsf{R}_{90}, \qquad \mathsf{R}_{90}^2, \qquad \mathsf{R}_{90}^3$$

to your points and in each case, draw the corresponding 7-line envelope of tangents.

Show all your work.

(20) The following points determine a polygonal-path:

$$(8, 0) \to (8, 8) \to (0, 8)$$

(a) Draw the corresponding 7-line envelope of tangents.
(b) Apply the matrices

$$\mathsf{R}_{90}, \qquad \mathsf{R}_{90}^2, \qquad \mathsf{R}_{90}^3$$

to your points and in each case, draw the corresponding 7-line envelope of tangents.

Show all your work.

(21) The following points determine a polygonal-path:

$$(-4, -4) \to (0, -4) \to (0, 0) \to (0, 4) \to (4, 4)$$

(a) Using a 7-line envelope of tangents, draw the corresponding Bézier curve.
(b) Apply the matrices

$$\mathsf{R}_{90}, \qquad \mathsf{R}_{90}^2, \qquad \mathsf{R}_{90}^3$$

to your points and in each case, draw the corresponding 7-line envelope of tangents.

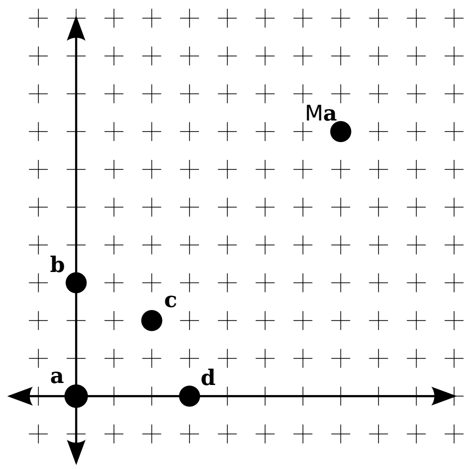
Show all your work.

Appendix A

Activities

A.1 What is an Isometry?

In this activity, we are going to explore the geometry of isometries.



1) Remind me, gentle reader, what is the definition of an isometry?

2) Suppose that some isometry M moves point $\mathbf{a}$ to $\mathsf{M}\mathbf{a}$. Carefully draw all possible locations for $\mathsf{M}\mathbf{b}$.

3) Now suppose that the isometry M moves point $\mathbf{c}$ to $\mathsf{M}\mathbf{c} = (5,5)$. Indicate all possible locations for $\mathsf{M}\mathbf{b}$.

4) Now suppose that the isometry M moves point $\mathbf{d}$ to $\mathsf{M}\mathbf{d} = (7,4)$. Indicate all possible locations for $\mathsf{M}\mathbf{b}$.

5) Finally, consider point $\mathbf{e} = (1,5)$. Where is $\mathsf{M}\mathbf{e}$?

A.2 Who Mapped the What Where?

Let M represent some mysterious matrix that maps the plane to itself. So M is of the form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

1) If I tell you that $\mathbf{p} = (3, 4)$ and

$$\mathsf{M}\mathbf{p} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix},$$

give 3 possible matrices for M.

2) Now suppose that in addition to the fact above, I tell that

$$\mathsf{M}\mathbf{o} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

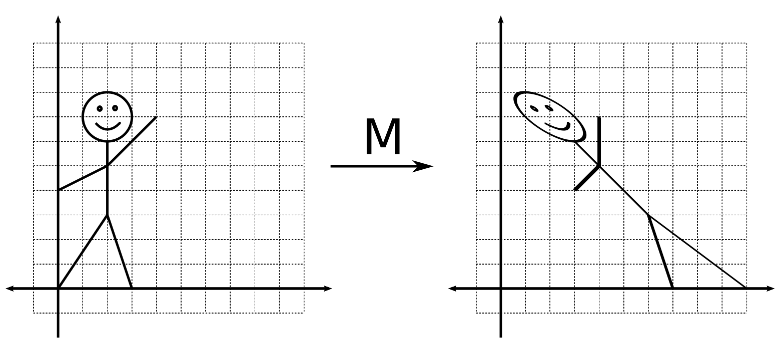
where **o** is the origin. Give 3 possible matrices for M.

3) Now suppose that in addition to the two facts above, I tell you that

$$\mathsf{M}\mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

where $\mathbf{q} = (1, 1)$. How many possibilities do you have for M now? What are they?

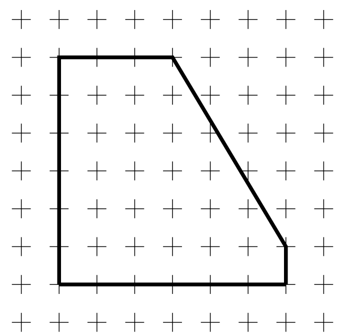
4) Here is a picture of my buddy *Sticky*:



As you can see, he's been dancing with some matrix M. Can you tell me which matrix it was? What are good points to pay attention to?

A.3 A Bigger Apartment Please

Marcy Matrix designs apartments. Here is her latest design:



In the picture above, one square has a side-length of $5'$. Marcy's client, Large Linus, is unhappy with the design. He insists that it must be three times bigger meaning he wants every dimension of his apartment bigger! For Marcy this is no problem at all because she knows about the following *dilation* matrix:

$$\mathsf{D}_s = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This handy matrix will dilate (think scale) pictures by a factor of s. However, there is one minor hiccup: Does Linus want the *perimeter* or the *area* of his apartment to be three times bigger?

1) What is the current perimeter of Linus' apartment? What is the current area of Linus' apartment?

2) Apply D_3 to every point of the apartment. What is the new perimeter? What is the new area?

3) What matrix should Marcy use to make the area of Linus' apartment three times bigger? Use this matrix to actually scale the apartment. What is the perimeter in this case?

4) Once someone told me that "five square feet" is the same as "five feet square." Do you agree or disagree with this statement—explain how you arrived at your conclusion.

5) Maverick Metric likes Linus' apartment, but he wants the measurements to be metric. He asks you for the measurement for the area of Linus' apartment in feet. Since there are around .3 feet per meter, Maverick Metric claims that the area of the apartment in meters is:

$$\text{area of apartment in feet} \times .3$$

Is this correct? Hint: No. Explain why not and solve the problem correctly.

A.4 Symmetries of the Square

Alright—let's write down the symmetries of the square!

A.5 Are You Getting Stairs?

Most building codes have very specific restrictions on how a staircase can be designed. Here are typical restrictions:

(a) The stair-step height can be at most $7\frac{3}{4}''$.

(b) The stair-step depth must be at least $10''$.

(c) From step-to-step, these dimensions cannot change (in real-life there is a minimum allowed variance for these dimensions).

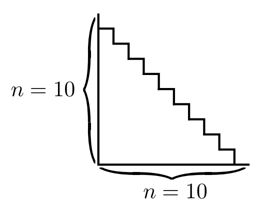
1) If you wish to build a staircase that is $15'$ wide, how tall can it be?

2) If you wish to build a staircase that is $12'$ tall, what is the minimum width that can it be?

Allow me to suggest using the following friendly matrix for the remainder of the problems:

$$\mathsf{D}_{s,t} = \begin{bmatrix} s & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a geometric transformation that will dilate a picture by a scale factor of s in the horizontal direction and a scale factor of t in the vertical direction. We'll use this to transform a "generic" staircase of n stairs to fit our needs:



The upshot is that in essence, to lay a staircase of n stairs that is up to code in a space that is w feet wide and h feet tall, you must solve the following equation:

$$\mathsf{D}_{s,t} \begin{bmatrix} n \\ n \\ 1 \end{bmatrix} = \begin{bmatrix} w \\ h \\ 1 \end{bmatrix}$$

where $t \leqslant \frac{31}{48}$ and $s \geqslant \frac{5}{6}$.

3) Where do the numbers

$$\frac{31}{48} \qquad \text{and} \qquad \frac{5}{6}$$

come from?

4) Carefully draw an (s,t)-plane, with s on the horizontal axis and t on the vertical axis.

(a) Shade in the region $t \leqslant \frac{31}{48}$ and $s \geqslant \frac{5}{6}$.

(b) Expand the left-hand side of the following equation:

$$\mathsf{D}_{s,t} \begin{bmatrix} n \\ n \\ 1 \end{bmatrix} = \begin{bmatrix} w \\ h \\ 1 \end{bmatrix}$$

(c) Now you should be able to write two equations. In each case, solve for n and set these two equations equal to each other.

(d) Solve for t. Now you should have a single equation $t = \ldots$.

(e) Now you can set values for w and h. Set $w = 15$ and $h = 11$. Plot your equation on the (s,t)-plane—what does this mean?

5) Suppose you wish to build a staircase that is $15'$ wide and $11'$ tall. What should be the dimensions of each step? How many different solutions can you find?

6) Suppose you wish to build a staircase that is $15'$ wide and $10'$ tall. What should be the dimensions of each step? How many different solutions can you find?

7) Suppose you wish to build a staircase that is $16'$ wide and $10'$ tall. What should be the dimensions of each step? How many different solutions can you find?

8) Smart Sam says he doesn't need matrices to solve Problem 5. Instead, he says:

> First you take the height of the staircase, $11'$ and divide it by $31/48'$ to get 18. Now take $15'$ and divide by 18 to get $5/6'$. So you have 18 steps that are $5/6'$ deep and $31/48'$ tall.

His arithmetic seems a bit strange to me, what is he doing? Is his solution different from what we did above? Hint: There are at least two errors in his solution, one that is minor and another that is major.

A.6 How Many?

In this activity, we'll investigate how many symmetries a regular n-gon has. To start us off, let's remember what it means for something to have symmetry. Roughly speaking, an object has symmetry if you can do something to it, and it somehow remains unchanged.

1) What does it mean for a polygon to be regular?

2) How many ways can you write down the letters A, B, and C in groups of 3? Explain your reasoning.

3) How many symmetries does a regular 3-gon have? Draw pictures and list them.

4) How many ways can you write down the letters A, B, C, and D in groups of 4? Explain your reasoning.

5) How many symmetries does a regular 4-gon have? Draw pictures and list them.

6) How many symmetries does a regular 5-gon have? Draw pictures and list them.

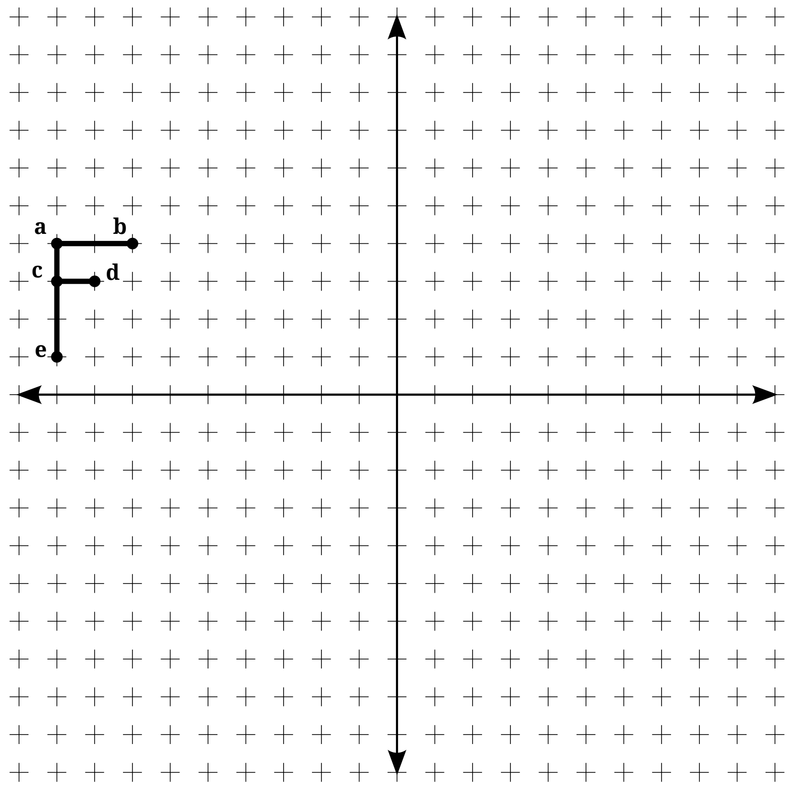
7) How many symmetries does a regular 6-gon have? Draw pictures and list them.

8) How many symmetries does a regular n-gon have when n is odd? Explain your reasoning.

9) How many symmetries does a regular n-gon have when n is even? Explain your reasoning.

A.7 A Moving Image

In this activity, we are going to explore how reflections are related to translations. Consider the following “F” shape:



1) Use two reflections to translate the “F” shape 8 units to the right. Don’t worry about the actual matrices involved.

2) What reflections would translate the original “F” shape 4 units to the right? Again, don’t worry about the actual matrices involved.

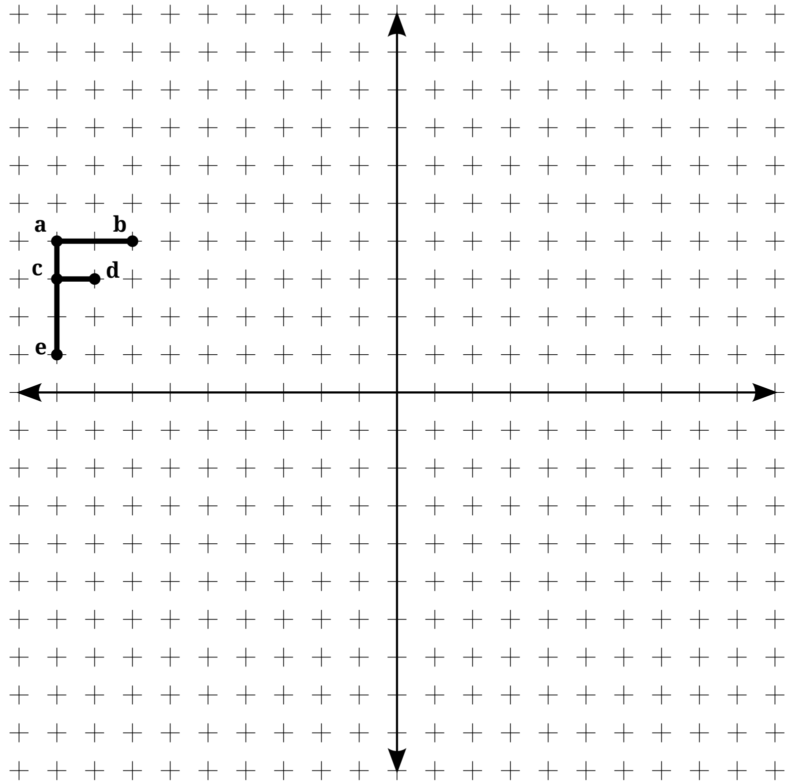
3) Can you generalize the two problems above? What composition of two reflections will give a horizontal translation of n units to the right?

4) Now suppose you wish to move the original “F” shape 4 units to the right, but you will first reflect via this matrix:

$$\mathsf{F}_{x=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

What is the other matrix that is needed?

5) Generalize the problem above: Suppose you wish to translate the original "F" shape n units to the right, and you start by reflecting by $\mathsf{F}_{x=0}$. What is the second matrix that you apply?



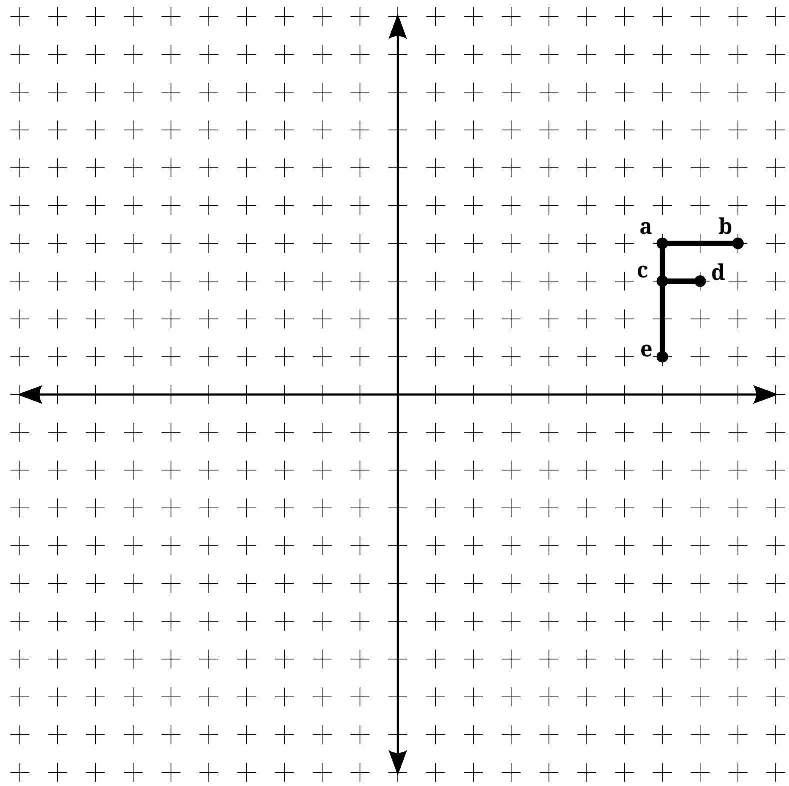
A.8 All You Need Are Reflections

In this activity, we are going to explore how reflections are related to rotations. When doing this activity, you don't need to worry about the actual matrices involved.

1) You may have noticed that 180° rotations and horizontal (or vertical) reflections seem kind of similar. Explain how they are similar, then explain how they are different.

2) Explain how to express a 180° as the composition of two reflections. Work out some examples to give evidence that you are correct.

Consider the following "F" shape:



3) Reflect our shape across the line $y = x$. Label the reflection of the points with a prime—the reflection of $\mathbf{a}$ across $y = x$ is $\mathbf{a}'$.

4) Reflect our new shape across $x = 0$. Label the reflection of the points with another prime—the reflection of $\mathbf{a}'$ across $x = 0$ is $\mathbf{a}''$.

5) Through how many degrees was our original shape rotated about the origin?

6) Letting **o** be the origin and "–" be a place-holder, use a protractor to fill in the described angles:

–	–, **o**, and –′	–′, **o**, and –″	–, **o**, and –″
a			
b			
c			
d			
e			

What do you notice?

7) Letting **o** be the origin and "–" be a place-holder, use a protractor to fill in the described angles:

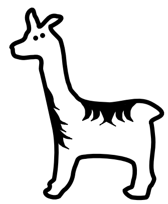
–	–, **o**, and $y = x$	–′, **o**, and $x = 0$	the sum
a			
b			
c			
d			
e			

What do you notice?

8) Suppose I want to rotate an object $\theta°$ around the origin using two reflections. Can you conjecture how the lines should be placed on the plane? Draw pictures to help with your explanation.

A.9 Louie Llama and the Triangle

We are going to investigate why the interior angles of a triangle sum to 180°. We won't be alone on this journey, we'll have help. Meet Louie Llama:



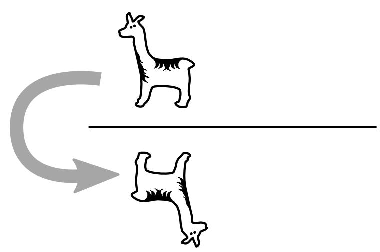
Louie Llama is rather radical for a llama and doesn't mind being rotated at all.

1) Draw a picture of Louie Llama rotated 90° counterclockwise.

2) Draw a picture of Louie Llama rotated 180° counterclockwise.

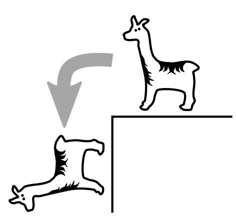
3) Draw a picture of Louie Llama rotated 360° counterclockwise.

4) Sometimes Louie Llama likes to walk around lines he finds:



Through what angle did Louie Llama just rotate?

Now we're going to watch Louie Llama go for a walk. Draw yourself any triangle, draw a crazy scalene triangle—those are the kind that Louie Llama likes best. Louie Llama is going to proudly parade around this triangle. When Louie Llama walks around corners he rotates. Check it out:



Take your triangle and denote the measure of its angles as a, b, and c. Start Louie Llama out along a side adjacent to the angle of measure a. He should be on the outside of the triangle, his feet should be pointing toward the triangle, and his face should be pointing toward the angle of measure b.

5) Sketch Louie Llama walking to the angle of measure b. Walk him around the angle. As he goes around the angle his feet should always be pointing toward the triangle. Through what angle did Louie Llama just rotate?

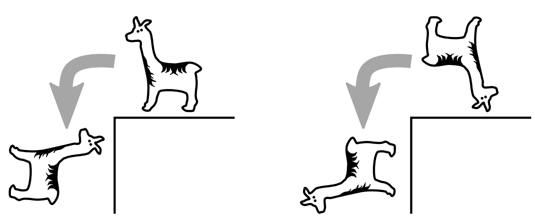
6) Sketch Louie Llama walking to the angle of measure c. Walk him around the angle. Through what angle did Louie Llama just rotate?

7) Finally sketch Louie Llama walking back to the angle of measure a. Walk him around the angle. He should be back at his starting point. Through what angle did Louie Llama just rotate?

8) All in all, how many degrees did Louie Llama rotate in his walk?

9) Write an equation where the right-hand side is Louie Llama's total rotation and the left-hand side is the sum of each rotation around the angle. Can you solve for $a + b + c$?

As you may have guessed, Louie Llama isn't your typical llama, for one thing he likes to walk backwards and on his head! He also like to do somersaults. Louie Llama can somersault around corners in two different ways:



10) What does Louie Llama's somersault have to do with the angle of the corner? Can you precisely explain how Louie Llama rotates when he somersaults around corners?

11) Can you walk Louie Llama around your original triangle allowing him to walk backwards (or even on his head!), letting him do somersaults as he pleases around corners, and **directly** arrive at the equation

$$a + b + c = 180°?$$

12) Can you rephrase what we did above in terms of *exterior angles* and *interior angles*?

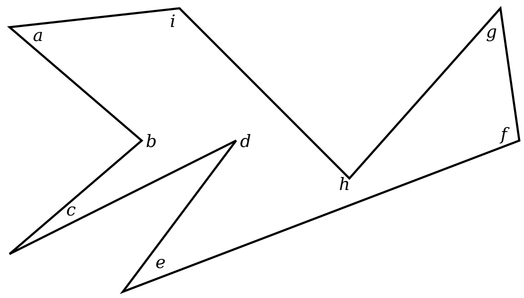
13) Can you walk Louie Llama around other shapes and figure out what the sum of their interior angles are? Let's do this with a table:

n-gon	sum of interior angles	interior angle of a regular n-gon
3		
4		
5		
6		
7		
8		
n		

A.10 Angles in a Funky Shape

We are going to investigate the sum of the interior angles of a funky shape.

1) Using a protractor, measure the interior angles of the crazy shape below:



Use this table to record your findings:

a	b	c	d	e	f	g	h	i

2) Find the sum of the interior angles of the polygon above.

3) What should the sum be? Explain your reasoning.

A.11 Rep-Tiles

A **rep-tile** is a polygon where several copies of a given rep-tile fit together to make a larger, similar, version of itself. If 2 copies are used, we call it a *rep-2-tile*, if 3 copies are used, we call it a *rep-3-tile*, and if n copies are used we call it a *rep-n-tile*.

1) With a separate sheet of paper, draw and cut-out:

(a) An isosceles right triangle whose sides have lengths $1''$, $1''$, and $\sqrt{2}''$.

(b) A rectangle whose sides have lengths $1''$ and $\sqrt{2}''$.

Working with partners, show that each of these polygons is a rep-2-tile.

2) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon?

3) With a fresh sheet of paper, start a table:

rep-n-tile	$\dfrac{\text{new perimeter}}{\text{old perimeter}}$	$\dfrac{\text{new area}}{\text{old area}}$
2	$\vdots$	$\vdots$
3	$\vdots$	$\vdots$

4) Geometry Giorgio suggests that a rectangle who sides have lengths $1''$ and $4''$ is also a rep-2-tile. Is he right? If you should happen to search the internet for other examples of rep-2-tiles, you might find a surprise.

5) With a separate sheet of paper, draw and cut-out:

(a) A 30-60-90 right triangle whose shortest side has length $1''$.

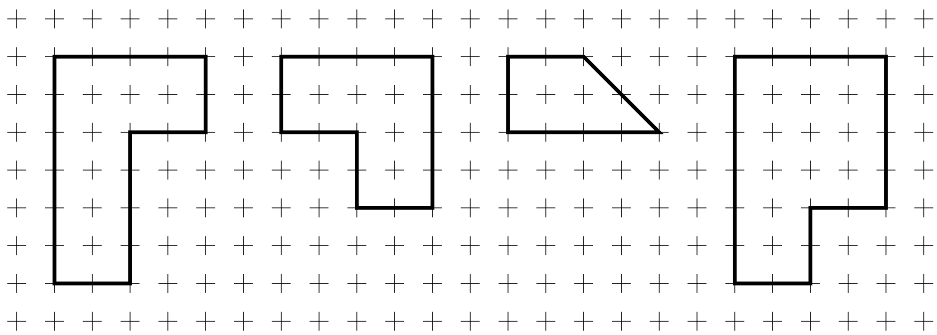
(b) A rectangle whose sides have lengths $1''$ and $\sqrt{3}''$.

Working with partners, show that each of these polygons is a rep-3-tile.

6) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon? Add this information to your table.

7) Explain why every triangle and every parallelogram is a rep-4-tile. Give an examples of both, and compute the perimeter and area. In both cases, compare the perimeter and area to that of the larger polygons.

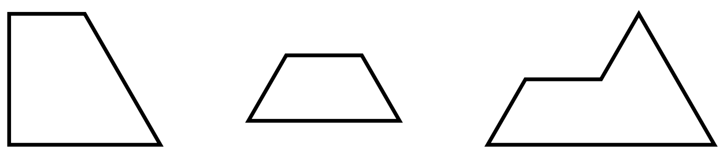
8) With a separate sheet of graph paper, draw and cut-out the following polygons:



Working with partners, show that each of these polygons is a rep-4-tile.

9) For each rep-tile above, compute the perimeter and area. In each case, how does this relate to the perimeter and area of the larger polygon? Add this information to your table.

10) With a separate sheet of paper, trace and cut-out the following polygons:



Working with partners, show that each of these polygons is a rep-4-tile.

11) Explain why every rectangle whose sides have ratio $1 : \sqrt{n}$ is a rep-n-tile.

12) Explain how you know that any rep-tile will tessellate the plane.

13) Give an example of a polygon that tessellates the plane that is not a rep-tile.

14) Every tessellation made by rep-tiles will have **symmetry of scale**. What does it mean to have *symmetry of scale*?

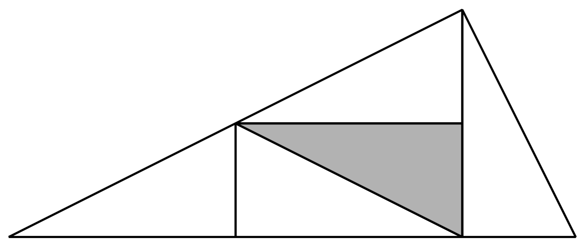
15) Consider the tessellations made by rep-tiles you've seen so far. What other symmetries do they have?

16) Do you think you can have a tessellation that has symmetry of scale but no other symmetries?

A.12 The Pinwheel Tiling

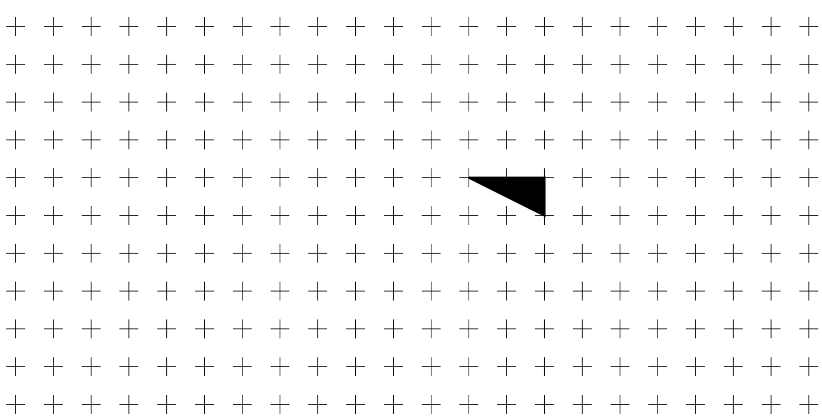
In this activity we are going to investigate a very special rep-tile, the *pinwheel tiling*.

1) The pinwheel tiling is based on the following triangle:



If the shortest side has a length of 1 unit and this is a rep-5-tile, what are the lengths of the other sides? What type of triangle is this?

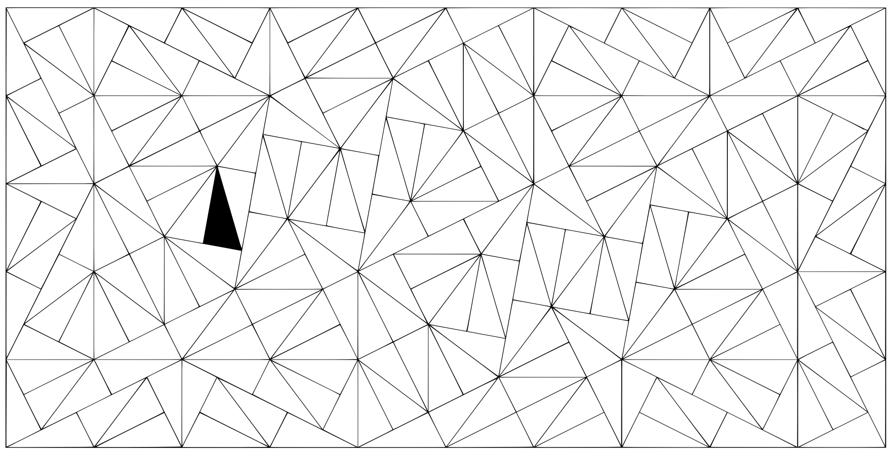
2) Each time we "inflate" the pinwheel rep-tile, we view the new larger triangle as being the shaded triangle in our base rep-tile. Inflate the triangle below two times.



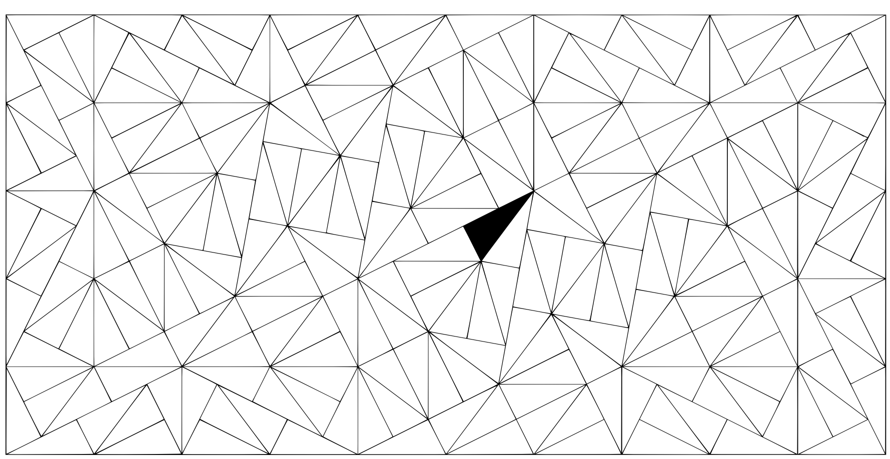
Check with your friend to see that you get the same result.

3) For the rep-tile above, compute the perimeter and area. How does this relate to the perimeter and area of the larger triangle?

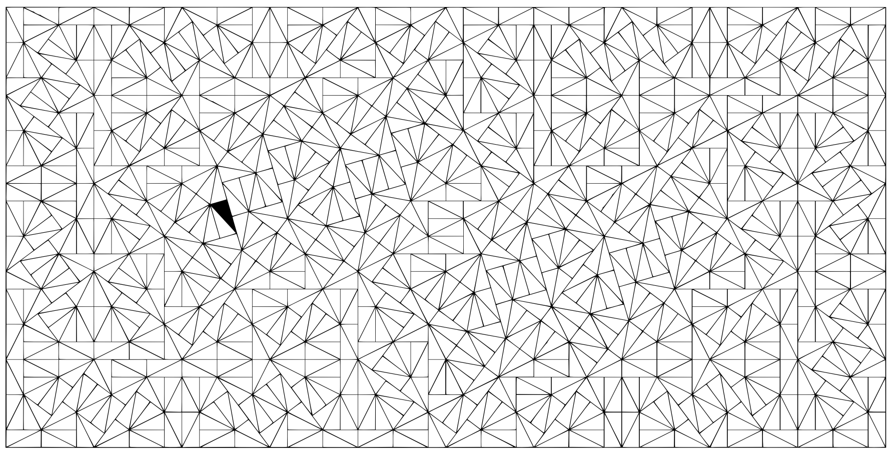
4) In the picture below, shade-in (using colored pencils) the various inflations of the shaded pinwheel rep-tile.



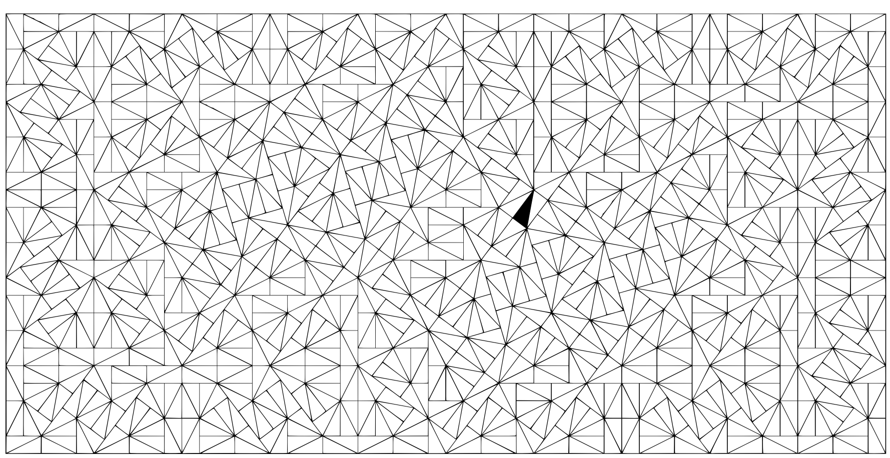
5) While only the shaded triangle above is used to "inflate" the pinwheel rep-tile, every triangle is part of **some** inflation. In the picture below, shade-in (using colored pencils) the various inflations containing the shaded pinwheel rep-tile.



6) In the picture below, shade-in (using colored pencils) the various inflations containing the shaded pinwheel rep-tile.



7) While only the shaded triangle above is used to "inflate" the pinwheel rep-tile, every triangle is part of **some** inflation. In the picture below, shade-in (using colored pencils) the various inflations containing the shaded pinwheel rep-tile.



We claim that the pinwheel tiling has no symmetry through any isometry—yet it still has symmetry of scale.

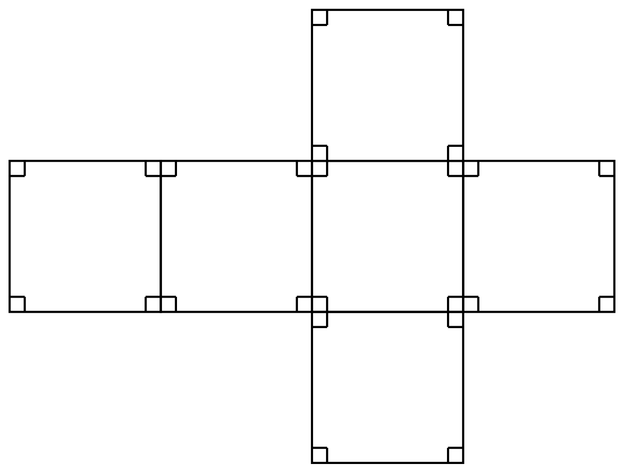
8) Explain why any triangle in a pinwheel tiling is necessarily a part of one, and only one, inflation.

9) Suppose there was an isometry that moved one triangle to another. What would that say about an inflation that contained them both? Why would that contradict the problem above?

10) The pinwheel tiling has been used in the design of several famous buildings—which ones?

A.13 Nothing but Nets

In this activity, we are going to build polyhedra out of poster board. To do this, we're going to draw *nets* of the various regular polyhedra. The **net** of a polyhedron is a single-piece arrangement of polygons that are connected along their edges so that they can be folded into the polyhedron. For example, here is the net of a cube:



1) Sketch another net for the cube. Share your sketch with your neighbor—does it look OK?

2) Sketch of a net for a tetrahedron. Share your sketch with your neighbor—does it look OK? Be sure to label pertinent angles with their correct measure.

3) Sketch of a net for an octahedron. Share your sketch with your neighbor—does it look OK? Be sure to label pertinent angles with their correct measure.

4) Sketch of a net for an icosahedron. Share your sketch with your neighbor—does it look OK? Be sure to label pertinent angles with their correct measure.

5) Sketch of a net for a dodecahedron. Share your sketch with your neighbor—does it look OK? Be sure to label pertinent angles with their correct measure.

6) Based on the work from your previous question, construct each of the 5 regular polyhedra.

- An edge of your tetrahedron should be 5 inches in length.
- An edge of your octahedron should be 4 inches in length.
- An edge of your cube should be 4 inches in length.
- An edge of your dodecahedron should be 4 inches in length.
- An edge of your icosahedron should be 3 inches in length.

A.14 Triangles on a Cone

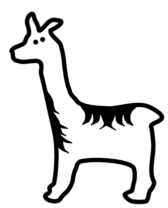
1) Put a dot at the center of a blank sheet of paper and call it **o**. Use a protractor to draw an angle of 50° with vertex at the point **o** and sides extending all the way out to the edge of the paper. Cut the paper along one side of the angle and one side only. Make a cone by moving the cut edge to the other side of the angle you drew. This cone (extended infinitely) is your universe.

2) Make a triangle in your universe that surrounds **o**. To do this, unfold your universe and lay it out flat on the desk and make the sides with your ruler. When a side gets to the cut side of your angle, put the other side of the angle on top and keep going.

3) You measure angles on your universe by laying the paper out flat and measuring the angles on the paper. Measure the angles in your triangle, what do they sum to?

4) Repeat the problems above, but this time cut an angle of 40° to make your cone. What do you notice?

Let's see if we can explain this. Do you know who is eager to help you? That's right: Louie Llama.



5) Take your triangle and denote the measure of its angles as a, b, and c. We would like to parade Louie around the triangle. There is only one catch: What happens to Louie when he passes over the "cut?" Draw some pictures and see if you can figure it out.

Start Louie Llama out along a side adjacent to the angle of measure a. He should be on the outside of the triangle, his feet should be pointing toward the triangle, and his face should be pointing toward the angle of measure b. Continue this process and walk him all around the triangle. When he gets to the "cut" put the paper together, and let him continue his walk.

6) Through what angle does Louie rotate when he strolls around a vertex?

7) How many degrees did the "cut" rotate Louie?

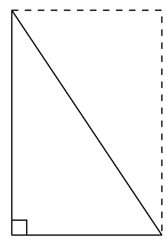
8) All in all, how many degrees did Louie Llama rotate in his walk?

9) If a cone is made on a sheet of paper with a cut of θ degrees, and a triangle is made surrounding the point of the cone, what is the sum of the degrees of this triangle?

A.15 Turn Up the Volume!

In this activity, we will investigate formulas for area and volume.

1) Explain how the following picture "proves" that the area of a right triangle is half the base times the height.

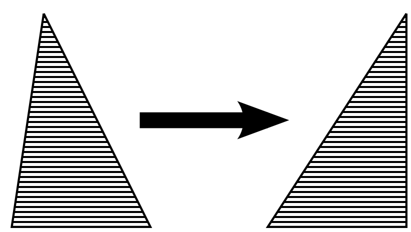


2) Cavalieri's principle states:

> Shearing parallel to a fixed direction does not change the n-dimensional measure of an object.

What is this saying?

3) Building on the first two problems, explain how the following picture "proves" that the area of any triangle is half the base times the height.



4) Give an intuitive argument explaining why Cavalieri's principle is true.

5) Explain how to use a picture to "prove" that a triangle of a given area could have an arbitrarily large perimeter.

6) Sketch a net for a right pyramid of height $2''$ with a $2'' \times 2''$ square base. Share your sketch with your neighbor—does it look OK?

7) Give detailed diagrams that show that a cube can be constructed from three equal pyramids.

8) Use your work above to derive a formula for the volume of a right-pyramid with a square base. The formula should be in terms of the side-length of the square base.

9) Give a formula for **every** pyramid with an $s \times s$ square base of height s in terms of s.

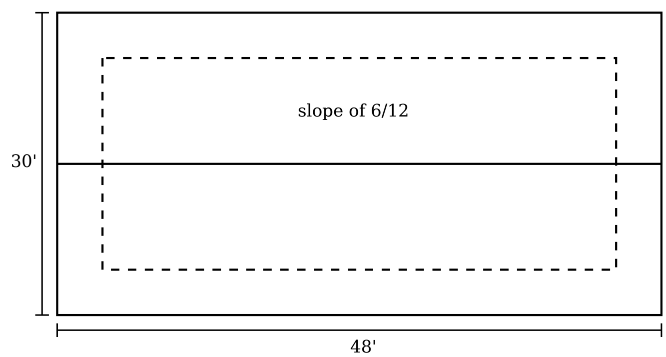
A.16 Here's the Pitch

The steepness of a roof is called its *slope* or *pitch*. This is a fraction where the numerator is the *rise* of the roof and the denominator is the *run*. The denominator is usually 12. Hence a slope of

$$\frac{7}{12}$$

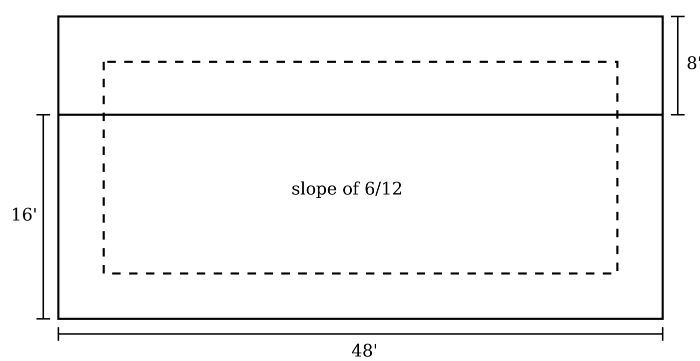
says that the roof goes up $7''$ every $12''$. Note there are several other notations for pitch. The one we are presenting here is not only rather common, but it is also mathematically correct!

1) Here is a diagram of a roof:



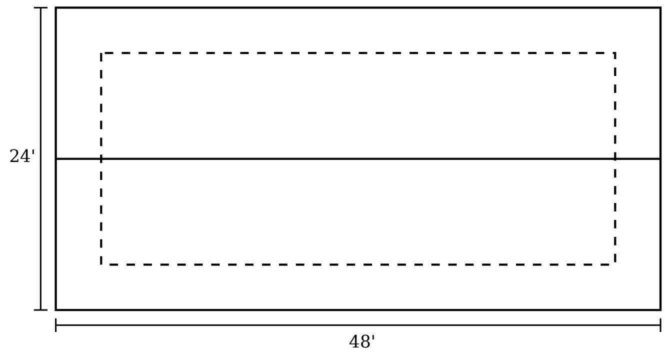
Find the area of the roof.

2) Here is a diagram of a roof with different slopes for each side:



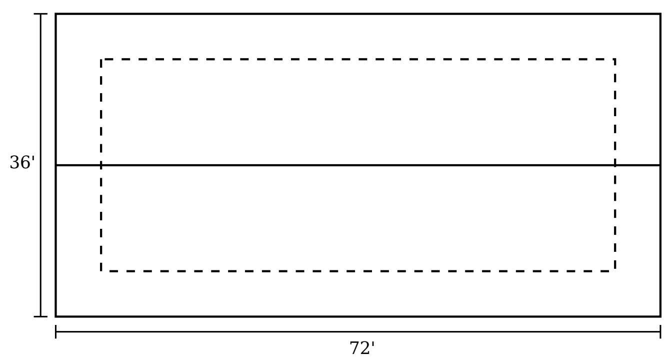
First find the missing slope, then find the area of the roof.

3) Here is a diagram of a roof:



The area of the roof is 1248 square feet. Find the slope of the roof.

4) Here is a diagram of a roof:



The area of the roof is 2808 square feet. Find the slope of the roof.

5) Explain how to measure the slope of a roof using two rulers and a level. Give examples of measurements that you could take, and say what the pitch would be in each case.

6) In Europe, it is more common to measure the steepness of a roof using the angle of the roof from horizontal. For example, a roof whose slope is 12/12 has an angle of 45°. Complete the conversion table below:

slope	angle
1/12	
2/12	
3/12	
4/12	
5/12	
6/12	
7/12	
8/12	
9/12	
10/12	
11/12	
12/12	

Hint: You'll need to remember something from a previous course.

A.17 Painted Cubes

In this activity we are going to test you skills at notating features of a three dimensional object, using only two dimensions

1) Suppose you have 8 cubes, all the same size. Two are red, two are blue, two are yellow, and two are green. These 8 cubes can be put together to form a larger cube. Without building a model, how many **different** larger cubes can be formed with these 8 cubes?

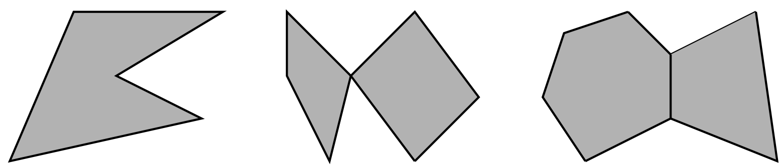
A.18 Euler Characteristic

In this activity we are going to investigate some fundamental properties of shapes in different dimensions.

Data and Observations

First Table

1) Consider the following shapes.



For each shape record: the number of vertexes, the number of edges, and the number of faces in a table:

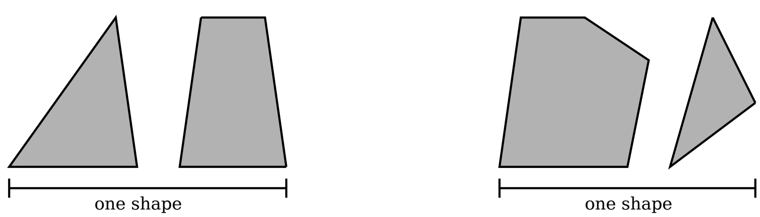
Vertexes	Edges	Faces
⋮	⋮	⋮

2) Draw five more shapes made from polygons. Record their number of vertexes, number of edges, and number of faces in your table too.

3) Can *any* set of three numbers appear in your table? Or is there some rule telling you when a set of numbers is possible?

Second Table

4) What if your shapes are disconnected? For example this is one shape:



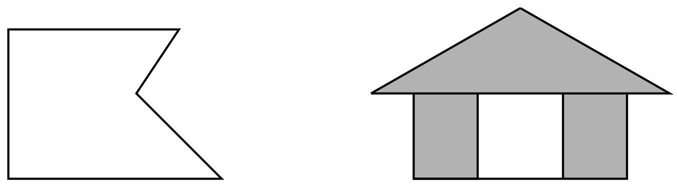
Draw five examples of disconnected shapes made from polygons. Record their number of vertexes, number of edges, number of faces, and pieces in a new table:

Vertexes	Edges	Faces	Pieces
⋮	⋮	⋮	⋮

5) Can *any* set of four numbers appear in your table? Or is there some rule telling you when a set of numbers is possible?

Third Table

6) What if your shapes have holes? For example, consider:



Draw five examples of shapes made from polygons with holes. Record their number of vertexes, number of edges, number of faces, and holes in a new table:

Vertexes	Edges	Faces	Holes
⋮	⋮	⋮	⋮

7) Can *any* set of four numbers appear in your table? Or is there some rule telling you when a set of numbers is possible?

Fourth Table

8) Consider the Platonic solids and the triangular dipyramid. For these, record the number of vertexes, edges, and faces in a table.

9) Can *any* set of three numbers appear in your table? Or is there some rule telling you when a set of numbers is possible?

The Why and How

10) Draw your favorite shape built from polygons in the plane—for fun make sure it has at least one non-triangular face. Compute $V - E + F$. Explain why this number doesn't change when:

(a) If there is a face that is not triangular, you add edges (but not vertexes) to break it into triangles.

(b) If you remove a triangle on the outer boarder of your shape. Note, there are three cases to consider!

11) Use your work above to explain your observations for $V - E + F$ for shapes made of polygons. Can you extend your argument to work for:

- Shapes that are disconnected?
- Shapes that have holes?
- Polyhedra? Hint: Use a sphere.

Into the Wild

12) Consider the hypercube and the hypertetrahedron. Record the number of vertexes, edges, faces, and solids. Now compute: $V - E + F - S$. What do you notice? Make wild conjectures and tell us about them.

A.19 Adventures Across Dimensions

In this activity we are going to investigate life in higher dimensions. We'll start off nice and slow.

The Second Dimension

If the universe was 2-dimensional, then we could represent it as the (x, y)-plane.

1) Give three example of points in this universe.

2) What is the formula for a line? If I give you two lines at random, how could they possibly relate to each other? Give examples of each possible case.

The Third Dimension

If the universe was 3-dimensional, then we could represent it as a a (x, y, z)-plane.

3) Give three example of points in this universe.

4) If I give you two lines at random, how could they possibly relate to each other? Give examples of each possible case.

5) What is the formula for a plane? If I give you two planes at random, how could they possibly relate to each other? Give examples of each possible case.

The Fourth Dimension

If the universe was 4-dimensional, then we could represent it as a (x, y, z, w)-plane.

6) Give three example of points in this universe.

7) If I give you two lines at random, how could they possibly relate to each other? Give examples of each possible case.

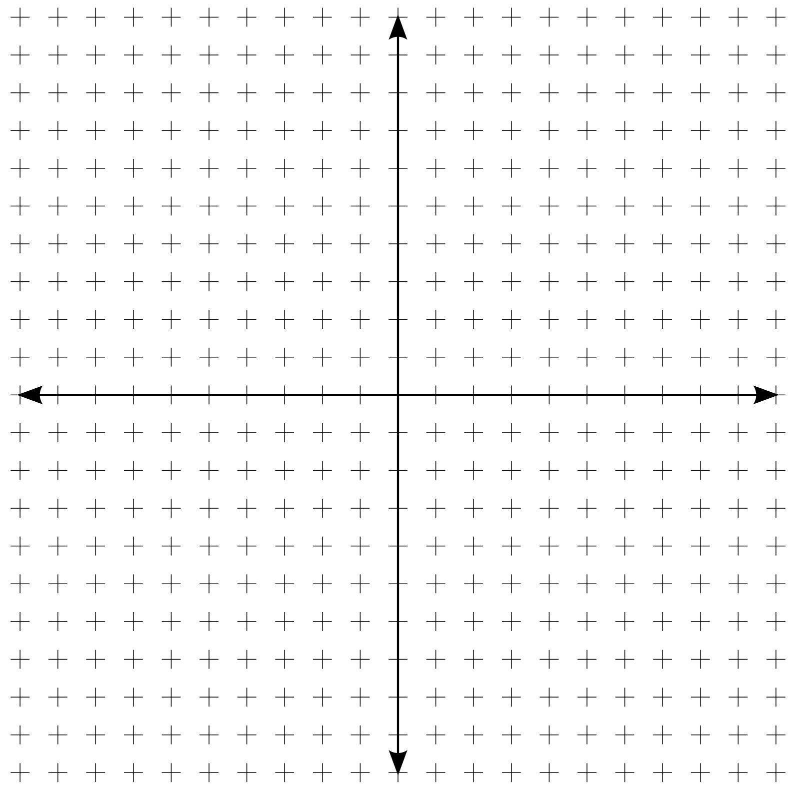
8) If I give you two planes at random, how could they possibly relate to each other? Give examples of each possible case.

9) What is the formula for a space? If I give you two spaces at random, how could they possibly relate to each other? Give examples of each possible case.

A.20 A Parabola from an Envelope of Tangents

In this activity, we will see that our "cross-connecting" method of drawing an envelope of tangents actually draws a parabola.

1) Consider the grid below:



Setting each square to be 1/2 a unit, carefully plot the lines $f(x) = 4x - 4$ and $g(x) = -4x - 4$.

2) Using the two lines above, consider the segments that go from $(2, 4)$ to $(0, -4)$ and from $(-2, 4)$ to $(0, -4)$. Carefully draw a 7-line envelope of tangents using these segments.

3) Here are two points:

$$\mathbf{a} = (t, f(t)) \qquad \text{and} \qquad \mathbf{b} = (t-2, g(t-2))$$

(a) What relevance do these points have with the plot above?

(b) Give three values of t that directly relate these points to what you have already drawn.

(c) Find the equation of the line connecting **a** and **b**. Call it $\ell_1(x)$.

4) Here are two more points:

$$\mathbf{c} = (t+\varepsilon, f(t+\varepsilon)) \qquad \text{and} \qquad \mathbf{d} = (t-2+\varepsilon, g(t-2+\varepsilon))$$

The Greek letter ε is our way of making a new line in our envelope of tangents. Set $\varepsilon = 1/2$ and see if you can answer the questions below:

(a) What relevance do these points have with the plot above?

(b) Give three values of t that directly relate these points to what you have already drawn.

(c) Forgetting that $\varepsilon = 1/2$, find the equation of the line connecting **c** and **d**. Call it $\ell_2(x)$

5) Solve

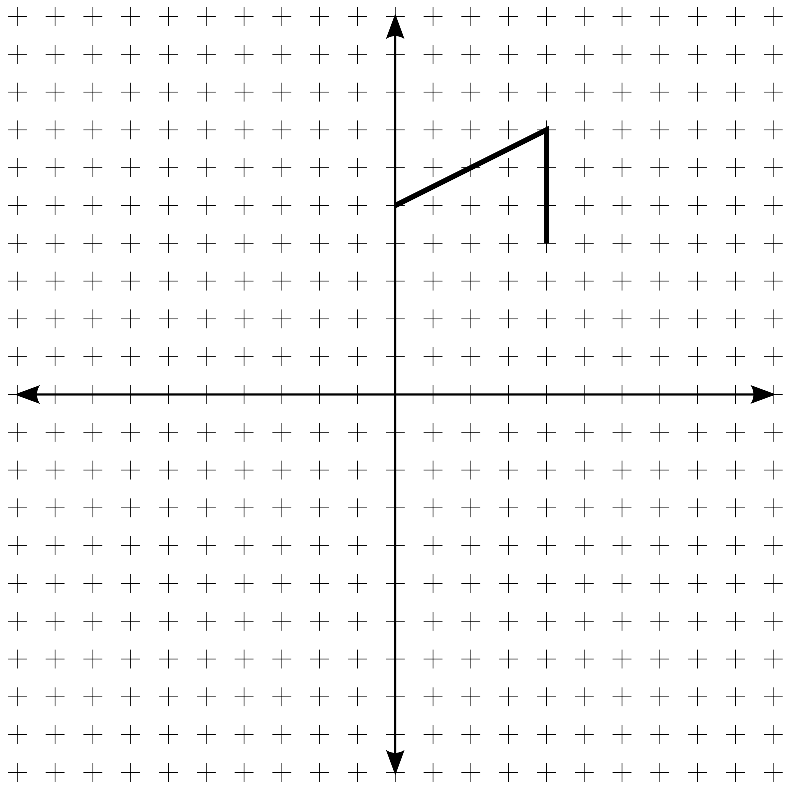
$$\ell_1(x) = \ell_2(x)$$

for x, and set $\varepsilon = 0$. What do you find? What do you get when you plot this?

A.21 Super Kaleidoscopes

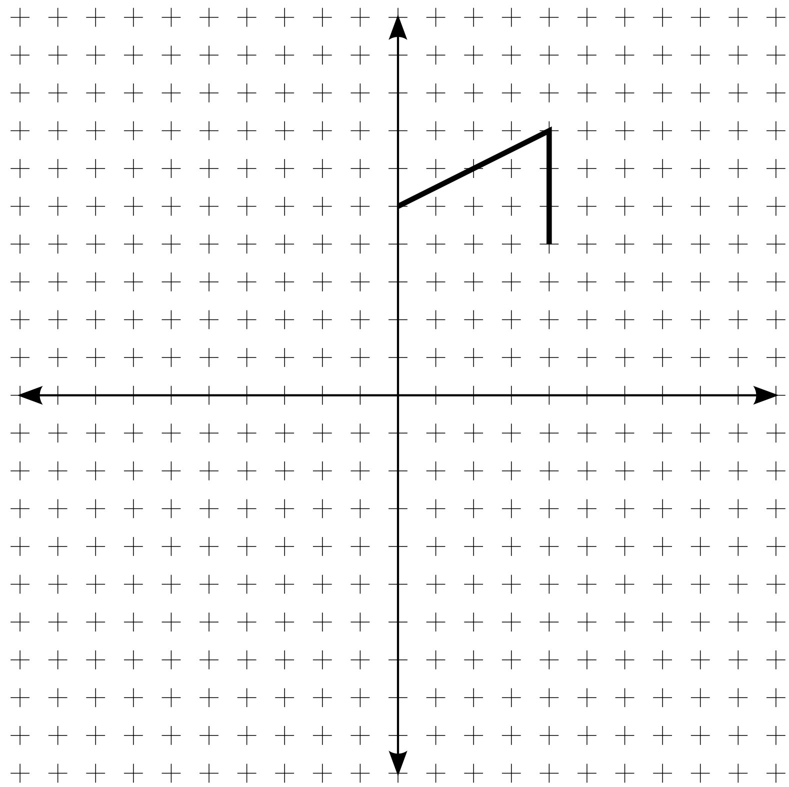
In this activity, we're going to investigate a different way to draw stars.

1) Apply every element of $\mathcal{R}_4$ to the picture below. What picture do you get?

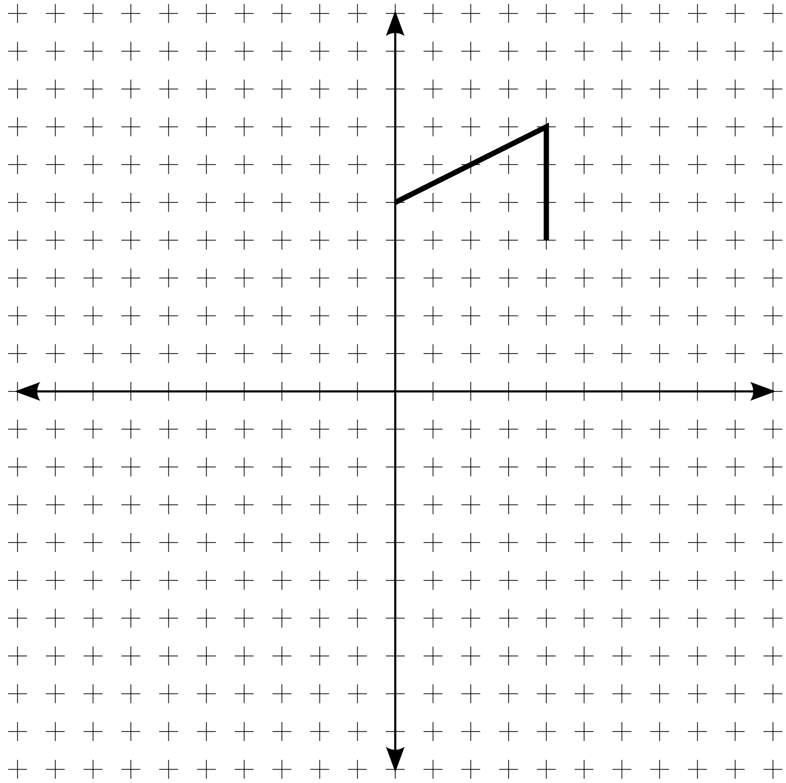


This is called the **orbit** of $\mathcal{R}_4$ on the figure above.

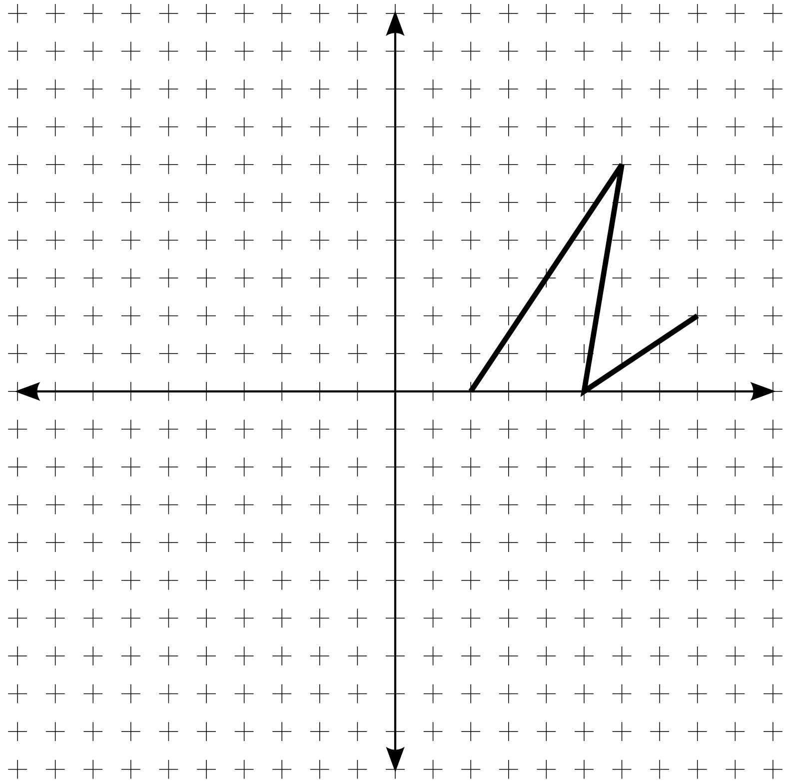
2) Apply every element of $\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{x=y}, \mathsf{F}_{x=-y}\}$ to the picture below. What picture do you get?



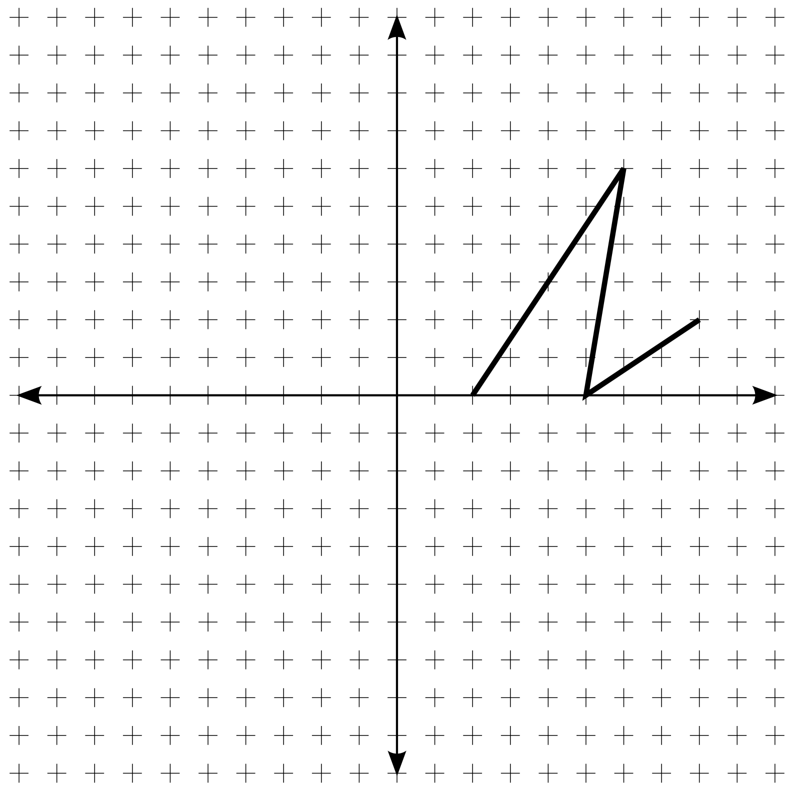
3) Apply every element of $\mathcal{D}_4$ to the picture below. What picture do you get?



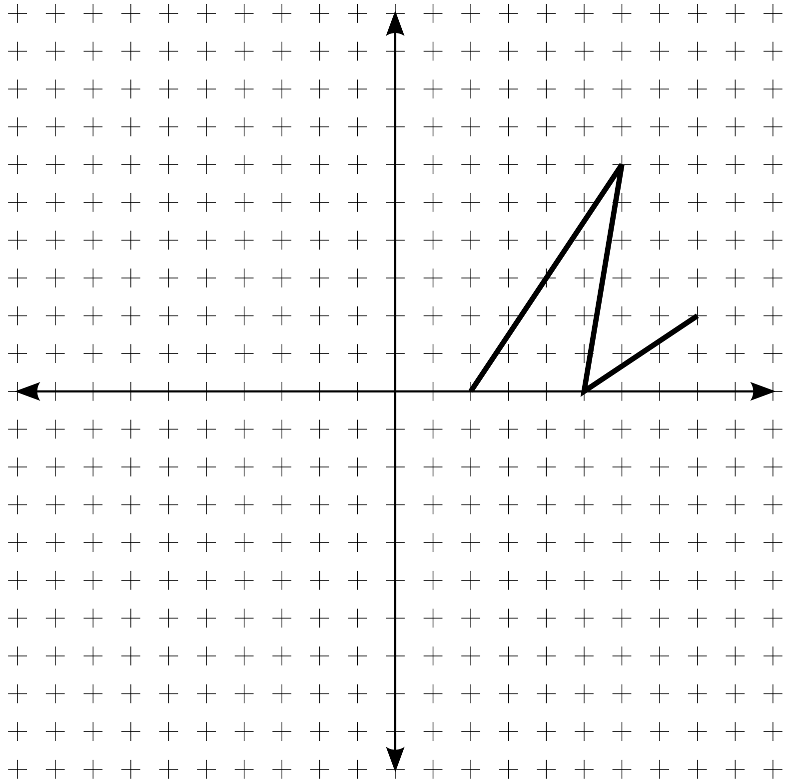
4) Apply every element of $\mathcal{R}_4$ to the picture below. What picture do you get?



5) Apply every element of $\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{x=y}, \mathsf{F}_{x=-y}\}$ to the picture below. What picture do you get?



6) Apply every element of $\mathcal{D}_4$ to the picture below. What picture do you get?



A.22 Pieces of Star Stuff

In this activity, we'll investigate how many stars $\left\{ \frac{n}{s} \right\}$ can be drawn without lifting your pencil—we'll consider $\left\{ \frac{3}{1} \right\}$ and $\left\{ \frac{3}{2} \right\}$ to be different stars. Call this number $\phi(n)$.

1) Compute $\phi(2)$, $\phi(3)$, $\phi(4)$, $\phi(5)$, $\phi(6)$, $\phi(7)$, and $\phi(8)$.

2) Given any value of n, can you give a simple description of how to compute $\phi(n)$ *without* drawing any stars?

3) Use your description to compute $\phi(11)$, $\phi(101)$, $\phi(5051)$.

4) If p is prime, what is $\phi(p)$? Explain your reasoning.

5) Compute $\phi(8)$, $\phi(16)$, $\phi(32)$, and $\phi(64)$.

6) Compute $\phi(9)$, $\phi(27)$, $\phi(81)$, and $\phi(243)$.

7) Compute $\phi(5)$, $\phi(25)$, $\phi(125)$, and $\phi(625)$.

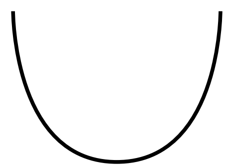
8) If p is prime, what is $\phi(p^n)$? Explain your reasoning.

9) Make a chart of values for $\phi(n)$ for $n = 1, \dots, 20$. Can you make a conjecture as to how $\phi(a)$ and $\phi(b)$ relate to $\phi(ab)$?

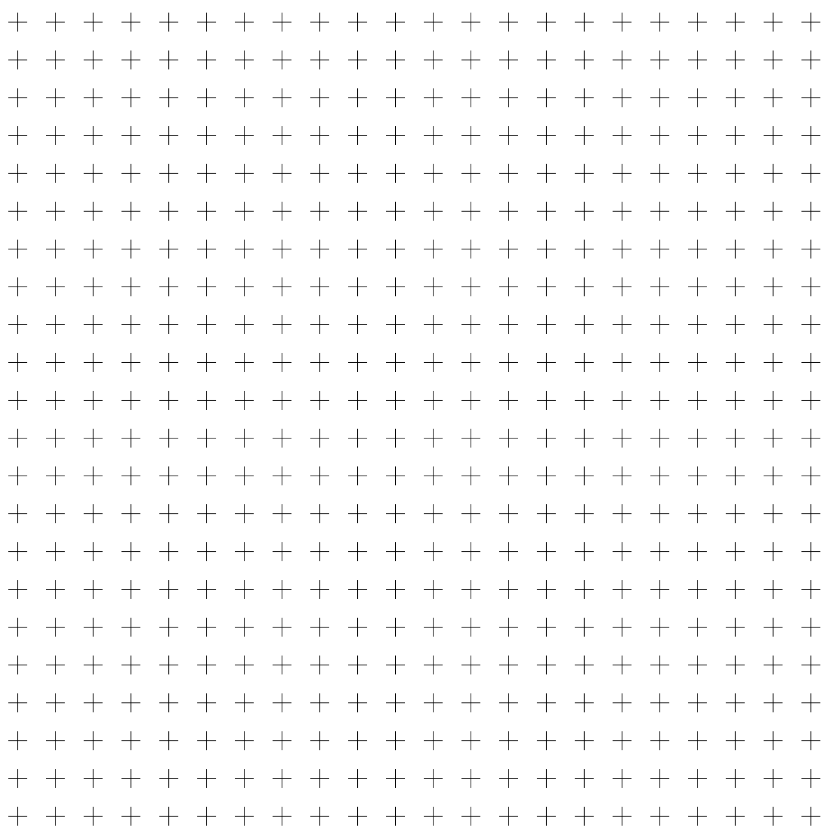
A.23 Envelopes for Curves

In this activity, we'll investigate how we actually make curves via envelopes of tangents.

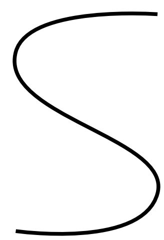
1) Consider the following curve:



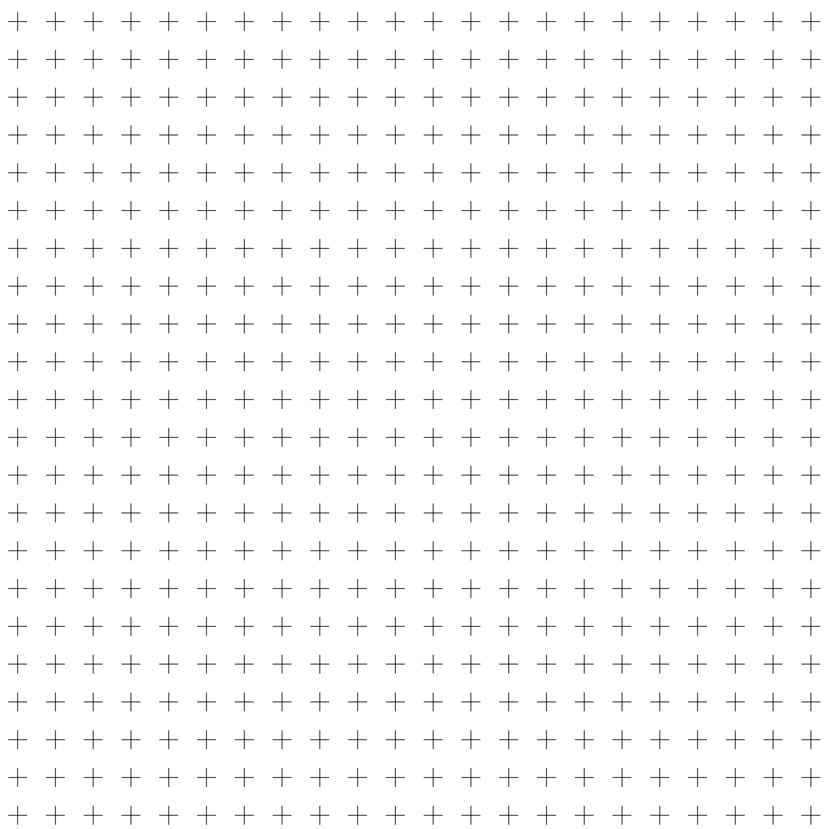
Use 7-line envelope of tangents to draw this curve in the provided grid:



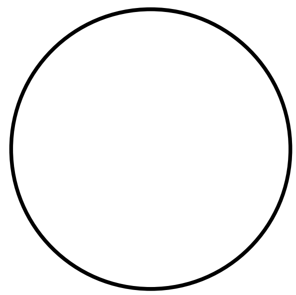
2) Consider the following curve:



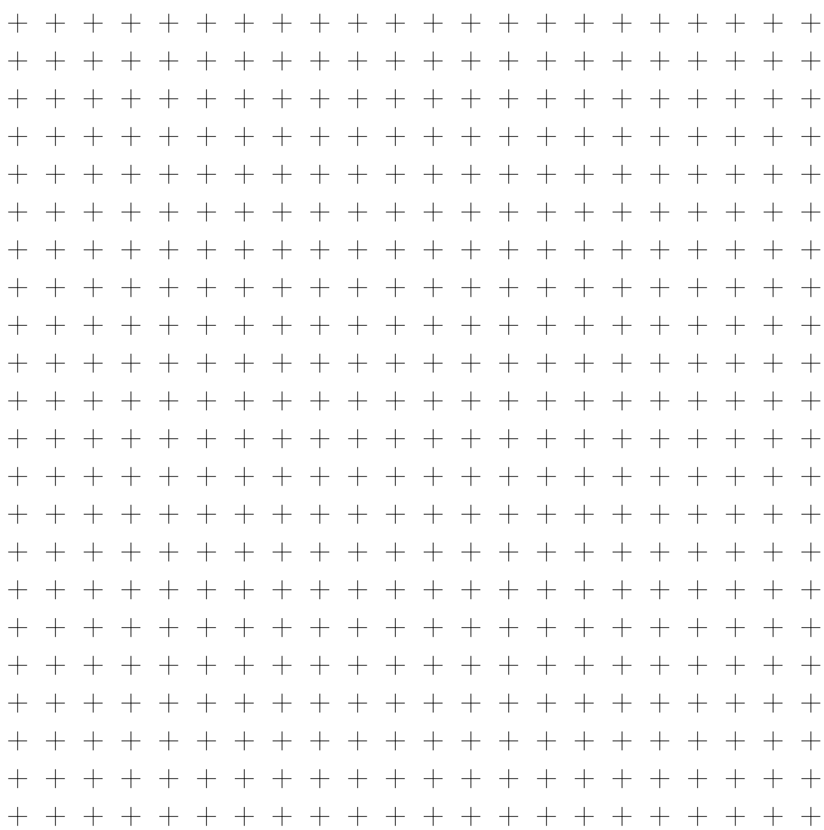
Use 7-line envelope of tangents to draw this curve in the provided grid:



3) Consider the following curve:

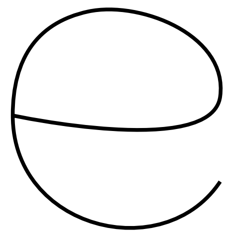


Use 7-line envelope of tangents to draw this curve in the provided grid:

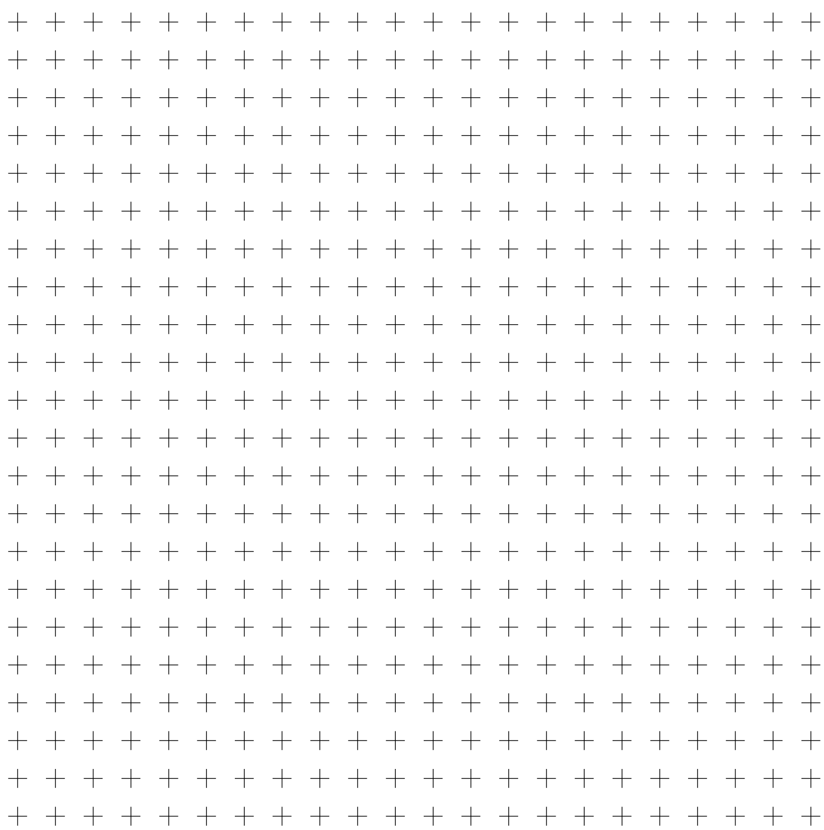


Check your solution with a compass—what do you find?

4) Consider the following curve:



Use 7-line envelope of tangents to draw this curve in the provided grid:



A.24 By Any Other Name

If you look up a rotation matrix you may find something like this:

> One rotates a point $\mathbf{p} = (x, y)$ counterclockwise around the origin through θ degrees by computing:
>
> $$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

1) How do we do the above computation?

2) Compute $\sin(90°)$ and $\cos(90°)$. Compare

$$\begin{bmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{bmatrix}$$

with R_{90}. Can you reconcile the "differences" in these functions?

3) Can you find a 2×2 matrix that will translate points? Specifically can you find a, b, c, d such that:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + u \\ y + v \end{bmatrix}$$

If so, give me an example. If not explain why not. Hint: Consider the origin, $(0, 0)$.

A.25 How Strange Could It Be?

In this activity, we are going to investigate just how strange a map given by

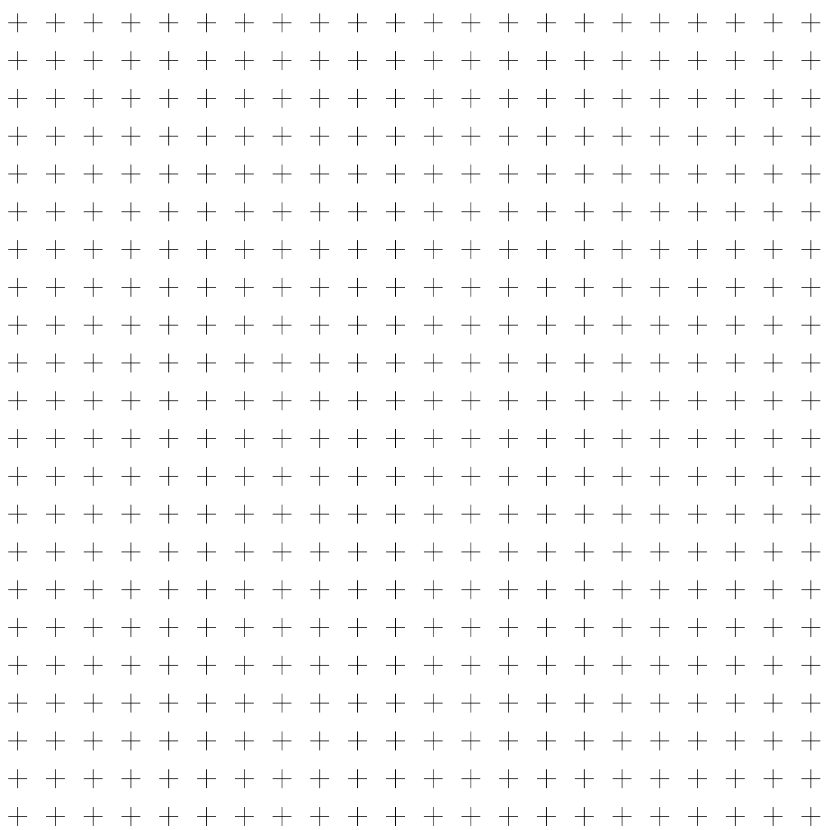
$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

could possibly be.

1) Let α, β, γ, and δ be real numbers, and let t be a variable. Consider the point:

$$(\alpha t + \beta, \gamma t + \delta)$$

Choose values for the Greek letters and plot this for varying values of t. What sort of curve do you get?



2) Now consider the line $y = mx + p$. Express its coordinates *without* using y.

3) Apply M to the coordinates you found above. What do you get? What does this tell you about what happens to lines after you apply a matrix to them?

4) Tell me some things about the line $y = mx + q$.

5) Apply M to the coordinates associated to $y = mx + q$. What does this tell you about what happens to parallel lines after you apply a matrix to them?

6) What's going to happen to a parallelogram after you apply a matrix to it?

References and Further Reading

[1] E.A. Abbott. *Flatland: A Romance of Many Dimensions.* Penguin Classics, 1998.

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[5] C.H. MacGillavry. *Fantasy&Symmetry: The Periodic Drawings of M.C. Escher.* Harry N. Abrams Inc, 1965.

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[7] J. Rosen. *Symmetry in Science.* Springer, 1995.

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[9] C. Sagan. *Cosmos.* Random House, 2002.

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