Parallels in Geometry

Math 1166: Spring 2013

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Preface

These notes are designed with future middle grades mathematics teachers in mind. While most of the material in these notes would be accessible to an accelerated middle grades student, it is our hope that the reader will find these notes both interesting and challenging. In some sense we are simply taking the topics from a middle grades class and pushing "slightly beyond" what one might typically see in schools. In particular, there is an emphasis on the ability to communicate mathematical ideas. Three goals of these notes are:

- To enrich the reader's understanding of both numbers and algebra. From the basic algorithms of arithmetic—all of which have algebraic underpinnings, to the existence of irrational numbers, we hope to show the reader that numbers and algebra are deeply connected.

- To place an emphasis on problem solving. The reader will be exposed to problems that "fight-back." Worthy minds such as yours deserve worthy opponents. Too often mathematics problems fall after a single "trick." Some worthwhile problems take time to solve and cannot be done in a single sitting.

- To challenge the common view that mathematics is a body of knowledge to be memorized and repeated. The art and science of doing mathematics is a process of reasoning and personal discovery followed by justification and explanation. We wish to convey this to the reader, and sincerely hope that the reader will pass this on to others as well.

In summary—you, the reader, must become a doer of mathematics. To this end, many questions are asked in the text that follows. Sometimes these questions are answered, other times the questions are left for the reader to ponder. To let the reader know which questions are left for cogitation, a large question mark is displayed:

?

The instructor of the course will address some of these questions. If a question is not discussed to the reader's satisfaction, then I encourage the reader to put on a thinking-cap and think, think, think! If the question is still unresolved, go to the World Wide Web and search, search, search!

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`http://www.math.osu.edu/~snapp/109/`

Please report corrections, suggestions, gripes, complaints, and criticisms to Bart Snapp at: `snapp@math.osu.edu`

Thanks and Acknowledgments

This document has a somewhat lengthy history. In the Fall of 2005 and Spring of 2006, Bart Snapp gave a set of lectures at the University of Illinois at Urbana-Champaign. His lecture notes were typed and made available as an open-source textbook. Each semester since, those notes have were revised and modified under the supervision of Alison Ahlgren and Bart Snapp. A growing number of people have made contributions, including Tom Cooney, Melissa Dennison, and Jesse Miller. A number of students also contributed to that document by either typing original hand-written notes or suggesting problems. They are: Camille Brooks, Michelle Bruno, Marissa Colatosti, Katie Colby, Anthony 'Tino' Forneris, Amanda Genovise, Melissa Peterson, Nicole Petschenko, Jason Reczek, Christina Reincke, David Seo, Adam Shalzi, Allice Son, Katie Strle, Beth Vaughn.

In 2009, Greg Williams, a Master of Arts in Teaching student at Coastal Carolina University, worked with Bart Snapp to produce an early draft of the chapter on isometries.

In the Winter of 2010 and 2011, Bart Snapp gave a new set of lectures at the Ohio State University. In this course the previous lecture notes were heavily modified, resulting in a new text *Parallels in Geometry*.

Contents

Chapter 1

Proof by Picture

A picture is worth a thousand words.

—Unknown

1.1 Basic Set Theory

The word *set* has more definitions in the dictionary than any other word. In our case we'll use the following definition:

Definition A **set** is any collection of elements for which we can always tell whether an element is in the set or not.

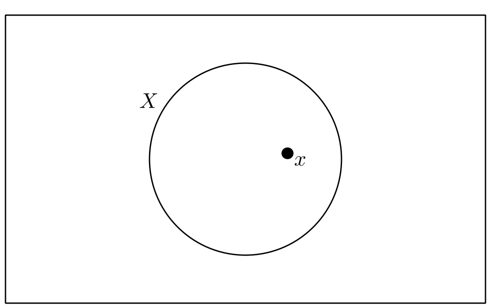
Question What are some examples of sets? What are some examples of things that are not sets?

?

If we have a set X and the element x is inside of X, we write:

$$x \in X$$

This notation is said "x in X." Pictorially we can imagine this as:



Definition A **subset** Y of a set X is a set Y such that every element of Y is also an element of X. We denote this by:

$$Y \subseteq X$$

If Y is contained in X, we will sometimes loosely say that X is *bigger* than Y.

Question Can you think of a set X and a subset Y where saying X is bigger than Y is a bit misleading?

?

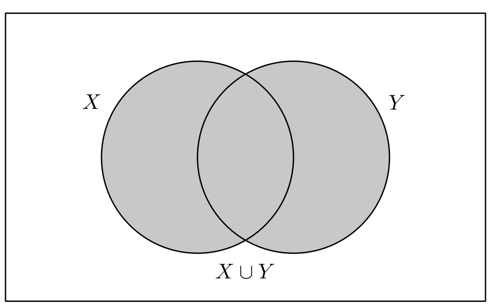
Question How is the meaning of the symbol $\in$ different from the meaning of the symbol $\subseteq$?

?

1.1.1 Union

Definition Given two sets X and Y, X **union** Y is the set of all the elements in X or Y. We denote this by $X \cup Y$.

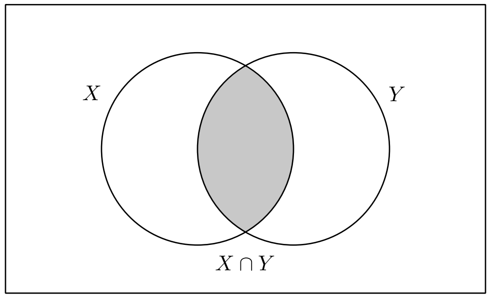
Pictorially, we can imagine this as:



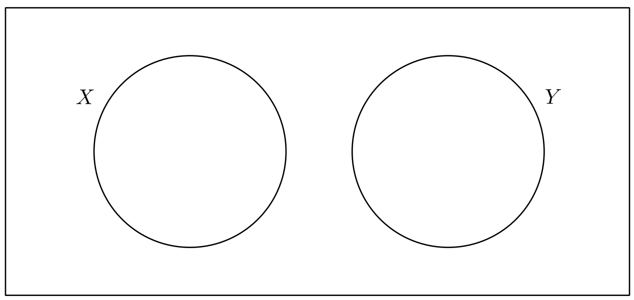
1.1.2 Intersection

Definition Given two sets X and Y, X **intersect** Y is the set of all the elements that are simultaneously in X and in Y. We denote this by $X \cap Y$.

Pictorially, we can imagine this as:



Question Consider the sets X and Y below:



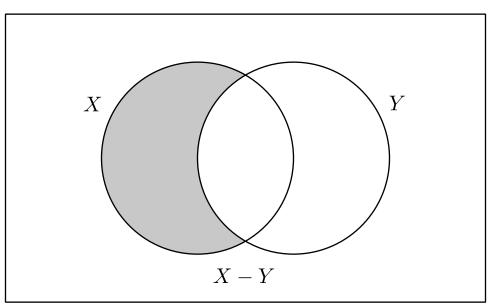
What is $X \cap Y$?

I'll take this one: Nothing! We have a special notation for the set with no elements, it is called the **empty set**. We denote the empty set by the symbol $\varnothing$.

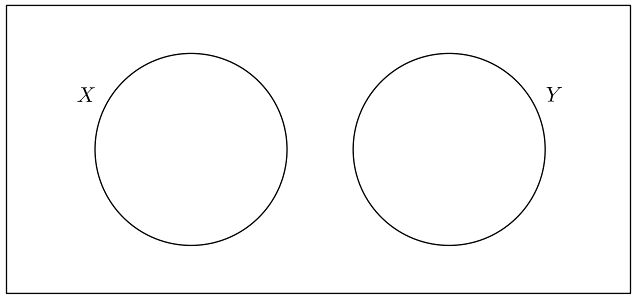
1.1.3 Complement

Definition Given two sets X and Y, X **complement** Y is the set of all the elements that are in X and are not in Y. We denote this by $X - Y$.

Pictorially, we can imagine this as:



Question Check out the two sets below:



What is $X - Y$? What is $Y - X$?

?

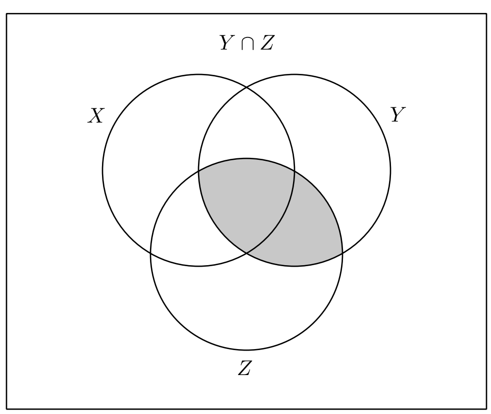
1.1.4 Putting Things Together

OK, let's try something more complex:

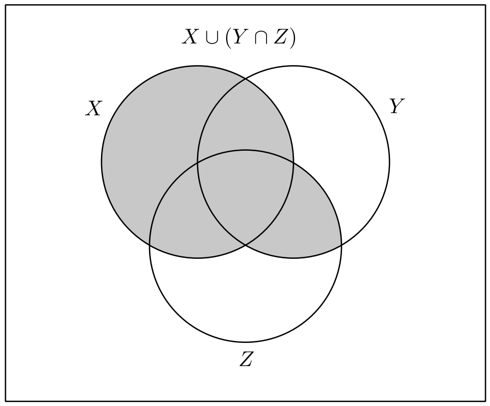
Question Prove that:

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

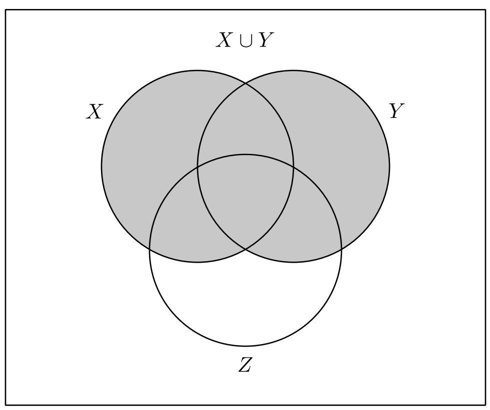
Proof Look at the left-hand side of the equation first:



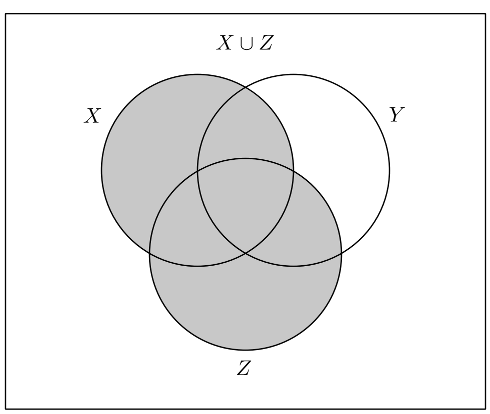
And so we see:



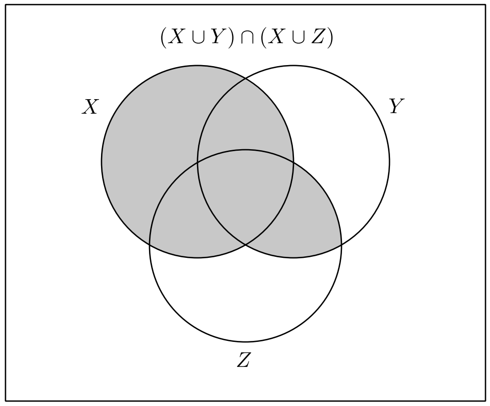
Now look at the right-hand side of the equation:



And:



So we see that:



Comparing the diagrams representing the left-hand and right-hand sides of the equation, we see that we are done. ■

Problems for Section 1.1

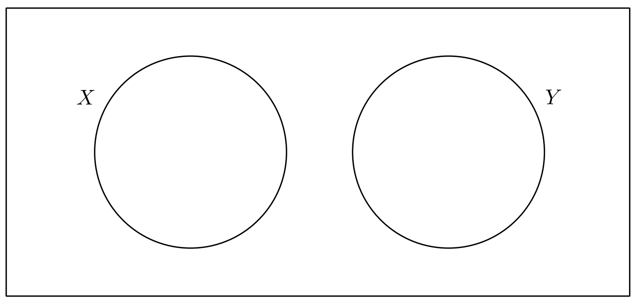
(1) Given two sets X and Y, explain what is meant by $X \cup Y$.

(2) Given two sets X and Y, explain what is meant by $X \cap Y$.

(3) Given two sets X and Y, explain what is meant by $X - Y$.

(4) Explain the difference between the symbols $\in$ and $\subseteq$.

(5) If we let X be the set of "right triangles" and we let Y be the set of "equilateral triangles" does the picture below show the relationship between these two sets?



Explain your reasoning.

(6) If $X = \{1, 2, 3, 4, 5\}$ and $Y = \{3, 4, 5, 6\}$ find:

(a) $X \cup Y$
(b) $X \cap Y$
(c) $X - Y$
(d) $Y - X$

In each case explain your reasoning.

(7) Let $n\mathbb{Z}$ represent the integer multiples of n. So for example:

$$3\mathbb{Z} = \{\dots, -12, -9, -6, -3, 0, 3, 6, 9, 12, \dots\}$$

Compute the following:

(a) $3\mathbb{Z} \cap 4\mathbb{Z}$
(b) $2\mathbb{Z} \cap 5\mathbb{Z}$
(c) $3\mathbb{Z} \cap 6\mathbb{Z}$
(d) $4\mathbb{Z} \cap 6\mathbb{Z}$
(e) $4\mathbb{Z} \cap 10\mathbb{Z}$

In each case explain your reasoning.

(8) Make a general rule for intersecting sets of the form $n\mathbb{Z}$ and $m\mathbb{Z}$. Explain why your rule works.

(9) Prove that:

$$X = (X \cap Y) \cup (X - Y)$$

(10) Prove that:

$$X - (X - Y) = (X \cap Y)$$

(11) Prove that:

$$X \cup (Y - X) = (X \cup Y)$$

(12) Prove that:

$$X \cap (Y - X) = \varnothing$$

(13) Prove that:

$$(X - Y) \cup (Y - X) = (X \cup Y) - (X \cap Y)$$

(14) Prove that:

$$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$$

(15) Prove that:

$$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$$

(16) Prove that:

$$X - (Y \cap Z) = (X - Y) \cup (X - Z)$$

(17) Prove that:

$$X - (Y \cup Z) = (X - Y) \cap (X - Z)$$

(18) If $X \cup Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(19) If $X \cup Y = Y$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(20) If $X \cap Y = X$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(21) If $X \cap Y = Y$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

(22) If $X - Y = \varnothing$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

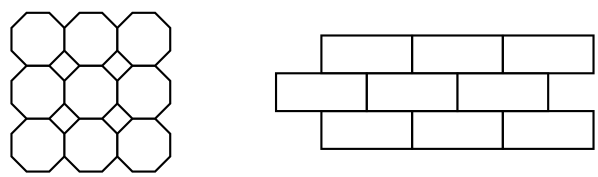
(23) If $Y - X = \varnothing$, what can we say about the relationship between the sets X and Y? Explain your reasoning.

1.2 Tessellations

Go to the internet and look up M.C. Escher. He was an artist. Look at some of his work. When you do your search be sure to include the word "tessellation" OK? Back already? Very good. Sometimes Escher worked with tessellations. What's a tessellation? I'm glad you asked:

Definition A **tessellation** is a pattern of polygons fitted together to cover the entire plane without overlapping.

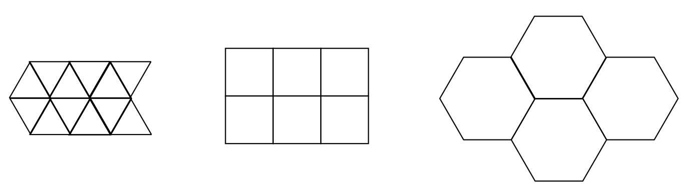
While it is impossible to actually cover the entire plane with shapes, if we give you enough of a tessellation, you should be able to continue it's pattern indefinitely. Here are pieces of tessellations:



On the left we have a tessellation of a square and an octagon. On the right we have a "brick-like" tessellation.

Definition A tessellation is called a **regular tessellation** if it is composed of copies of a single regular polygon and these polygons meet vertex to vertex.

Example *Here are some examples of regular tessellations:*



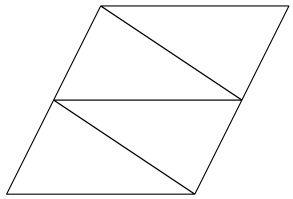
Johannes Kepler, who lived from 1571–1630, was one of the first people to study tessellations. He certainly knew the next theorem:

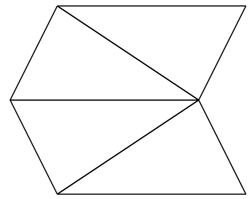
Theorem 1 *There are only* 3 *regular tessellations.*

Question Why is the theorem above true?

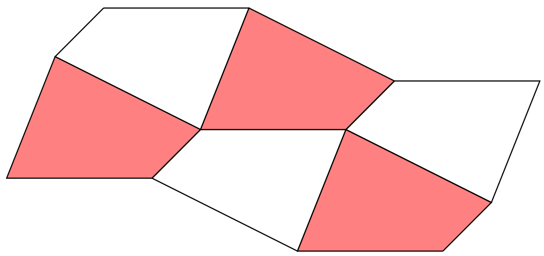
?

Since one can prove that there are only three regular tessellations, and we have shown three above, then that is all of them. On the other hand there are lots of nonregular tessellations. Here are two different ways to tessellate the plane with a triangle:





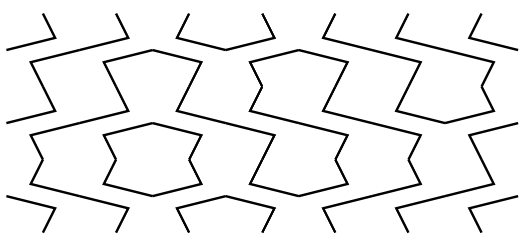
Here is a way that you can tessellate the plane with any old quadrilateral:



1.2.1 Tessellations and Art

How does one make art with tessellations? To start, a little decoration goes a long way. Check this out: Decorate two squares as such:

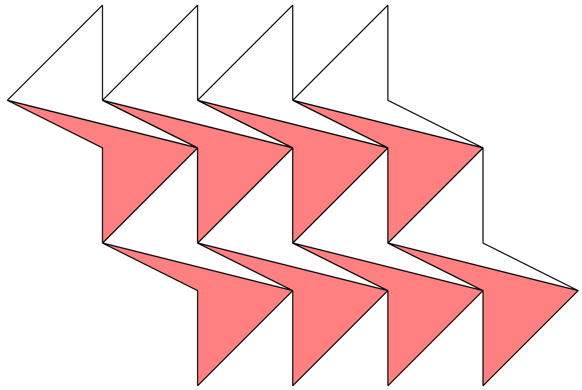
Tessellate them randomly in the plane to get this lightning-like picture:



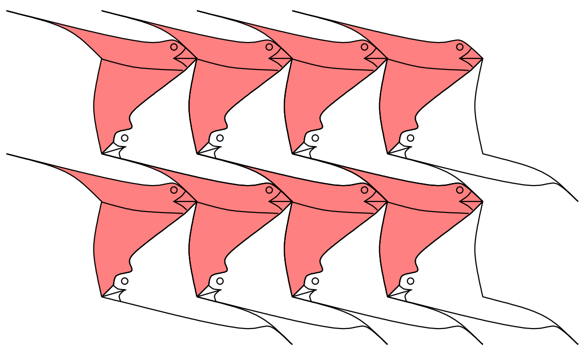
Question What sort of picture do you get if you tessellate these decorated squares randomly in a plane?

?

Another way to go is to start with your favorite tessellation:



Then you modify it a bunch to get something different:



Question What kind of art can you make with tessellations?

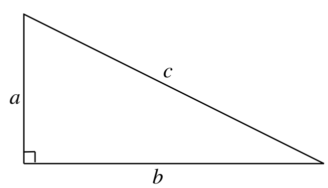
?

I'm not a very good artist, but I am a mathematician. So let's use a tessellation to give a proof! Let me ask you something:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

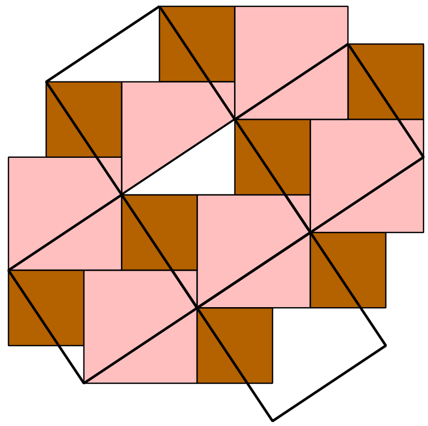
Theorem 2 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

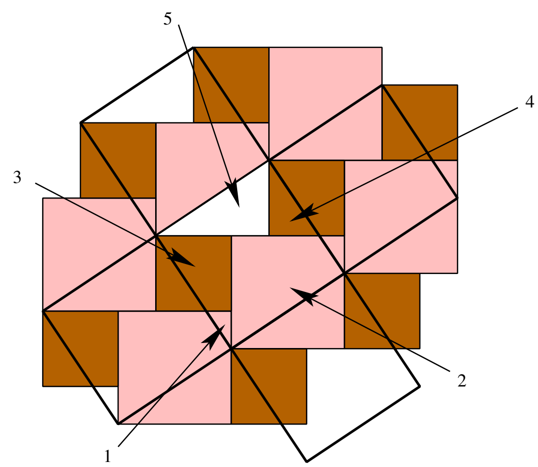
$$a^2 + b^2 = c^2.$$

Let's give a proof! Check out this tessellation involving 2 squares:



Question How does the picture above "prove" the Pythagorean Theorem?

Solution The white triangle is our right triangle. The area of the middle overlaid square is c^2, the area of the small dark squares is a^2, and the area of the medium lighter square is b^2. Now label all the "parts" of the large overlaid square:



From the picture we see that

$$\begin{aligned} a^2 &= \{3 \text{ and } 4\} \\ b^2 &= \{1, 2, \text{ and } 5\} \\ c^2 &= \{1, 2, 3, 4, \text{ and } 5\} \end{aligned}$$

Hence

$$c^2 = a^2 + b^2$$

Since we can always put two squares together in this pattern, this proof will work for any right triangle. ■

Question Can you use the above tessellation to give a dissection proof of the Pythagorean Theorem?

?

Problems for Section 1.2

(1) Show two different ways of tessellating the plane with a given scalene triangle. Label your picture as necessary.

(2) Show how to tessellate the plane with a given quadrilateral. Label your picture.

(3) Show how to tessellate the plane with a nonregular hexagon. Label your picture.

(4) Give an example of a polygon with 9 sides that tessellates the plane.

(5) Give examples of polygons that tessellate and polygons that do not tessellate.

(6) Give an example of a triangle that tessellates the plane where both 4 and 8 angles fit around each vertex.

(7) True or False: Explain your conclusions.

- (a) There are exactly 5 regular tessellations.
- (b) Any quadrilateral tessellates the plane.
- (c) Any triangle will tessellate the plane.
- (d) If a triangle is used to tessellate the plane, then it is always the case that exactly 6 angles will fit around each vertex.
- (e) If a polygon has more than 6 sides, then it cannot tessellate the plane.

(8) Given a regular tessellation, what is the sum of the angles around a given vertex?

(9) Given that the regular octagon has 135 degree angles, explain why you cannot give a regular tessellation of the plane with a regular octagon.

(10) Fill in the following table:

Regular n-gon	Does it tessellate?	Measure of an angle	If it tessellates, how many surround each vertex?
3-gon			
4-gon			
5-gon			
6-gon			
7-gon			
8-gon			
9-gon			
10-gon			

Hint: A regular n-gon has interior angles of $180(n-2)/n$ degrees.

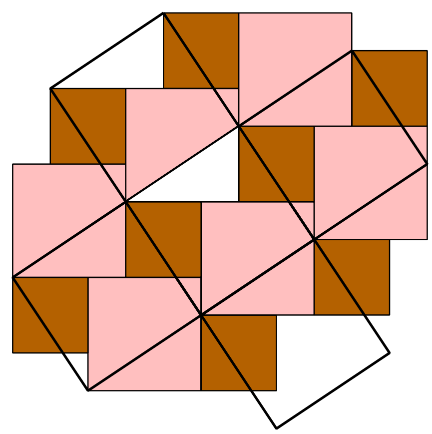
(a) What do the shapes that tessellate have in common?

(b) Make a graph with the number of sides of an n-gon on the horizontal axis and the measure of a single angle on the vertical axis. Briefly describe the relationship between the number of sides of a regular n-gon and the measure of one of its angles.

(c) What regular polygons *could* a bee use for building hives? Give some reasons that bees seem to use hexagons.

(11) Considering that the regular n-gon has interior angles of $180(n-2)/n$ degrees, and Problem (10) above, prove that there are only 3 regular tessellations of the plane.

(12) Explain how the following picture "proves" the Pythagorean Theorem.



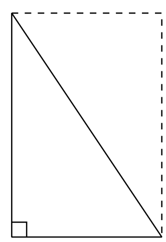
1.3 Proof by Picture

Pictures generally do not constitute a proof on their own. However, a good picture can show insight and communicate concepts better than words alone. In this section we will show you pictures giving the idea of a proof and then ask you to supply the words to finish off the argument.

1.3.1 Proofs Involving Right Triangles

Let's start with something easy:

Question Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



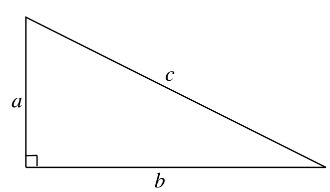
?

That wasn't so bad was it? Now for a game of *whose-who*:

Question What is the most famous theorem in mathematics?

Probably the Pythagorean Theorem comes to mind. Let's recall the statement of the Pythagorean Theorem:

Theorem 3 (Pythagorean Theorem) *Given a right triangle, the sum of the squares of the lengths of the two legs equals the square of the length of the hypotenuse. Symbolically, if a and b represent the lengths of the legs and c is the length of the hypotenuse,*



then

$$a^2 + b^2 = c^2.$$

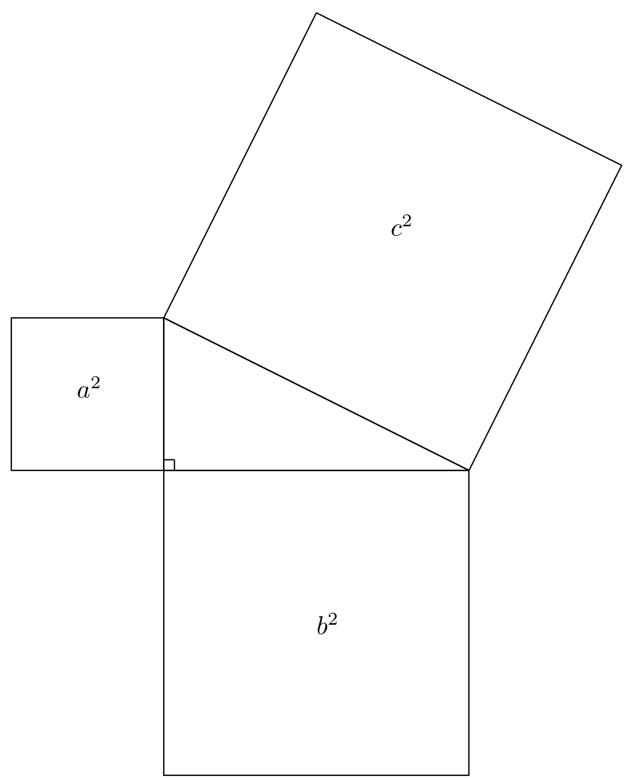
Most of the *pictures* from this section are adapted from the wonderful source books: [17] and [18].

Question What is the converse to the Pythagorean Theorem? Is it true? How do you prove it?

?

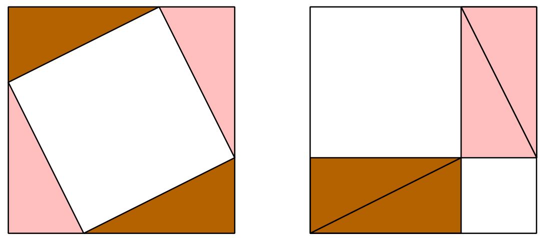
While everyone may know the Pythagorean Theorem, not as many know how to prove it. Euclid's proof goes kind of like this:

Consider the following picture:



Now, cut up the squares a^2 and b^2 in such a way that they fit into c^2 perfectly. When you give a proof that involves cutting up the shapes and putting them back together, it is called a **dissection proof**. The trick to ensure that this is actually a proof is in making sure that your dissection will work no matter what right triangle you are given. Does it sound complicated? Well it can be.

Is there an easier proof? Sure, look at:



Question How does the picture above "prove" the Pythagorean Theorem?

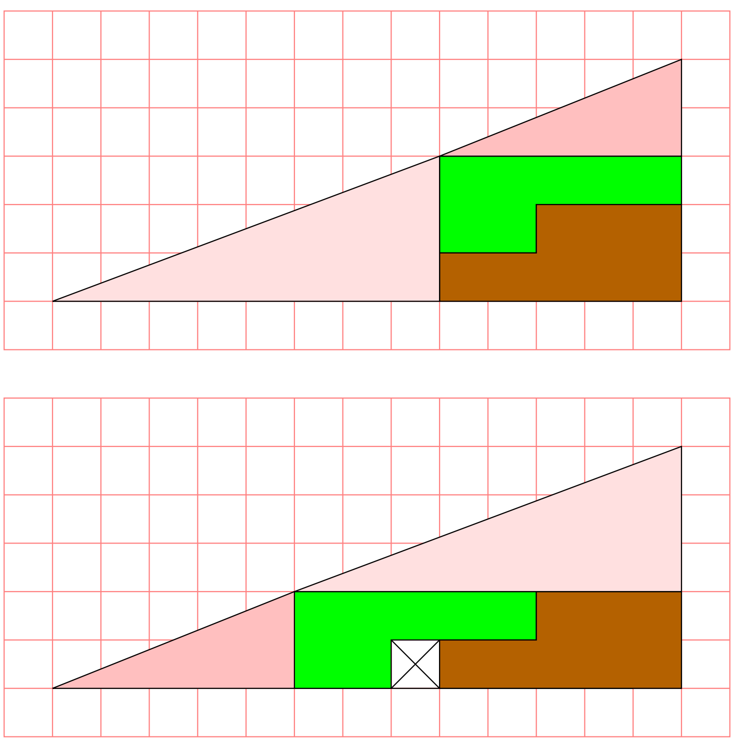
Solution Both of the large squares above are the same size. Moreover both the unshaded regions above must have the same area. The large white square on the left has an area of c^2 and the two white squares on the right have a combined area of $a^2 + b^2$. Thus we see that:

$$c^2 = a^2 + b^2$$

$\blacksquare$

Now a paradox:

Paradox *What is wrong with this picture?*



Question How does this happen[1]?

?

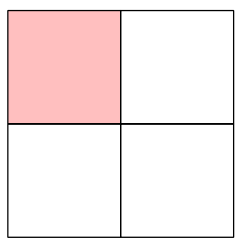
1.3.2 Proofs Involving Boxy Things

Consider the problem of *Doubling the Cube*. If a mathematician asks us to double a cube, he or she is asking us to double the **volume** of a given cube. One may be tempted to merely double each side, but this doesn't double the volume!

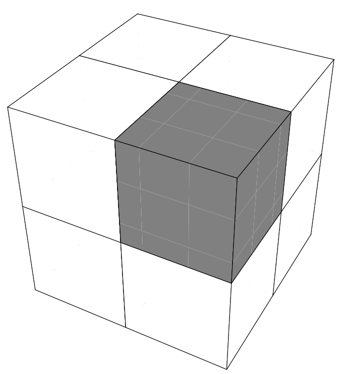
Question Why doesn't doubling each side of the cube double the volume of the cube?

?

Well, let's answer an easier question first. How do you double the area of a square? Does taking each side and doubling it work?



No! You now have four times the area. So you **cannot** double the area of a square merely by doubling each side. What about for the cube? Can you double the volume of a cube merely by doubling the length of every side? Check this out:



Ah, so the answer is again no. If you double each side of a cube you have 8 times the volume.

[1]See [7] Chapter 8, for a wonderful discussion of puzzling pictures like this one.

Question What happens to the area of a square if you multiply the sides by an arbitrary integer? What about the volume of a cube? Can you explain what is happening here?

?

1.3.3 Proofs Involving Infinite Sums

As is our style, we will start off with a question:

Question Can you add up an infinite number of terms and still get a finite number?

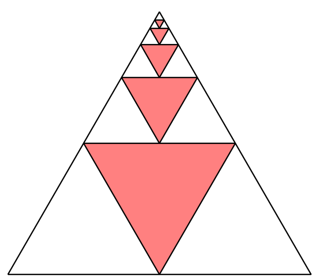
Consider 1/3. Actually, consider the decimal notation for 1/3:

$$\frac{1}{3} = .3333333333333333333333333333333\ldots$$

But this is merely the sum:

$$.3 + .03 + .003 + .0003 + .00003 + .000003 + \cdots$$

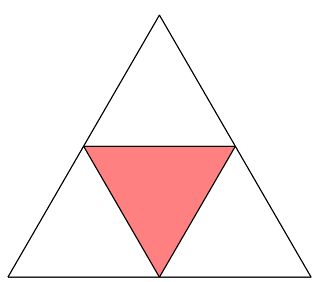
It stays less than 1 because the terms get so small so quickly. Are there other infinite sums of this sort? You bet! Check out this picture:



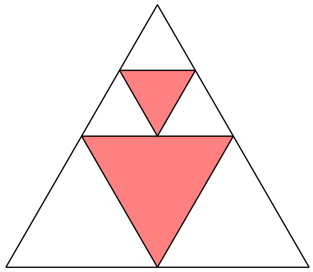
Question Explain how the picture above "proves" that:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots = \frac{1}{3}$$

Solution Let's take it in steps. If the big triangle has area 1, the area of the shaded region below is 1/4.



We also see that the area of the shaded region below



is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2$$

Continuing on in this fashion we see that the area of all the shaded regions is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots$$

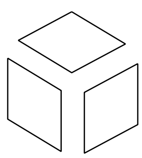
But look, the unshaded triangles have twice as much area as the shaded triangle. Thus the shaded triangles must have an area of 1/3. ■

1.3.4 Thinking Outside the Box

A *calisson* is a French candy that sort of looks like two equilateral triangles stuck together. They usually come in a hexagon-shaped box.

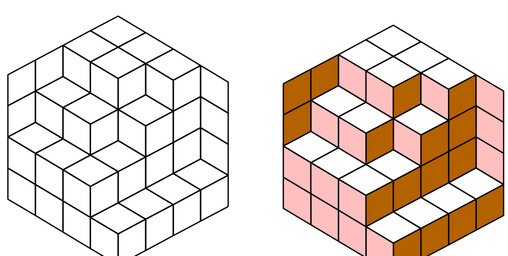
Question How do the calissons fit into their hexagon-shaped box?

If you start to put the calissons into a box, you quickly see that they can be placed in there with exactly three different orientations:



Theorem 4 *In any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.*

Look at this picture:



Question How does the picture above "prove" Theorem 4? Hint: Think outside the box!

?

Problems for Section 1.3

(1) Explain the rule

$$\text{even} + \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(2) Explain the rule

$$\text{odd} + \text{even} = \text{odd}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(3) Explain the rule

$$\text{odd} + \text{odd} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(4) Explain the rule

$$\text{even} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(5) Explain the rule

$$\text{odd} \cdot \text{odd} = \text{odd}$$

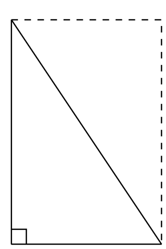
in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

(6) Explain the rule

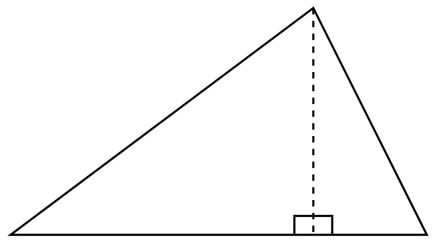
$$\text{odd} \cdot \text{even} = \text{even}$$

in two different ways. First give an explanation based on pictures. Second give an explanation based on algebra.

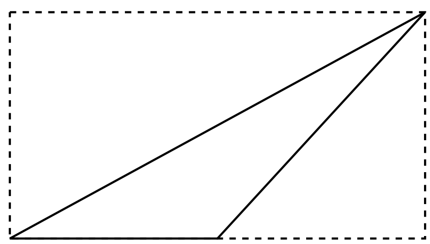
(7) Explain how the following picture "proves" that the area of a right triangle is half the base times the height.



(8) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of **every** triangle is half the base times the height.

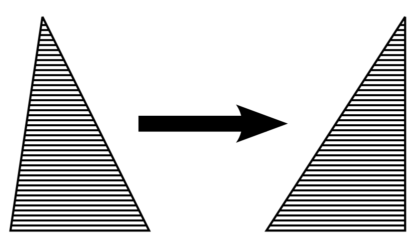


Now suppose that a student, say *Geometry Giorgio* attempts to solve a similar problem. Again knowing that the area of a right triangle is half the base times the height, he draws the following picture:



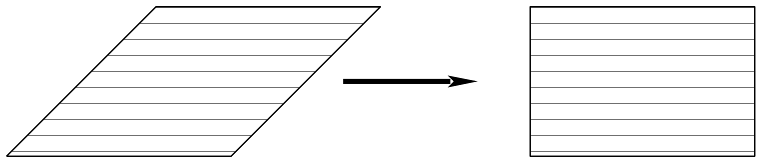
Geometry Giorgio states that the diagonal line cuts the rectangle in half, and thus the area of the triangle is half the base times the height. Is this correct reasoning? If so, give a complete explanation. If not, give correct reasoning based on *Geometry Giorgio*'s picture.

(9) Suppose you know that the area of a **right** triangle is half the base times the height. Explain how the following picture "proves" that the area of any triangle is half the base times the height. Note, this way of thinking is the basis for Cavalieri's Principle.



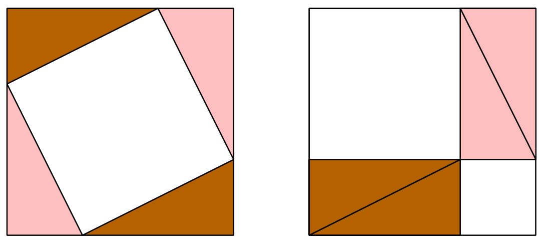
(10) Explain how the following picture "proves" that the area of any parallelogram is base times height. Note, this way of thinking is the basis for

Cavalieri's Principle.

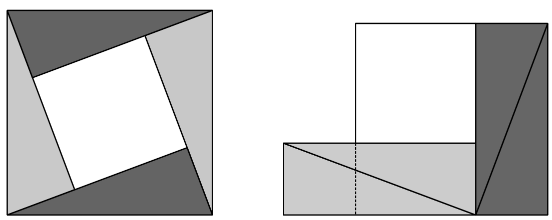


(11) Explain how to use a picture to “prove” that a triangle of a given area could have an arbitrarily large perimeter.

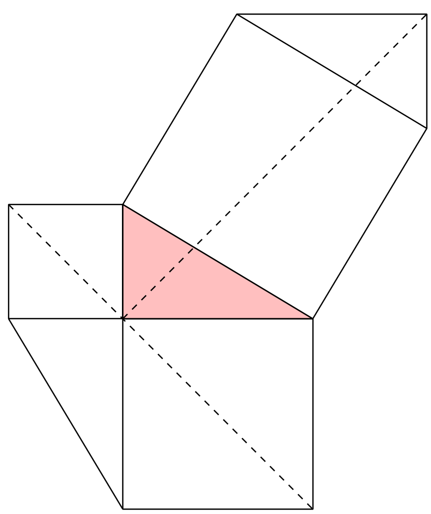
(12) Explain how the following picture “proves” the Pythagorean Theorem.



(13) Explain how the following picture “proves” the Pythagorean Theorem.

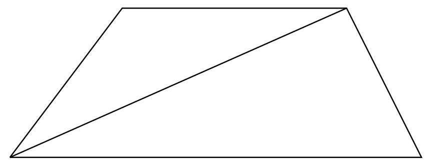


(14) Explain how the following picture “proves” the Pythagorean Theorem.



Note: This proof is due to Leonardo da Vinci.

(15) Recall that a trapezoid is a quadrilateral with two parallel sides. Consider the following picture:

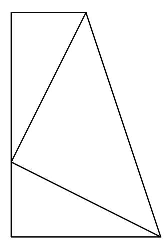


How does the above picture prove that the area of a trapezoid is

$$\text{area} = \frac{h(b_1 + b_2)}{2},$$

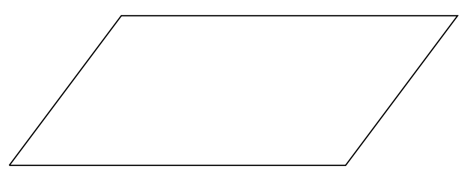
where h is the height of the trapezoid and b_1, b_2, are the lengths of the parallel sides?

(16) Explain how the following picture "proves" the Pythagorean Theorem.



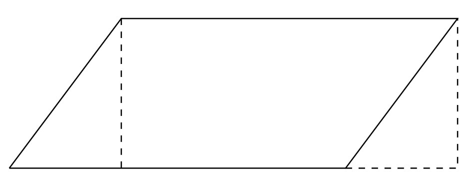
Note: This proof is due to James A. Garfield, the 20th President of the United States.

(17) Look at Problem (15). Can you use a similar picture to prove that the area of a parallelogram

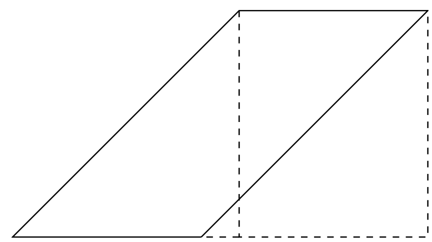


is the length of the base times the height?

(18) Explain how the following picture "proves" that the area of a parallelogram is base times height.



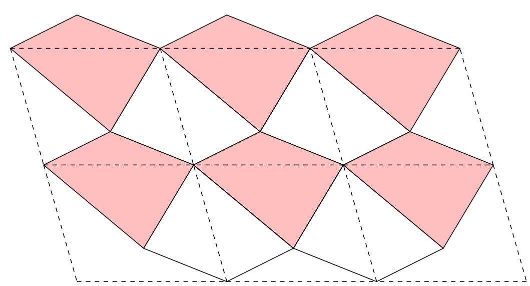
Now suppose that a student, say *Geometry Giorgio* attempts to solve a similar problem. In an attempt to prove the formula for the area of a parallelogram, *Geometry Giorgio* draws the following picture:



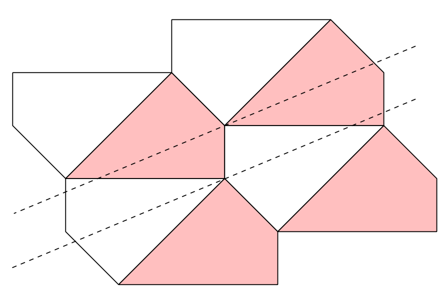
At this point *Geometry Giorgio* says that he has proved the formula for area of a parallelogram. What do you think of his picture? Give a complete argument based on his picture.

(19) Which of the above “proofs” for the formula for the area of a parallelogram is your favorite? Explain why.

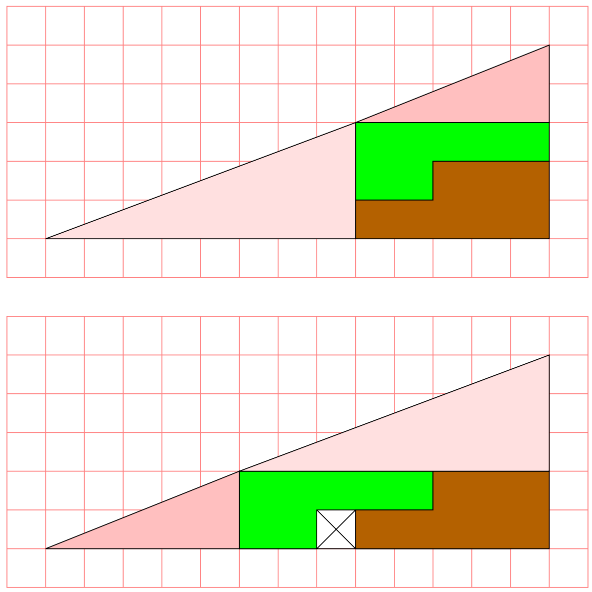
(20) Explain how the following picture “proves” that the area of a quadrilateral is equal to half of the area of the parallelogram whose sides are parallel to and equal in length to the diagonals of the original quadrilateral.



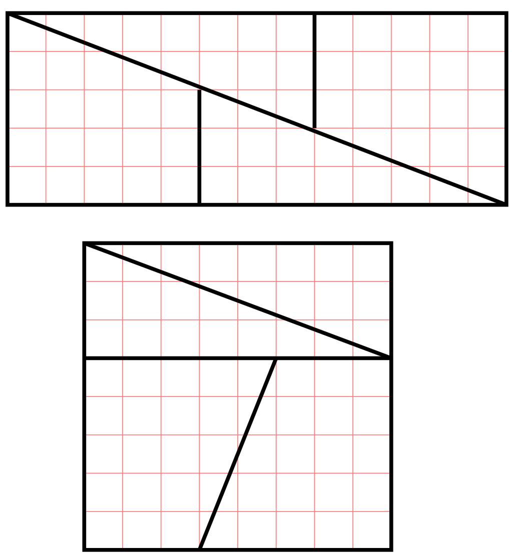
(21) Explain how the following picture “proves” that if a quadrilateral has two opposite angles that are equal, then the bisectors of the other two angles are parallel or on top of each other.



(22) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?



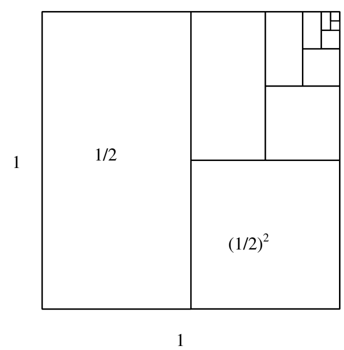
(23) Why might someone find the following picture disturbing? How would you assure them that actually everything is good and well in the geometrical world?



(24) How could you explain to someone that doubling the lengths of each side of a cube does not double the volume of the cube?

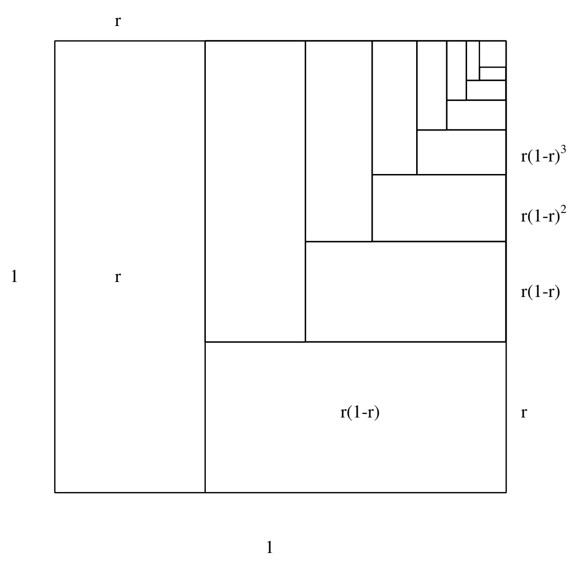
(25) Explain how the following picture "proves" that:

$$\frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \cdots = 1$$



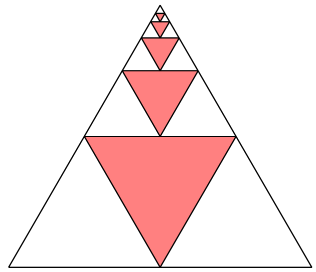
(26) Explain how the following picture "proves" that if $0 < r < 1$:

$$r + r(1-r) + r(1-r)^2 + r(1-r)^3 + \cdots = 1$$



(27) Explain how the following picture "proves" that:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \left(\frac{1}{4}\right)^4 + \left(\frac{1}{4}\right)^5 + \cdots = \frac{1}{3}$$

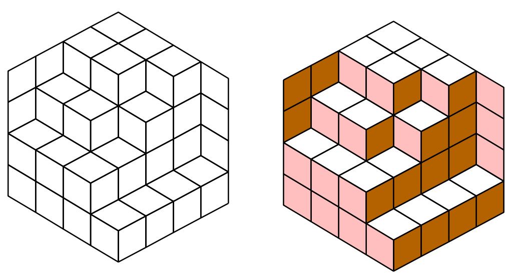


(28) Considering Problem (25), Problem (26), and Problem (27) can you give a new picture "proving" that:

$$\frac{1}{4}+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\left(\frac{1}{4}\right)^4+\left(\frac{1}{4}\right)^5+\cdots=\frac{1}{3}$$

Carefully explain the connection between your picture and the mathematical expression above.

(29) Explain how the following picture "proves" that in any packing, the number of calissons with a given orientation is exactly one-third the total number of calissons in the box.



Chapter 2

Compass and Straightedge Constructions

Mephistopheles:	I must say there is an obstacle That prevents my leaving: It's the pentagram on your threshold.
Faust:	The pentagram impedes you? Tell me then, you son of hell, If this stops you, how did you come in?
Mephistopheles:	Observe! The lines are poorly drawn; That one, the outer angle, Is open, the lines don't meet.

—Göthe, *Faust* act I, scene III

2.1 Constructions

About a century before the time of Euclid, Plato—a student of Socrates—declared that the compass and straightedge should be the only tools of the geometer. Why would he do such a thing? For one thing, both the the compass and straightedge are fairly simple instruments. One draws circles, the other draws lines—what else could possibly be needed to study geometry? Moreover, rulers and protractors are far more complex in comparison and people back then couldn't just walk to the campus bookstore and buy whatever they wanted. However, there are other reasons:

(1) Compass and straightedge constructions are **independent of units**.

(2) Compass and straightedge constructions are **theoretically correct**.

(3) Combined, the compass and straightedge seem like **powerful tools**.

Compass and straightedge constructions are independent of units. Whether you are working in centimeters or miles, compass and straightedge constructions work just as well. By not being locked to set of units, the constructions given by a compass and straightedge have certain generality that is appreciated even today.

Compass and straightedge constructions are theoretically correct. In mathematics, a correct method to solve a problem is more valuable than a correct solution. In this sense, the compass and straightedge are ideal tools for the mathematician. Easy enough to use that the rough drawings that they produce can be somewhat relied upon, yet simple enough that the tools themselves can be described theoretically. Hence it is usually not too difficult to connect a given construction to a formal proof showing that the construction is correct.

Combined, the compass and straightedge seem like powerful tools. No tool is useful unless it can solve a lot of problems. Without a doubt, the compass and straightedge combined form a powerful tool. Using a compass and straightedge, we are able to solve many problems exactly. Of the problems that we cannot solve exactly, we can always produce an approximate solution.

We'll start by giving the rules of compass and straightedge constructions:

Rules for Compass and Straightedge Constructions

(1) You may only use a compass and straightedge.

(2) You must have two points to draw a line.

(3) You must have a point and a line segment to draw a circle. The point is the center and the line segment gives the radius.

(4) Points can only be placed in two ways:

  (a) As the intersection of lines and/or circles.

  (b) As a **free point**, meaning the location of the point is not important for the final outcome of the construction.

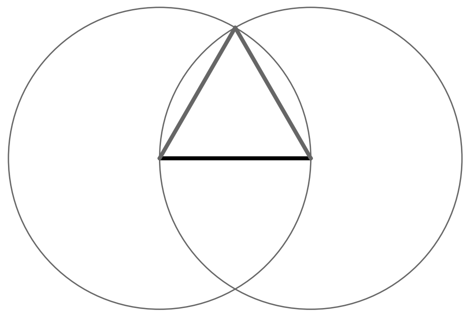
Our first construction is also Euclid's first construction:

Construction (Equilateral Triangle) *We wish to construct an equilateral triangle given the length of one side.*

(1) *Open your compass to the width of the line segment.*

(2) *Draw two circles, one with the center being each end point of the line segment.*

(3) *The two circles intersect at two points. Choose one and connect it to both of the line segment's endpoints.*



Euclid's second construction will also be our second construction:

Construction (Transferring a Segment) *Given a segment, we wish to move it so that it starts on a given point, on a given line.*

(1) *Draw a line through the point in question.*

(2) *Open your compass to the length of the line segment and draw a circle with the given point as its center.*

(3) *The line segment consisting of the given point and the intersection of the circle and the line is the transferred segment.*

If you read *The Elements*, you'll see that Euclid's construction is much more complicated than ours. Apparently, Euclid felt the need to justify the ability to move a distance. Many sources say that Euclid used what is called a *collapsing compass*, that is a compass that collapsed when it was picked up. However, I do not believe that such an invention ever existed. Rather this is something that lives in the conservative geometer's head.

Regardless of whether the difficulty of transferring distances was theoretical or physical, we need not worry when we do it. In fact, Euclid's proof of the above theorem proves that our modern way of using the compass to transfer distances is equivalent to using the so-called collapsing compass.

Question Exactly how would one prove that the modern compass is equivalent to the collapsing compass? Hint: See Euclid's proof.

?

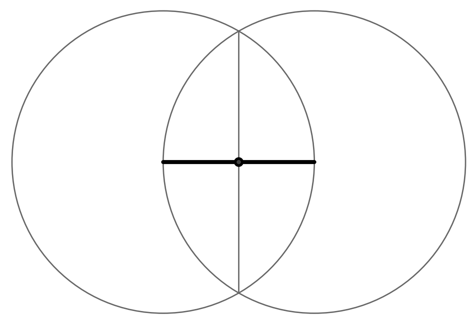
Construction (Bisecting a Segment) *Given a segment, we wish to cut it in half.*

(1) *Open your compass to the width of the segment.*

(2) *Draw two circles, one with the center being at each end point of the line segment.*

(3) *The circles intersect at two points. Draw a line through these two points.*

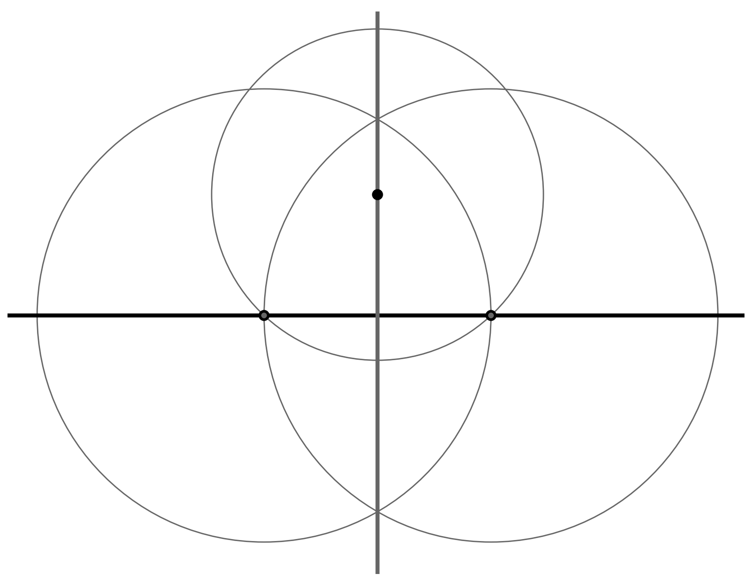
(4) *The new line bisects the original line segment.*



Construction (Perpendicular to a Line through a Point) *Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.*

(1) *Draw a circle centered at the point large enough to intersect the line in two distinct points.*

(2) *Bisect the line segment. The line used to do this will be the desired line.*

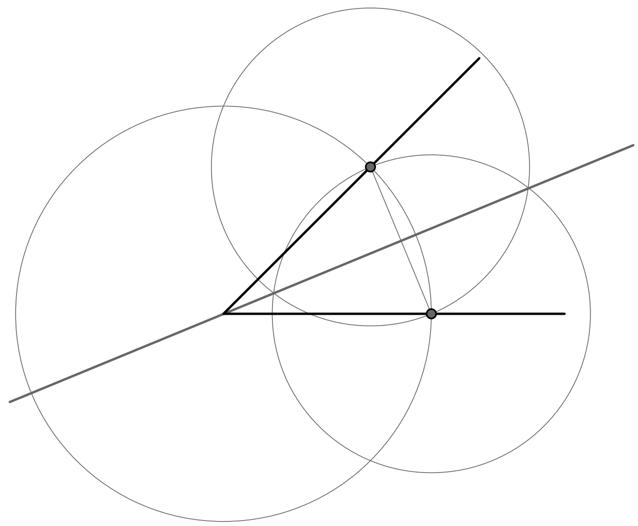


Construction (Bisecting an Angle) *We wish to divide an angle in half.*

(1) *Draw a circle with its center being the vertex of the angle.*

(2) *Draw a line segment where the circle intersects the lines.*

(3) *Bisect the new line segment. The bisector will bisect the angle.*



We now come to a very important construction:

Construction (Copying an Angle) *Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.*

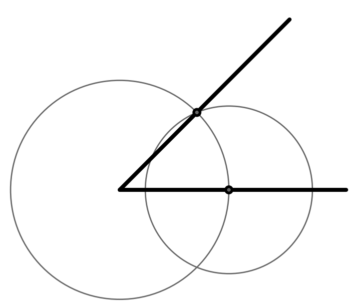
(1) *Open the compass to a fixed width and make a circle centered at the vertex of the angle.*

(2) *Make a circle of the same radius on the line with the point.*

(3) *Open the compass so that one end touches the 1st circle where it hits an edge of the original angle, with the other end of the compass extended to where the 1st circle hits the other edge of the original angle.*

(4) *Draw a circle with the radius found above with its center where the second circle hits the line.*

(5) *Connect the point to where the circles meet. This is the other leg of the angle we are constructing.*

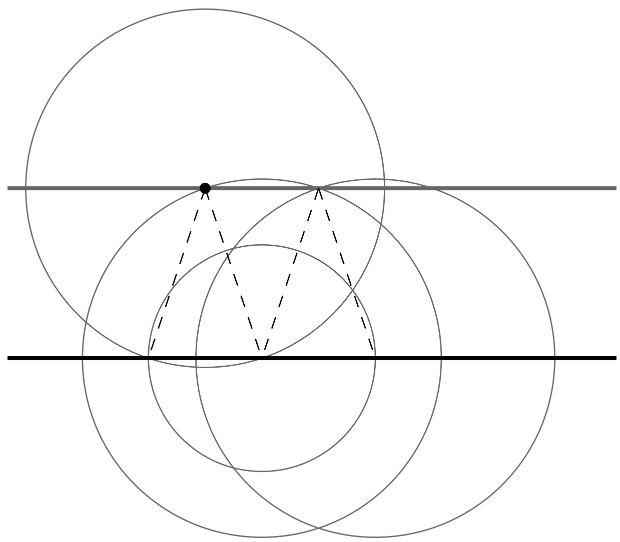


Construction (Parallel to a Line through a Point) *Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.*

(1) *Draw a circle around the given point that passes through the given line at two points.*

(2) *We now have an isosceles triangle, duplicate this triangle.*

(3) *Connect the top vertexes of the triangles and we get a parallel line.*



Question Can you give another different construction?

?

Problems for Section 2.1

(1) What are the rules for compass and straightedge constructions?

(2) What is a collapsing compass? Why don't we use them or worry about them any more?

(3) Prove that the collapsing compass is equivalent to the modern compass.

(4) Given a line segment, construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction.

(5) Use a compass and straightedge to bisect a given line segment. Explain the steps in your construction.

(6) Given a line segment with a point on it, construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction.

(7) Use a compass and straightedge to bisect a given angle. Explain the steps in your construction.

(8) Given an angle and some point, use a compass and straightedge to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction.

(9) Given a point and line, construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(10) Given a point and line, construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

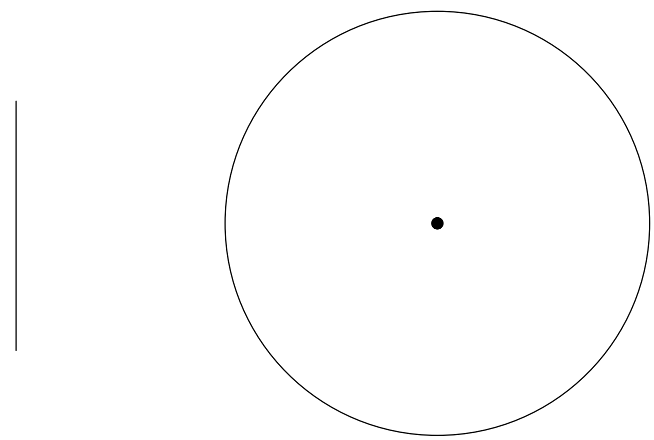
(11) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction.

(12) Construct a 30-60-90 right triangle. Explain the steps in your construction.

(13) Given a length of 1, construct a triangle with a perimeter of $3+\sqrt{5}$. Explain the steps in your construction.

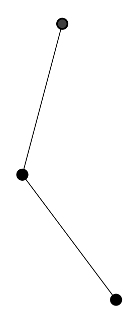
(14) Given a length of 1, construct a triangle with a perimeter that is a multiple of $2+\sqrt{2}$. Explain the steps in your construction.

(15) Here is a circle and here is the side length of an inscribed regular 5-gon.



Construct the regular 5-gon. Explain the steps in your construction.

(16) Here is a piece of a regular 7-gon.



Construct the entire regular 7-gon. Explain the steps in your construction.

2.2 Anatomy of Figures

In studying geometry we seek to discover the points that can be obtained given a set of rules. In our case the set of rules consists of the rules for compass and straightedge constructions.

Question In regards to compass and straightedge constructions, what is a *point*?

?

Question In regards to compass and straightedge constructions, what is a *line*?

?

Question In regards to compass and straightedge constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them:

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

2.2.1 Lines Related to Triangles

Believe it or not, in mathematics we often try to study the simplest objects as deeply as possible. After the objects listed above, triangles are among the most basic of geometric figures, yet there is much to know about them. There are several lines that are commonly associated to triangles. Here they are:

- Perpendicular bisectors of the sides.
- Bisectors of the angles.
- Altitudes of the triangle.
- Medians of the triangle.

The first two lines above are self-explanatory. The next two need definitions.

Definition An **altitude** of a triangle is a line segment originating at a vertex of the triangle that meets the line containing the opposite side at a right angle.

Definition A **median** of a triangle is a line segment that connects a vertex to the midpoint of the opposite side.

Question The intersection of any two lines containing the altitudes of a triangle is called an **orthocenter**. How many orthocenters does a given triangle have?

?

Question The intersection of any two medians of a triangle is called a **centroid**. How many centroids does a given triangle have?

?

Question What is the physical meaning of a centroid?

?

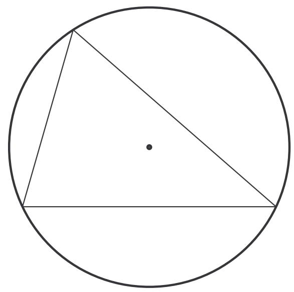
2.2.2 Circles Related to Triangles

There are also two circles that are commonly associated to triangles. Here they are:

- The circumcircle.
- The incircle.

These aren't too bad. Check out the definitions.

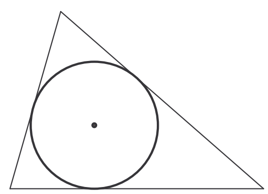
Definition The **circumcircle** of a triangle is the circle that contains all three vertexes of the triangle. Its center is called the **circumcenter** of the triangle.



Question Does every triangle have a circumcircle?

?

Definition The **incircle** of a triangle is the largest circle that will fit inside the triangle. Its center is called the **incenter** of the triangle.



Question Does every triangle have an incircle?

?

Question Are any of the lines described above related to these circles and/or centers? Clearly articulate your thoughts.

?

Problems for Section 2.2

(1) Compare and contrast the idea of "intersecting sets" with the idea of "intersecting lines."

(2) Place three points in the plane. Give a detailed discussion explaining how they may or may not be on a line.

(3) Place three lines in the plane. Give a detailed discussion explaining how they may or may not intersect.

(4) Explain how a perpendicular bisector is different from an altitude. Draw an example to illustrate the difference.

(5) Explain how a median is different from an angle bisector. Draw an example to illustrate the difference.

(6) What is the name of the point that is the same distance from all three sides of a triangle? Explain your reasoning.

(7) What is the name of the point that is the same distance from all three vertexes of a triangle? Explain your reasoning.

(8) Could the circumcenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(9) Could the orthocenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(10) Could the incenter be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(11) Could the centroid be outside the triangle? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(12) Are there shapes that do not contain their centroid? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(13) Draw an equilateral triangle. Now draw the lines containing the altitudes of this triangle. How many orthocenters do you have as intersections of lines in your drawing? Hints:

   (a) More than one.

   (b) How many triangles are in the picture you drew?

(14) Given a triangle, construct the circumcenter. Explain the steps in your construction.

(15) Given a triangle, construct the orthocenter. Explain the steps in your construction.

(16) Given a triangle, construct the incenter. Explain the steps in your construction.

(17) Given a triangle, construct the centroid. Explain the steps in your construction.

(18) Given a triangle, construct the incircle. Explain the steps in your construction.

(19) Given a triangle, construct the circumcircle. Explain the steps in your construction.

(20) Where is the circumcenter of a right triangle? Explain your reasoning.

(21) Where is the orthocenter of a right triangle? Explain your reasoning.

(22) Can you draw a triangle where the circumcenter, orthocenter, incenter, and centroid are all the same point? If so, draw a picture and explain. If not, explain why not using pictures as necessary.

(23) True or False: Explain your conclusions.

   (a) An altitude of a triangle is always perpendicular to a line containing some side of the triangle.
   (b) An altitude of a triangle always bisects some side of the triangle.
   (c) The incenter is always inside the triangle.
   (d) The circumcenter, the centroid, and the orthocenter always lie in a line.
   (e) The circumcenter can be outside the triangle.
   (f) The orthocenter is always inside the triangle.
   (g) The centroid is always inside the incircle.

(24) Given 3 distinct points not all in a line, construct a circle that passes through all three points. Explain the steps in your construction.

2.3 Trickier Constructions

Question How do you construct regular polygons? In particular, how do you construct regular: 3-gons, 4-gons, 5-gons, 6-gons, 7-gons, 8-gons, 10-gons, 12-gons, 17-gons, 24-gons, and 144-gons?

?

Well the equilateral triangle is easy. It was the first construction that we did. What about squares? What about regular hexagons? It turns out that they aren't too difficult. What about pentagons? Or say n-gons? We'll have to think about that. Let's leave the difficult land of n-gons and go back to thinking about nice, three-sided triangles.

Construction (SAS Triangle) *Given two sides with an angle between them, we wish to construct the triangle with that angle and two adjacent sides.*

(1) *Transfer the one side so that it starts at the vertex of the angle.*

(2) *Transfer the other side so that it starts at the vertex.*

(3) *Connect the end points of all moved line segments.*

The "SAS" in this construction's name spawns from the fact that it requires two sides with an angle *between* them. The SAS Theorem states that we can obtain a unique triangle given two sides and the angle between them.

Construction (SSS Triangle) *Given three line segments we wish to construct the triangle that has those three sides if it exists.*

(1) *Choose a side and select one of its endpoints.*

(2) *Draw a circle of radius equal to the length of the second side around the chosen endpoint.*

(3) *Draw a circle of radius equal to the length of the third side around the other endpoint.*

(4) *Connect the end points of the first side and the intersection of the circles. This is the desired triangle.*

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Remember earlier when we asked about the converse to the Pythagorean Theorem? Can you use the construction above to prove the converse of the Pythagorean Theorem?

?

Question Can you state the SSS Theorem?

?

Construction (SAA Triangle) *Given a side and two angles, where the given side does not touch one of the angles, we wish to construct the triangle that has this side and these angles if it exists.*

(1) *Start with the given side and place the adjacent angle at one of its endpoints.*

(2) *Move the second angle so that it shares a leg with the leg of the first angle–not the leg with the side.*

(3) *Extend the side past the first angle, forming a new angle with the leg of the second angle.*

(4) *Move this new angle to the other endpoint of the side, extending the legs of this angle and the first angle will produce the desired triangle.*

Question Can this construction fail to produce a triangle? If so, show how. If not, why not?

?

Question Can you state the SAA Theorem?

?

Question What about other combinations of S's and A's?

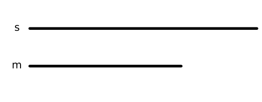
SSS, SSA, SAS, SAA, ASA, AAA

?

2.3.1 Challenge Constructions

Question How can you construct a triangle given the length of one side s, the length of the median to that side m, and the length of the altitude of the opposite angle a?

Follow-Along Use these lengths and follow the directions below.



(1) Start with the given side.

(2) Since the median hits our side at the center, bisect the given side.

(3) Make a circle of radius equal to the length of the median centered at the bisector of the given side.

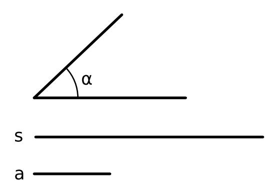
(4) Construct a line parallel to our given line of distance equal to the length of the given altitude away.

(5) Where the line and the circle intersect is the third point of our triangle. Connect the endpoints of the given side and the new point to get the triangle we want.

■

Question How can you construct a triangle given one angle α, the length of an adjacent side s, and the altitude to that side a?

Follow-Along Use these and follow the directions below.



(1) Start with a line containing the side.

(2) Put the angle at the end of the side.

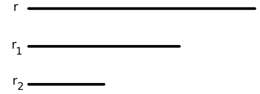
(3) Draw a parallel line to the side of the length of the altitude away.

(4) Connect the angle to the parallel side. This is the third vertex. Connect the endpoints of the given side and the new point to get the triangle we want.

■

Question How can you construct a circle with a given radius tangent to two other circles?

Follow-Along Use these and follow the directions below.



(1) Let r be the given radius, and let r_1 and r_2 be the radii of the given circles.

(2) Draw a circle of radius $r_1 + r$ around the center of the circle of radius r_1.

(3) Draw a circle of radius $r_2 + r$ around the center of the circle of radius r_2.

(4) Where the two circles drawn above intersect is the center of the desired circle.

■

Question Place two tacks in a wall. Insert a sheet of paper so that the edges hit the tacks and the corner passes through the imaginary line between the tacks. Mark where the corner of the piece of paper touches the wall. Repeat this process, sliding the paper around. What curve do you end up drawing?

?

Question How can you construct a triangle given an angle and the length of the opposite side?

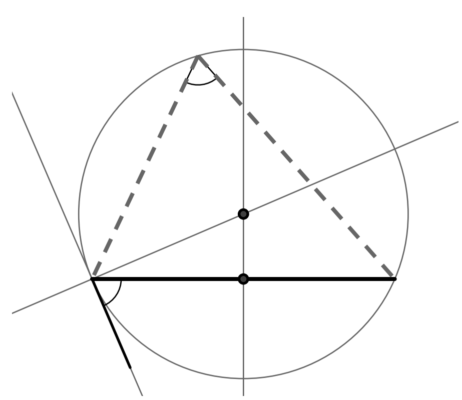
Solution We really can't solve this problem completely because the information given doesn't uniquely determine a triangle. However, we can still say something. Here is what we can do:

(1) Put the known angle at one end of the line segment. Note in the picture below, it is at the left end of the line segment and it is opening downwards.

(2) Construct the perpendicular bisector of the given segment.

(3) See where the bisector in Step 2 intersects the perpendicular of the other leg of the angle drawn from the vertex of the angle.

(4) Draw arc centered at the point found in Step 3 that touches the endpoints of the original segment.



Every point on the arc is a valid choice for the vertex of the triangle. ■

Question Why does the above method work?

?

Question You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays between the lighthouses as measured from the boat. How do you figure out where you are?

?

2.3.2 Problem Solving Strategies

The harder constructions discussed in this section can be difficult to do. There is no rote method to solve these problems, hence you must rely on your brain. Here are some hints that you may find helpful:

Construct what you can. You should start by constructing anything you can, even if you don't see how it will help you with your final construction. In doing so you are "chipping away" at the problem just as a rock-cutter chips away at a large boulder. Here are some guidelines that may help when constructing triangles:

(1) If a side is given, then you should draw it.

(2) If an angle is given and you know where to put it, draw it.

(3) If an altitude of length ℓ is given, then draw a line parallel to the side that the altitude is perpendicular to. This new line must be distance ℓ from the side.

(4) If a median is given, then bisect the segment it connects to and draw a circle centered around the bisector, whose radius is the length of the median.

(5) If you are working on a figure, construct any "mini-figures" inside the figure you are trying to construct. For example, many of the problems below ask you to construct a triangle. Some of these constructions have right-triangles inside of them, which are easier to construct than the final figure.

Sketch what you are trying to find. It is a good idea to try to sketch the figure that you are trying to construct. Sketch it accurately and label all pertinent parts. If there are special features in the figure, say two segments have the same length or there is a right-angle, make a note of it on your sketch. Also mark what is unknown in your sketch. We hope that doing this will help organize your thoughts and get your "brain juices" flowing.

Question Why are the above strategies good?

?

Problems for Section 2.3

(1) Construct a square. Explain the steps in your construction.

(2) Construct a regular hexagon. Explain the steps in your construction.

(3) Your friend Margy is building a clock. She needs to know how to align the twelve numbers on her clock so that they are equally spaced on a circle. Explain how to use a compass and straightedge construction to help her out. Illustrate your answer with a construction and explain the steps in your construction.

(4) Construct a triangle given two sides of a triangle and the angle between them. Explain the steps in your construction.

(5) State the SAS Theorem.

(6) Construct a triangle given three sides of a triangle. Explain the steps in your construction.

(7) State the SSS Theorem.

(8) Construct a triangle given a side and two angles where one of the angles does not touch the given side. Explain the steps in your construction.

(9) State the SAA Theorem.

(10) Construct a triangle given a side between two given angles. Explain the steps in your construction.

(11) State the ASA Theorem.

(12) Explain why when given an isosceles triangle, that two of its angles have equal measure. Hint: Use the SAS Theorem.

(13) Construct a figure showing that a triangle cannot always be uniquely determined when given an angle, a side adjacent to that angle, and the side opposite the angle. Explain the steps in your construction and explain how your figure shows what is desired. Explain what this says about the possibility of a SSA theorem. Hint: Draw many pictures to help yourself out.

(14) Give a construction showing that a triangle is uniquely determined if you are given a right-angle, a side touching that angle, and another side not touching the angle. Explain the steps in your construction and explain how your figure shows what is desired.

(15) Construct a triangle given two adjacent sides of a triangle and a median to one of the given sides. Explain the steps in your construction.

(16) Construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

(17) Construct a triangle given a side, the median to the side, and the angle opposite to the side. Explain the steps in your construction.

(18) Construct a triangle given an altitude, and two angles not touching the altitude. Explain the steps in your construction.

(19) Construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(20) Construct a triangle, given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(21) Construct a circle with a given radius tangent to two other given circles. Explain the steps in your construction.

(22) Does a given angle and a given opposite side uniquely determine a triangle? Explain your answer.

(23) You are on the bank of a river. There is a tree directly in front of you on the other side of the river. Directly left of you is a friend a known distance away. Your friend knows the angle starting with them, going to the tree, and ending with you. How wide is the river? Explain your work.

(24) You are on a boat at night. You can see three lighthouses, and you know their position on a map. Also you know the angles of the light rays from the lighthouses. How do you figure out where you are? Explain your work.

(25) Construct a triangle given an angle, the length of a side adjacent to the given angle, and the length of the angle's bisector to the opposite side. Explain the steps in your construction.

(26) Construct a triangle given an angle, the length of the opposite side, and the length of the altitude of the given angle. Explain the steps in your construction.

(27) Construct a triangle given one side, the length of the altitude of the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(28) Construct a triangle given one side, the length of the altitude of an adjacent angle, and the radius of the circumcircle. Explain the steps in your construction.

(29) Construct a triangle given one side, the length of the median connecting that side to the opposite angle, and the radius of the circumcircle. Explain the steps in your construction.

(30) Construct a triangle given one angle and the lengths of the altitudes to the two other angles. Explain the steps in your construction.

(31) Construct a circle with a given radius tangent to two given intersecting lines. Explain the steps in your construction.

(32) Given a circle and a line, construct another circle of a given radius that is tangent to both the original circle and line. Explain the steps in your construction.

(33) Construct a circle with three smaller circles of equal size inside such that each smaller circle is tangent to the other two and the larger outside circle. Explain the steps in your construction.

Chapter 3

Folding and Tracing Constructions

> We don't even know if Foldspace introduces us to one universe or many...
>
> —Frank Herbert

3.1 Constructions

While origami as an art form is quite ancient, folding and tracing constructions in mathematics are relatively new. The earliest mathematical discussion of folding and tracing constructions that I know of appears in T. Sundara Row's book *Geometric Exercises in Paper Folding*, [21], first published near the end of the Nineteenth Century. In the Twentieth Century it was shown that every construction that is possible with a compass and straightedge can be done with folding and tracing. Moreover, there are constructions that are possible via folding and tracing that are *impossible* with compass and straightedge alone. This may seem strange as you can draw a circle with a compass, yet this seems impossible to do via paper-folding. We will address this issue in due time. Let's get down to business—here are the rules of folding and tracing constructions:

Rules for Folding and Tracing Constructions

(1) You may only use folds, a marker, and semi-transparent paper.

(2) Points can only be placed in two ways:

   (a) As the intersection of two lines.

   (b) By marking "through" folded paper onto a previously placed point. Think of this as when the ink from a permanent marker "bleeds" through the paper.

(3) Lines can only be obtained in three ways:

   (a) By joining two points—either with a drawn line or a fold.
   (b) As a crease created by a fold.
   (c) By marking "through" folded paper onto a previously placed line.

(4) One can only fold the paper when:

   (a) Matching up points with points.
   (b) Matching up a line with a line.
   (c) Matching up two points with two intersecting lines.

Now we are going to present several basic constructions. Compare these to the ones done with a compass and straightedge. We will proceed by the order of difficulty of the construction.

Construction (Transferring a Segment) *Given a segment, we wish to move it so that it starts on a given point, on a given line.*

Construction (Copying an Angle) *Given a point on a line and some angle, we wish to copy the given angle so that the new angle has the point as its vertex and the line as one of its edges.*

Transferring segments and copying angles using folding and tracing without a "bleeding marker" can be tedious. Here is an easy way to do it:

Use 2 sheets of paper and a pen that will mark through multiple sheets.

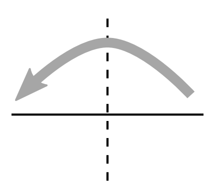
Question Can you find a way to do the above constructions without using a marker whose ink will pass through paper?

?

Construction (Bisecting a Segment) *Given a segment, we wish to cut it in half.*

(1) *Fold the paper so that the endpoints of the segment meet.*

(2) *The crease will bisect the given segment.*



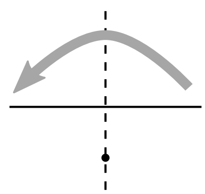
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Perpendicular through a Point) *Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.*

(1) *Fold the given line onto itself so that the crease passes though the given point.*

(2) *The crease will be the perpendicular line.*



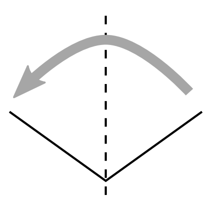
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Bisecting an Angle) *We wish to divide an angle in half.*

(1) *Fold a point on one leg of the angle to the other leg so that the crease passes though the vertex of the angle.*

(2) *The crease will bisect the angle.*



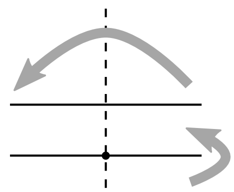
Question Which rule for folding and tracing constructions are we using above?

?

Construction (Parallel through a Point) *Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.*

(1) *Fold a perpendicular line through the given point.*

(2) *Fold a line perpendicular to this new line through the given point.*



Now there may be a pressing question in your head:

Question How the heck are we going to fold a circle?

First of all, remember the definition of a circle:

Definition A **circle** is the set of points that are a fixed distance from a given point.

Question Is the center of a circle part of the circle?

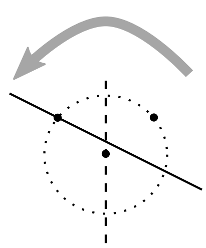
?

Secondly, remember that when doing compass and straightedge constructions we can **only** mark points that are intersections of lines and lines, lines and circles, and circles and circles. Thus while we technically draw circles, we can only actually mark certain points on circles. When it comes to folding and tracing constructions, drawing a circle amounts to marking points a given distance away from a given point—that is exactly what we can do with compass and straightedge constructions.

Construction (Intersection of a Line and a Circle) *We wish to construct the points where a given line meets a given circle. Note: A circle is given by a point on the circle and the central point.*

(1) *Fold the point on the circle onto the given line so that the crease passes through the center of the circle.*

(2) *Mark this point though both sheets of paper onto the line.*



Question Which rule for folding and tracing constructions are we using above?

?

Question How could you check that your folding and tracing construction is correct?

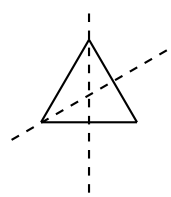
?

Construction (Equilateral Triangle) *We wish to construct an equilateral triangle given the length of one side.*

(1) *Bisect the segment.*

(2) *Fold one end of the segment onto the bisector so that the crease passes through the other end of the segment. Mark this point onto the bisector.*

(3) *Connect the points.*



Question Which rules for folding and tracing constructions are we using above?

?

Construction (Intersection of Two Circles) *We with to intersect two circles, each given by a center point and a point on the circle.*

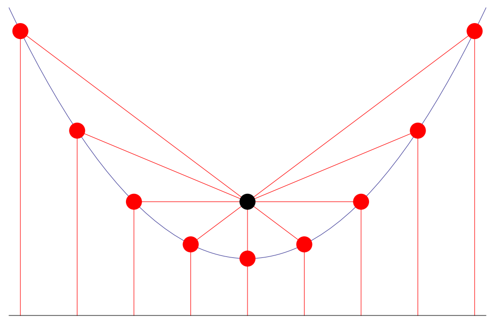
(1) *Use four sheets of tracing paper. On the first sheet, mark the centers of both circles. On the next two sheets, mark the center and point on each of the circle—one circle per sheet.*

(2) *Simply move the two sheets with the centers and points on the circles, so that the centers are over the centers from the first sheet, and the points on the circles coincide. Now on the fourth sheet, mark all points.*

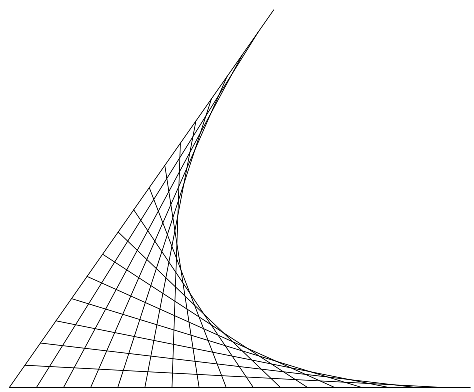
?

Think about the definition of a circle. In a similar fashion we can define other common geometric figures:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



We can also form a parabola from an *envelope of tangents*:



Using a similar idea we can essentially obtain a parabola using folding and tracing.

Construction (Parabola) *Given a point and a line we wish to construct a parabola.*

(1) *Make a series of equally spaced marks on your line.*

(2) *Fold the point onto the marks.*

(3) *Repeat the above step until an envelope of tangents forms.*

Question Considering the definition of the parabola, can you explain why the above construction makes sense?

?

Question Can you give a compass and straightedge construction of a parabola?

?

Our final basic folding and tracing construction is one that **cannot** be done with compass and straightedge alone.

Construction (Angle Trisection[1]) *We wish to divide an angle into thirds.*

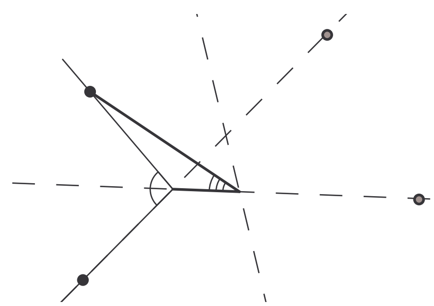
(1) *Bisect the given angle.*

(2) *Find two points (one on each leg of the angle) equidistant from the vertex of the angle.*

(3) *Fold the two points found above so that one of them lands on the extension (behind the angle) of the angle bisector and one lands on the line containing the other leg of the triangle—this will be behind the vertex. You are basically folding the angle back over itself.*

(4) *The crease from the last step will intersect the angle bisector at some point, mark it.*

(5) *The angle with the above mark as its vertex, the bisector found above as one of its legs, and the line to either of the points found in step 2 above will be one third of the starting angle.*



[1] This construction was discovered by S.T. Gormsen and verified by S.H. Kung.

Problems for Section 3.1

(1) What are the rules for folding and tracing constructions?

(2) Use folding and tracing to bisect a given line segment. Explain the steps in your construction.

(3) Given a line segment with a point on it, use folding and tracing to construct a line perpendicular to the segment that passes through the given point. Explain the steps in your construction.

(4) Use folding and tracing to bisect a given angle. Explain the steps in your construction.

(5) Given a point and line, use folding and tracing to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(6) Given a point and line, use folding and tracing to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a circle (a center and a point on the circle) and line, use folding and tracing to construct the intersection. Explain the steps in your construction.

(8) Given a line segment, use folding and tracing to construct an equilateral triangle whose edge has the length of the given segment. Explain the steps in your construction.

(9) Explain how to use folding and tracing to transfer a segment.

(10) Given an angle and some point, use folding and tracing to copy the angle so that the new angle has as its vertex the given point. Explain the steps in your construction.

(11) Explain how to use folding and tracing to construct envelope of tangents for a parabola.

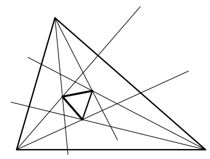
(12) Explain how to use folding and tracing to trisect a given angle.

(13) Use folding and tracing to construct a square. Explain the steps in your construction.

(14) Use folding and tracing to construct a regular hexagon. Explain the steps in your construction.

(15) Morley's Theorem states: If you trisect the angles of any triangle with lines, then those lines form a new equilateral triangle inside the original

triangle.



Give a folding and tracing construction illustrating Morley's Theorem. Explain the steps in your construction.

(16) Given a length of 1, construct a triangle whose perimeter is a multiple of 6. Explain the steps in your construction.

(17) Construct a 30-60-90 right triangle. Explain the steps in your construction.

(18) Given a length of 1, construct a triangle with a perimeter of $3 + \sqrt{5}$. Explain the steps in your construction.

3.2 Anatomy of Figures Redux

Remember, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for folding and tracing constructions.

Question In regards to folding and tracing constructions, what is a *point*?

?

Question In regards to folding and tracing constructions, what is a *line*?

?

Question In regards to folding and tracing constructions, what is a *circle*?

?

OK, those are our basic figures, pretty easy right? Now I'm going to quiz you about them (I know we've already gone over this, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

Problems for Section 3.2

(1) In regards to folding and tracing constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle.

(2) Explain how a perpendicular bisector is different from an altitude. Use folding and tracing to illustrate the difference.

(3) Explain how a median different from an angle bisector. Use folding and tracing to illustrate the difference.

(4) Given a triangle, use folding and tracing to construct the circumcenter. Explain the steps in your construction.

(5) Given a triangle, use folding and tracing to construct the orthocenter. Explain the steps in your construction.

(6) Given a triangle, use folding and tracing to construct the incenter. Explain the steps in your construction.

(7) Given a triangle, use folding and tracing to construct the centroid. Explain the steps in your construction.

(8) Could the circumcenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(9) Could the orthocenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

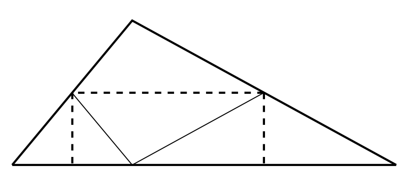
(10) Could the incenter be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(11) Could the centroid be outside the triangle? If so explain how and use folding and tracing to give an example. If not, explain why not using folding and tracing to illustrate your ideas.

(12) Where is the circumcenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(13) Where is the orthocenter of a right triangle? Explain your reasoning and illustrate your ideas with folding and tracing.

(14) The following picture shows a triangle that has been folded along the dotted lines:



Explain how the picture "proves" the following statements:

(a) The angles in a triangle sum to 180°.

(b) The area of a triangle is given by $bh/2$.

(15) Use folding and tracing to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(16) Use folding and tracing to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(17) Use folding and tracing to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(18) Use folding and tracing to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

3.3 Similar Triangles

In geometry, we may have several different segments all with the same length. We don't want to say that the segments are *equal* because that would mean that they are *exactly* the same—position included. Hence we need a new concept:

Definition When different segments have the same length, we say they are **congruent segments**.

Definition In a similar fashion, if different angles have the same measure, we say they are **congruent angles**.

Put these together and we have the definition for triangles:

Definition Two triangles are said to be **congruent triangles** if they have the same side lengths and same angle measures.

After working with triangles for short time, one quickly sees that the notion of congruence is not the only "equivalence" one wants to make between triangles. In particular, we want the notion of *similarity.*

One day when aloof old Professor Rufus was trying to explain similar triangles to his class, he merely wrote

$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{array}{c} \angle A \simeq \angle A' \\ \angle B \simeq \angle B' \\ \angle C \simeq \angle C' \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of similar triangles?

?

Question Devise a way to use folding and tracing constructions to help explore this notion of similar triangles.

?

Another day when aloof old Professor Rufus was trying to explain similar triangles to his class, he merely wrote

$$\triangle ABC \sim \triangle A'B'C' \qquad \Leftrightarrow \qquad \begin{array}{c} AB = k \cdot A'B' \\ BC = k \cdot B'C' \\ CA = k \cdot C'A' \end{array}$$

and walked out of the room.

Question Can you give 3 much needed examples of similar triangles?

?

Question Devise a way to use folding and tracing constructions to help explore this notion of similar triangles.

?

Question What's going on in aloof old Professor Rufus' head[2]—why are his explanations so different?

?

3.3.1 Theorems for Similar Triangles

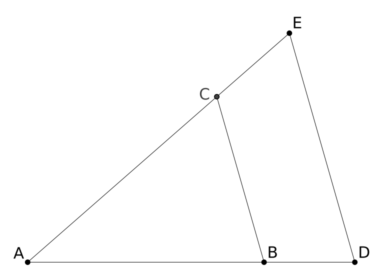
In this section we will show that that **both** definitions of similar triangles given above are **equivalent**. We'll start with a gentle question:

Question What is the formula for the area of a triangle?

?

Using merely the formula for the area of a triangle, we (meaning you) will explain why the following important theorem is true. Note, throughout this discussion we will use the convention that when we write AB we mean the *length* of the segment AB.

Theorem 5 (Parallel-Side) *Given:*



If side BC is parallel to side DE, then

$$\frac{AB}{AD} = \frac{AC}{AE}.$$

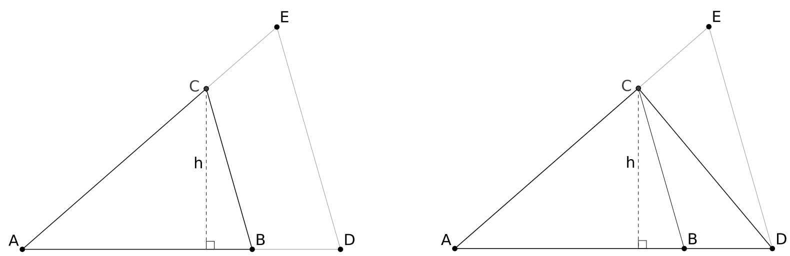
Question Can you tell me in English what this theorem says? Provide some examples of this theorem in action.

?

[2] I realize that this is a dangerous question!

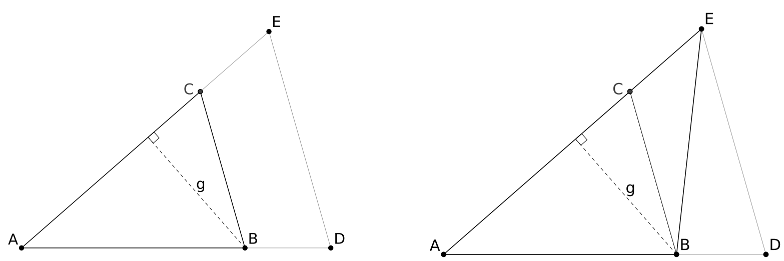
Now we (meaning you) are going to explore a bit. See if answering these questions sheds light on this.

Question If h is the height of $\triangle ABC$, find a formula for the areas of $\triangle ABC$ and $\triangle ADC$.



?

Question If g is the height of $\triangle ACB$, find a formula for the areas of $\triangle ACB$ and $\triangle AEB$.



?

Question Explain why

$$\text{Area}(\triangle ABC) = \text{Area}(\triangle ACB).$$

?

Question Explain why

$$\text{Area}(\triangle CBE) = \text{Area}(\triangle CBD).$$

Big hint: Use the fact that you have two parallel sides! Draw a picture to help clarify your explanation.

?

Question Explain why

$$\text{Area}(\triangle ADC) = \text{Area}(\triangle AEB).$$

?

Question Explain why

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} = \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question Compute and simplify both of the following expressions:

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle ADC)} \qquad \text{and} \qquad \frac{\text{Area}(\triangle ACB)}{\text{Area}(\triangle AEB)}$$

?

Question How can you conclude that:

$$\frac{AB}{AD} = \frac{AC}{AE}$$

?

Question Why is it important that line DE is parallel to line CB?

?

Question Can you sketch out (in words) how the questions above prove the Parallel-Side Theorem?

?

Now comes the moment of truth.

Question Can you use the Parallel-Side Theorem to explain why if you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$\angle A \simeq \angle A'$$
$$\angle B \simeq \angle B'$$
$$\angle C \simeq \angle C'$$

then we must have that

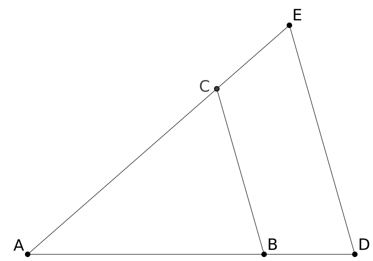
$$AB = k \cdot A'B'$$
$$BC = k \cdot B'C'$$
$$CA = k \cdot C'A'$$

?

The Converse

The converse of the Parallel-Side Theorem states:

Theorem 6 (Split-Side) *Given:*



If side BC intersects (splits) the sides of $\triangle ADE$ so that

$$\frac{AB}{AD} = \frac{AC}{AE},$$

then side BC is parallel to side DE.

Question How could you investigate this theorem using any of the construction techniques above?

?

Now we (meaning you) will answer questions in the hope that they will help us see why the above theorem is true.

Question Suppose that you **doubt** that side BC is parallel to side DE. Explain how to place a point C' on side AE so that side BC' is parallel to line DE. Be sure to sketch the situation(s).

?

Question You now have a triangle $\triangle ADE$ whose sides are split by a line BC' such that the line BC' is parallel to line DE. What does the Parallel-Side Theorem have to say about this?

?

Question What can you conclude about points C and C'?

?

Question What does this tell you about the Split-Side Theorem?

?

Let's see if you can put this all together:

Question Can you use the Split-Side Theorem to explain why if you know that if you have two triangles, $\triangle ABC$ and $\triangle A'B'C'$ with:

$$\begin{aligned} AB &= k \cdot A'B' \\ BC &= k \cdot B'C' \\ CA &= k \cdot C'A' \end{aligned}$$

then we must have that

$$\begin{aligned} \angle A &\simeq \angle A' \\ \angle B &\simeq \angle B' \\ \angle C &\simeq \angle C' \end{aligned}$$

?

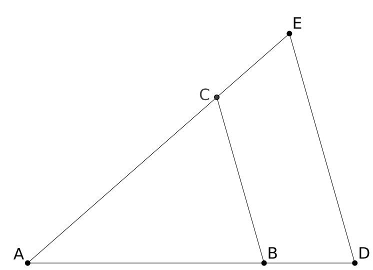
Putting all of our work above together, we may now say the following:

Definition Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are said to be **similar** if either equilvalent condition holds:

$$\begin{array}{ccc} \angle A \simeq \angle A' & & AB = k \cdot A'B' \\ \angle B \simeq \angle B' & \text{or} & BC = k \cdot B'C' \\ \angle C \simeq \angle C' & & CA = k \cdot C'A' \end{array}$$

SAS-Similarity Theorem

Theorem 7 (SAS-Similarity Theorem) *Knowing the ratio of the lengths of two sides and the measure of the angle between them, determines a triangle up to similarity. In pictures, we have something like:*



$$\frac{AB}{AC} = \frac{AD}{AE} \qquad \Rightarrow \qquad \triangle ABC \simeq \triangle ADE.$$

Question What does this mean, "up to similarity?"

?

Let's see if we (meaning you) can get to the bottom of why this theorem is true. This time, you're going to produce the illustrations. Use folding and tracing why not!

Question Fold any triangle. Now fold another triangle sharing one of the angles so that the ratio of the lengths of the sides are the same in both triangles. The sides touching the angle should share folds. You should see some parallel lines. Which theorem above says that this should happen?

?

Question What do we know about parallel lines crossing another line?

?

Question Can you sketch out (in words) how the questions above prove the SAS-Similarity Theorem?

?

3.3.2 A Meaning of Multiplication

Do you ever sit around asking yourself what different things *mean*? For example, "What does being happy really mean?" In mathematics we don't dare try to tackle a brain-buster like that one. Instead we try to focus on simple problems.

Question What can multiplication mean?

?

I have no idea how you might have answered that question. Anyhow, maybe you can answer the next questions:

Question Can you give multiplication meaning involving groups of groups or something of the sort?

?

Question Can you give multiplication meaning involving areas or something of the sort?

?

Question Compare and contrast the two meanings of multiplication given above.

OK—all of this is fine and good, but we one little problem. If numbers by themselves "mean" lengths and numbers multiplied together "mean" areas, how do we do something like this:

$$\underbrace{3}_{\text{length}} + \underbrace{4 \cdot 5}_{\text{area}}?$$

I say we can use similar triangles to save the day!

Question Can you somehow give "meaning" to multiplication using similar triangles? Hint: One of them should have a side of length 1.

?

Problems for Section 3.3

(1) Compare and contrast the ideas of *equal triangles*, *congruent triangles*, and *similar triangles*.

(2) Explain why all equilateral triangles are similar to each other.

(3) Explain why all isosceles right triangles are similar to each other.

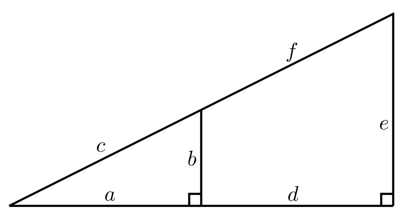
(4) Explain why when given a right triangle, the altitude of the right angle divides the triangle into two smaller triangles each similar to the original right triangle.

(5) The following sets contain lengths of sides of similar triangles. Solve for all unknowns—give all solutions. In each case explain your reasoning.

(a) $\{3,4,5\}$, $\{6,8,x\}$
(b) $\{3,3,5\}$, $\{9,9,x\}$
(c) $\{5,5,x\}$, $\{10,4,y\}$
(d) $\{5,5,x\}$, $\{10,8,y\}$
(e) $\{3,4,x\}$, $\{4,5,y\}$

(6) A *Pythagorean Triple* is a set of three positive integers $\{a,b,c\}$ such that $a^2+b^2=c^2$. Write down an infinite list of Pythagorean Triples. Explain your reasoning and justify all claims.

(7) Here is a right triangle, note it is **not** drawn to scale:



Solve for all unknowns in the following cases.

(a) $a=3$, $b=?$, $c=?$, $d=12$, $e=5$, $f=?$
(b) $a=?$, $b=3$, $c=?$, $d=8$, $e=13$, $f=?$
(c) $a=7$, $b=4$, $c=?$, $d=?$, $e=11$, $f=?$
(d) $a=5$, $b=2$, $c=?$, $d=6$, $e=?$, $f=?$

In each case explain your reasoning.

(8) Suppose you have two similar triangles. What can you say about the area of one in terms of the area of the other? Be specific and explain your reasoning.

(9) During a solar eclipse we see that the apparent diameter of the Sun and Moon are nearly equal. If the Moon is around 240000 miles from Earth, the Moon's diameter is about 2000 miles, and the Sun's diameter is about 865000 miles how far is the Sun from the Earth?

  (a) Draw a relevant (and helpful) picture showing the important points of this problem.
  (b) Solve this problem, be sure to explain your reasoning.

(10) When jets fly above 8000 meters in the air they form a vapor trail. Cruising altitude for a commercial airliner is around 10000 meters. One day I reached my arm into the sky and measured the length of the vapor trail with my hand—my hand could just span the entire trail. If my hand spans 9 inches and my arm extends 25 inches from my eye, how long is the vapor trail? Explain your reasoning.

  (a) Draw a relevant (and helpful) picture showing the important points of this problem.
  (b) Solve this problem, be sure to explain your reasoning.

(11) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Dave sits comfortably with his dog Fritz at a distance of 10 feet. How far must Michael stand from his TV to have the "same" viewing experience? Explain your reasoning.

  (a) Draw a relevant (and helpful) picture showing the important points of this problem.
  (b) Solve this problem, be sure to explain your reasoning.

(12) You love IMAX movies. While the typical IMAX screen is 72 feet by 53 feet, your TV is only a 32 inch screen—it has a 32 inch diagonal. How close do you have to sit to your screen to simulate the IMAX format? Explain your reasoning.

  (a) Draw a relevant (and helpful) picture showing the important points of this problem.
  (b) Solve this problem, be sure to explain your reasoning.

(13) David proudly owns a 42 inch (measured diagonally) flat screen TV. Michael proudly owns a 13 inch (measured diagonally) flat screen TV. Michael stands and watches his TV at a distance of 2 feet. Dave sits comfortably with his dog Fritz at a distance of 10 feet. Whose TV appears bigger to the respective viewer? Explain your reasoning.

  (a) Draw a relevant (and helpful) picture showing the important points of this problem.

(b) Solve this problem, be sure to explain your reasoning.

(14) Here is a personal problem: Suppose you are out somewhere and you see that when you stretch out your arm, the width of your thumb is the same apparent size as a distant object. How far away is the object if you know the object is:

(a) 6' long (as tall as a person).

(b) 16' long (as long as a car).

(c) 40' long (as long as a school bus).

(d) 220' long (as long as a large passenger airplane).

(e) 340' long (as long as an aircraft carrier).

Explain your reasoning.

(15) I was walking down Woody Hayes Drive, standing in front of St. John Arena when a car pulled up and the driver asked, "Where is Ohio Stadium?" At this point I was a bit perplexed, but nevertheless I answered, "Do you see the enormous concrete building on the other side of the street that looks like the Roman Colosseum? That's it."

The person in the car then asked, "Where are the Twin-Towers then?" Looking up, I realized that the towers were in fact just covered by top of Ohio Stadium. I told the driver to just drive around the stadium until they found two enormous identical towers—that would be them. They thanked me and I suppose they met their destiny.

I am about 2 meters tall, I was standing about 100 meters from the Ohio Stadium and Ohio Stadium is about 40 meters tall. If the Towers are around 500 meters from the rotunda (the front entrance of the stadium), how tall could they be and still be obscured by the stadium? Explain your reasoning—for the record, the towers are about 80 meters tall.

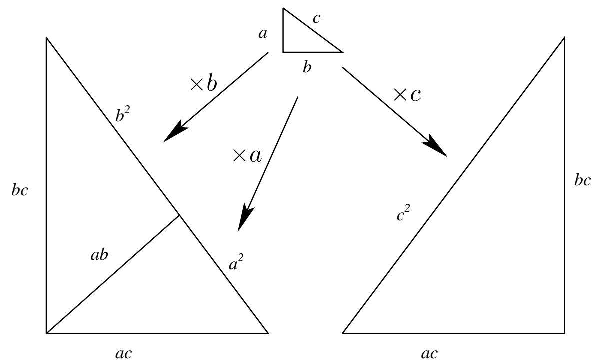
(16) Explain how to use the notion of similar triangles to multiply numbers with your answer expressed as a segment of the appropriate length.

(17) Explain how to use the notion of similar triangles to divide numbers with your answer expressed as a segment of the appropriate length.

(18) Consider the following combinations of S's and A's. Which of them produce a *Congruence Theorem*? Which of them produce a *Similarity Theorem*? Explain your reasoning.

SSS, SSA, SAS, SAA, ASA, AAA

(19) Explain how the following picture "proves" the Pythagorean Theorem.



Chapter 4

Coordinate Constructions

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> —Joseph Louis Lagrange

4.1 Constructions

One of the deepest and powerful aspects of mathematics is that it allows one to see connections between disparate areas. So far we have used different physical techniques (compass and straightedge constructions along with origami constructions) to solve similar problems. Take a minute and reflect upon that—isn't it cool that similar problems can be solved by such different methods? You back? OK—so let's see if we can solidify these connections through abstraction and in the process, make a third connection. We are going to see the algebra behind the geometry we've done. Making these connections isn't easy and can be scary. Thankfully, you are a fearless (yet gentle) reader.

Rules for Coordinate Constructions

(1) Points can only be placed as the intersection of lines and/or circles.

(2) Lines are defined as all solutions to equations of the form

$$ax + by = c \qquad \text{for given } a, b, c.$$

(3) Circles centered at (a, b) of radius c are defined as all solutions to equations of the form

$$(x - a)^2 + (y - b)^2 = c^2 \qquad \text{for given } a, b, c.$$

(4) The distance between two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$ is given by

$$d(A, B) = \sqrt{(a_x - b_x)^2 + (a_y - b_y)^2}.$$

Just as we have done before, we will present several basic constructions. Compare these to the ones done with a compass and straightedge and the ones done by origami. We will proceed by the order of difficulty of the construction.

Construction (Bisecting a Segment) *Given a segment, we wish to cut it in half.*

(1) *Let (x_1, y_1) and (x_2, y_2) be the endpoints of your segment.*

(2) *We claim the midpoint is:*

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Question Can you explain why this works?

?

Construction (Parallel through a Point) *Given a line and a point, we wish to construct another line parallel to the first that passes through the given point.*

(1) *Let $ax + by = c$ be the line and let (x_0, y_0) be the point.*

(2) *Set $c_0 = ax_0 + by_0$.*

(3) *The line $ax + by = c_0$ is the desired parallel line.*

Question Can you explain why this works?

?

Construction (Perpendicular through a Point) *Given a point and a line, we wish to construct a line perpendicular to the original line that passes through the given point.*

(1) *Let (x_0, y_0) be the given point and let $ax + by = c$ be the given line.*

(2) *Find $c_0 = bx_0 - ay_0$.*

(3) *The desired line is $bx + (-a)y = c_0$.*

Question Can you explain why this works? Can you give some examples of it in action?

?

Construction (Line between two Points) *Given two points, we wish to give the line connecting them.*

(1) *Call the two points* (x_1, y_1) *and* (x_2, y_2).

(2) *Write*

$$ax_1 + by_1 = c,$$
$$ax_2 + by_2 = c.$$

(3) *Solve for* $-a/b$ *and* c.

Example *Suppose you want to find the line between the points* $(3, 1)$ *and* $(2, 5)$. *Write*

$$a \cdot 3 + b \cdot 1 = c,$$
$$a \cdot 2 + b \cdot 5 = c,$$

and subtract these equations to get:

$$a - b \cdot 4 = 0$$

Now we see

$$-b \cdot 4 = -a,$$
$$-4 = -a/b.$$

Now we can take ***any*** *values of* a *and* b *that make the equation above true, and plug them back in to* $a \cdot 3 + b = c$ *to obtain* c. ***You should explain why this works!*** *I choose* $a = 4$ *and* $b = 1$. *From this I see that* $c = 13$ *so the line we desire is:*

$$4x + y = 13$$

Construction (Intersection of a Line and a Circle) *We wish to find the points where a given line meets a given circle.*

(1) *Let* $ax + by = c$ *be the given line.*

(2) *Let* $(x - x_0)^2 + (y - y_0)^2 = r^2$ *be the given circle.*

(3) *Solve for* x *and* y.

Question Can you give an example and draw a picture of this construction?

?

Construction (Bisecting an Angle) *We wish to divide an angle in half.*

(1) *Find two points on the angle equidistant from the vertex.*

(2) *Bisect the segment connecting the point above.*

(3) *Find the line connecting the vertex to the bisector above.*

Question Can you give an example and draw a picture of this construction?

?

Construction (Intersection of Two Circles) *Given two circles, we wish to find the points where they meet.*

(1) *Let* $(x - a_1)^2 + (y - b_1)^2 = c_1^2$ *be the first circle.*

(2) *Let* $(x - a_2)^2 + (y - b_2)^2 = c_2^2$ *be the second circle.*

(3) *Solve for* x *and* y.

Question Can you give an example and draw a picture of this construction? How many examples should you give for "completeness" sake?

?

Question We wish to construct an equilateral triangle given the length of one side. Can you do this?

?

Problems for Section 4.1

(1) What are the rules for coordinate constructions?

(2) Explain how to transfer a segment using coordinate constructions.

(3) Explain how to copy an angle using coordinate constructions (but don't actually do it!)

(4) Given two points, use coordinate constructions to construct a line between both points. Explain the steps in your construction.

(5) Given segment, use coordinate constructions to bisect the segment. Explain the steps in your construction.

(6) Given a point and line, use coordinate constructions to construct a line parallel to the given line that passes through the given point. Explain the steps in your construction.

(7) Given a point and line, use coordinate constructions to construct a line perpendicular to the given line that passes through the given point. Explain the steps in your construction.

(8) Given a line and a circle, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(9) Use coordinate constructions to bisect a given angle. Explain the steps in your construction.

(10) Given two circles, use coordinate constructions to construct the intersection of these figures. Explain the steps in your construction.

(11) Use algebra to help explain why lines intersect in zero, one, or infinitely many points.

(12) Use algebra to help explain why circles and lines intersect in zero, one, or two points.

(13) Use algebra to help explain why circles intersect in zero, one, two, or infinitely many points.

(14) Use coordinate constructions to construct an equilateral triangle. Explain the steps in your construction.

(15) Use coordinate constructions to construct a square. Explain the steps in your construction.

(16) Use coordinate constructions to construct a regular hexagon. Explain the steps in your construction.

4.2 Brave New Anatomy of Figures

Once more, in studying geometry we seek to discover the points that can be obtained given a set of rules. Now the set of rules consists of the rules for coordinate constructions.

Question In regards to coordinate constructions, what is a *point*?

?

Question In regards to coordinate constructions, what is a *line*?

?

Question In regards to coordinate constructions, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

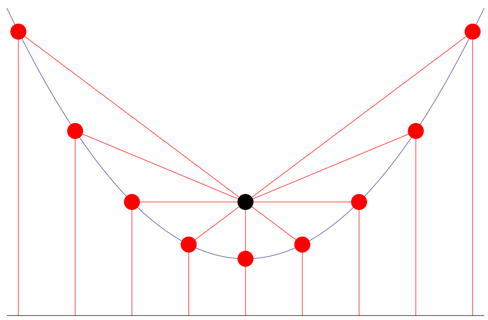
Question Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

?

4.2.1 Parabolas

Recall the definition of a *parabola*:

Definition Given a point and a line, a **parabola** is the set of points such that each of these points is the same distance from the given point as it is from the given line.



Fancy folks call the point the **focus** and they call the line the **directrix**.

However I know that you—being rather cosmopolitan in your knowledge and experience—know that from a coordinate geometry point of view that the formula for a parabola should be *something* like:

$$y = ax^2 + bx + c$$

Question How do you rectify these two different notions of a parabola?

I'm feeling chatty, so let me take this one. What would be really nice is if we could extract the focus and directrix from any formula of the form $y = ax^2+bx+c$. I think we'll work it for a specific example. Consider:

$$y = 3x^2 + 6x - 7$$

Step 1 Complete the square. Write:

$$\begin{aligned} y &= 3x^2 + 6x - 7 \\ &= 3(x^2 + 2x) - 7 \\ &= 3(x^2 + 2x + 1 - 1) - 7 \\ &= 3(x^2 + 2x + 1) - 3 - 7 \\ &= 3(x + 1)^2 - 10 \end{aligned}$$

Step 2 Compare with the following basic form:

$$y = u(x - v)^2 + w$$

Given a parabola in the form above, we have that

$$\text{focus}: \left(v, w + \frac{1}{4u}\right) \qquad \text{and} \qquad \text{directrix}: y = w - \frac{1}{4u}.$$

So in our case the focus is at

$$\left(-1, -10 + \frac{1}{12}\right)$$

and our directrix is the line

$$y = -10 - \frac{1}{12}.$$

Question Can you use the distance formula to show that every point on the parabola is the same distance from focus as it is from the directrix?

?

Problems for Section 4.2

(1) In regards to coordinate constructions, what is a *point*? Compare and contrast this to a naive notion of a point.

(2) In regards to coordinate constructions, what is a *line*? Compare and contrast this to a naive notion of a line.

(3) In regards to coordinate constructions, what is a *circle*? Compare and contrast this to a naive notion of a circle. In particular, explain how the formula for the circle arises.

(4) Explain what is meant by the *focus* of a parabola.

(5) Explain what is meant by the *directrix* of a parabola.

(6) Will the following formula

$$y = ax^2 + bx + c$$

really plot *any* parabola in the plane? If so why? If not, can you give a formula that will? Explain your reasoning.

(7) For each parabola given, find the focus and directrix:

(a) $y = x^2$
(b) $y = 7x^2$
(c) $y = -2x^2$
(d) $y = x^2 - 4x$
(e) $y = x^2 - 12$
(f) $y = x^2 - x + 1$
(g) $y = x^2 + 2x - 5$
(h) $y = 2x^2 - 3x - 7$
(i) $y = -17x^2 + 42x - 3$
(j) $x = y^2 - 5y$
(k) $x = 3y^2 - 23y + 17$

In each case explain your reasoning.

(8) Explain in general terms (without appealing to an example) how to find the focus and directrix of a parabola $y = ax^2 + bx + c$.

(9) Use coordinate constructions to construct the circle that passes through the points:

$$A = (0, 0), \qquad B = (3, 3), \qquad C = (4, 0).$$

Sketch this situation and explain your reasoning.

(10) Consider the points

$$A = (1,1) \qquad \text{and} \qquad B = (5,3).$$

(a) Find the midpoint between A and B.
(b) Find the line the connects A and B. Use algebra to show that the midpoint found above is actually on this line.
(c) Use algebra to show that this midpoint is equidistant from both A and B.

Sketch this situation and explain your reasoning in each step above.

(11) Consider the parabola $y = x^2/4 + x + 2$.

(a) Find the focus and directrix of this parabola.
(b) Sketch the parabola by plotting points.
(c) Use origami to fold the envelope of tangents of the parabola.

Present the above items simultaneously on a single graph. Explain the steps in your work.

(12) Consider the following line and circle:

$$x - y = -1 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(13) Consider the following two circles:

$$x^2 + y^2 = 5 \qquad \text{and} \qquad (x-1)^2 + (y-1)^2 = 5$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(14) Consider the following two circles:

$$(x+1)^2 + (y-1)^2 = 9 \qquad \text{and} \qquad (x-3)^2 + (y-2)^2 = 4$$

Use algebra to find their points of intersection. What were the degrees of the equations you solved to find these points? Sketch this situation and explain your reasoning.

(15) Explain how to find the minimum or maximum of a parabola of the form:

$$y = ax^2 + bx + c$$

(16) Given a triangle, use coordinate constructions to construct the circumcenter. Explain the steps in your construction.

(17) Given a triangle, use coordinate constructions to construct the orthocenter. Explain the steps in your construction.

(18) Given a triangle, use coordinate constructions to construct the incenter. Explain the steps in your construction.

(19) Given a triangle, use coordinate constructions to construct the centroid. Explain the steps in your construction.

(20) Use coordinate constructions to construct a triangle given the length of one side, the length of the the median to that side, and the length of the altitude of the opposite angle. Explain the steps in your construction.

(21) Use coordinate constructions to construct a triangle given one angle, the length of an adjacent side and the altitude to that side. Explain the steps in your construction.

(22) Use coordinate constructions to construct a triangle given one angle and the altitudes to the other two angles. Explain the steps in your construction.

(23) Use coordinate constructions to construct a triangle given two sides and the altitude to the third side. Explain the steps in your construction.

4.3 Constructible Numbers

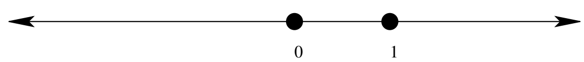
We've now practiced three types of constructions:

(1) Compass and straightedge constructions.

(2) Origami constructions.

(3) Coordinate constructions.

You may be wondering what is meant by the words "constructible numbers." Imagine a line with two points on it:



Label the left point 0 and the right point 1. If we think of this as a starting point for a number line, then a **constructible number** is nothing more than a point we can obtain on the above number line using one of the construction techniques above starting with the points 0 and 1.

(1) Denote the set of numbers constructible by compass and straightedge with $\mathcal{C}$. We'll call $\mathcal{C}$ the set of *constructible numbers*.

(2) Denote the set of numbers constructible by origami with $\mathcal{O}$. We'll call $\mathcal{O}$ the set of *origami numbers*.

(3) Denote the set of numbers constructible by coordinate constructions with $\mathcal{D}$. We'll call $\mathcal{D}$ the set of *Descartes numbers*[1].

Mostly in this chapter we'll be talking about $\mathcal{C}$. You'll have to deal with $\mathcal{O}$ and $\mathcal{D}$ yourself.

Question Exactly what numbers are in $\mathcal{C}$?

?

How do we attack this question? Well first let's get a bit of notation. Recall that we use the symbol "$\in$" to mean *is in*. So we know that 0 and 1 are *in* the set of constructible numbers. So we write

$$0 \in \mathcal{C} \qquad \text{and} \qquad 1 \in \mathcal{C}.$$

Question Is this true for $\mathcal{O}$, the set of origami numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

If we could use constructions to make the operations $+$, $-$, $\cdot$, and $\div$, then we would be able to say a lot more. In fact we will do just this.

[1] Be warned, this notion of so-called "Descartes numbers" is unique to these pages.

Question How does one add and subtract using a compass and straightedge?

?

Question Starting with 0 and 1, what numbers could we add to our number line by simply adding and subtracting?

At this point we have all the positive whole numbers, zero, and the negative whole numbers. We have a special name for this set, we call it the **integers** and denote it by the letter $\mathbb{Z}$:

$$\mathbb{Z} = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}.$$

Question Are the integers contained in $\mathcal{O}$, the set of origami numbers? Are the integers continued in $\mathcal{D}$, the set of Descartes numbers?

?

We still have some more operations:

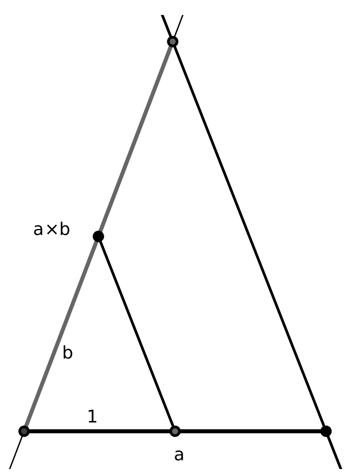
Construction (Multiplication) *This construction is based on the idea of similar triangles. Start with given segments of length a, b, and 1:*

(1) *Make a small triangle with the segment of length 1 and segment of length b.*

(2) *Now place the segment of length a on top of the unit segment with one end at the vertex.*

(3) *Draw a line parallel to the segment connecting the unit to the segment of length b starting at the other end of segment of length a.*

(4) *The length from the vertex to the point that the line containing b intersects the line drawn in Step 3 is of length $a \cdot b$.*



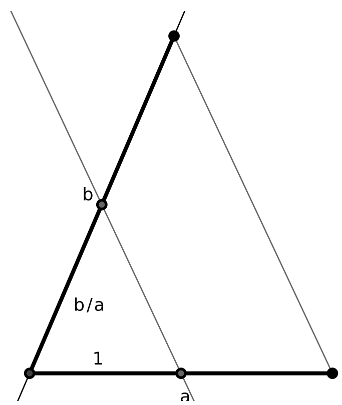
Construction (Division) *This construction is also based on the idea of similar triangles. Again, you start with given segments of length a, b, and 1:*

(1) *Make a triangle with the segment of length a and the segment of length b.*

(2) *Put the unit along the segment of length a starting at the vertex where the segment of length a and the segment of length b meet.*

(3) *Make a line parallel to the third side of the triangle containing the segment of length a and the segment of length b starting at the end of the unit.*

(4) *The distance from where the line drawn in Step 3 meets the segment of length b to the vertex is of length b/a.*



Question What does our number line look like at this point?

Currently we have $\mathbb{Z}$, the integers, and all of the fractions. In other words:

$$\mathbb{Q} = \left\{ \frac{a}{b} \text{ such that } a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

Fancy folks will replace the words *such that* with a colon ":" to get:

$$\mathbb{Q} = \left\{ \frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z} \text{ with } b \neq 0 \right\}$$

We call this set the **rational numbers**. The letter $\mathbb{Q}$ stands for the word *quotient*, which should remind us of fractions.

In mathematics we study sets of numbers. In any field of science, the first step to understanding something is to classify it. One sort of classification that we have is the notion of a *field*.

Definition A **field** is a set of numbers, which we will call F, that is closed under two associative and commutative operations $+$ and $\cdot$ such that:

(1) (a) There exists an additive identity $0 \in F$ such that for all $x \in F$,

$$x + 0 = x.$$

(b) For all $x \in F$, there is an additive inverse $-x \in F$ such that

$$x + (-x) = 0.$$

(2) (a) There exists a multiplicative identity $1 \in F$ such that for all $x \in F$,

$$x \cdot 1 = x.$$

(b) For all $x \in F$ where $x \neq 0$, there is a multiplicative inverse x^{-1} such that

$$x \cdot x^{-1} = 1.$$

(3) Multiplication distributes over addition. That is, for all $x, y, z \in F$

$$x \cdot (y + z) = x \cdot y + x \cdot z.$$

Now, a word is in order about three tricky words I threw in above: *closed*, *associative*, and *commutative*:

Definition A set F is **closed** under an operation $\star$ if for all $x, y \in F$, $x \star y \in F$.

Example *The set of integers, $\mathbb{Z}$, is closed under addition, but is not closed under division.*

Definition An operation $\star$ is **associative** if for all x, y, and z

$$x \star (y \star z) = (x \star y) \star z.$$

Definition An operation $\star$ is **commutative** if for all x, y

$$x \star y = y \star x.$$

Question Is $\mathbb{Z}$ a field? Is $\mathbb{Q}$ a field? Can you think of other fields? What about the set of constructible numbers $\mathcal{C}$? What about the origami numbers $\mathcal{O}$? What about the Descartes numbers $\mathcal{D}$?

?

From all the constructions above we see that the set of constructible numbers $\mathcal{C}$ is a field. However, which field is it? In fact, the set of constructible numbers is bigger than $\mathbb{Q}$!

Construction (Square-Roots) *Start with given segments of length a and 1:*

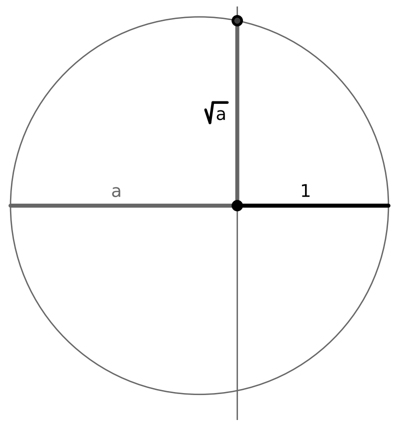
(1) *Put the segment of length a immediately to the left of the unit segment on a line.*

(2) *Bisect the segment of length $a + 1$.*

(3) *Draw an arc centered at the bisector that starts at one end of the line segment of length $a + 1$ and ends at the other end.*

(4) *Construct the perpendicular at the point where the segment of length a meets the unit.*

(5) *The line segment connecting the meeting point of the segment of length a and the unit to the arc drawn in Step 3 is of length $\sqrt{a}$.*



This tells us that square-roots are constructible. In particular, the square-root of two is constructible. But the square-root of two is not rational! That is, there is no fraction

$$\frac{a}{b} = \sqrt{2} \qquad \text{such that } a, b \in \mathbb{Z}.$$

Question Can you remind me, how do we know that $\sqrt{2}$ is not rational?

?

Question Are square-roots found in $\mathcal{O}$, the set of origami numbers? What about $\mathcal{D}$, the set of Descartes numbers?

?

OK, so how do we talk about a field that contains both $\mathbb{Q}$ and $\sqrt{2}$? Simple, use this notation:

$$\mathbb{Q}(\sqrt{2}) = \{\text{the smallest field containing both } \mathbb{Q} \text{ and } \sqrt{2}\}$$

So the set of constructible numbers contains all of $\mathbb{Q}(\sqrt{2})$. Does the set of constructible numbers contain even more numbers? Yes! In fact the $\sqrt{3}$ is also not rational, but is constructible. So here is our situation:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathcal{C}$$

So all the numbers in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ are also in $\mathcal{C}$. But is this all of $\mathcal{C}$? Hardly! We could keep on going, adding more and more square-roots 'til the cows come home, and we still will not have our hands on all of the constructible numbers. But all is not lost. We can still say something:

Theorem 8 *The use of compass and straightedge alone on a field F can at most produce numbers in a field $F(\sqrt{\alpha})$ where $\alpha \in F$.*

Question Can you explain why the above theorem is true? Big hint: What is the relationship between $\mathcal{C}$ and $\mathcal{D}$?

?

The upshot of the above theorem is that the only numbers that are constructible are expressible as a combination of rational numbers and the symbols:

$$+ \quad - \quad \cdot \quad \div \quad \sqrt{}$$

So what are examples of numbers that are not constructible? Well to start $\sqrt[3]{2}$ is not constructible. Also π is not constructible. While both of these facts can be carefully explained, we will spare you gentle reader—for now[2].

Question Which of the following numbers are constructible?

$$3.1415926, \qquad \sqrt[16]{5}, \qquad \sqrt[3]{27}, \qquad \sqrt[6]{27}.$$

?

[2]The most accessible discussion of this fact that I know of can be found in [4].

Problems for Section 4.3

(1) Explain what the set denoted by $\mathbb{Z}$ is.

(2) Explain what the set denoted by $\mathbb{Q}$ is.

(3) Explain what the set $\mathcal{C}$ of constructible numbers is.

(4) Given two line segments a and b, construct $a+b$. Explain the steps in your construction.

(5) Given two line segments a and b, construct $a-b$. Explain the steps in your construction.

(6) Given three line segments 1, a, and b, construct $a \cdot b$. Explain the steps in your construction.

(7) Given three line segments 1, a, and b, construct a/b. Explain the steps in your construction.

(8) Given a unit, construct $4/3$. Explain the steps in your construction.

(9) Given a unit, construct $3/4$. Explain the steps in your construction.

(10) Use the construction for multiplication to explain why when multiplying two numbers between 0 and 1, the product is always still between 0 and 1.

(11) Explain why the construction for multiplication works.

(12) Use the construction for division to explain why when dividing a positive number by a number between 0 and 1, the quotient is always larger than the initial positive number.

(13) Explain why the construction for division works.

(14) Given a unit, construct $\sqrt{2}$. Explain the steps in your construction.

(15) Use algebra to help explain why the construction for square-roots works.

(16) Give relevant and revealing examples of numbers in the set $\mathbb{Z}$.

(17) Give relevant and revealing examples of numbers in the set $\mathbb{Q}$.

(18) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2})$.

(19) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3})$.

(20) Give relevant and revealing examples of numbers in the set $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$.

(21) Which of the following are constructible numbers? Explain your answers.

   (a) 3.141

   (b) $\sqrt[3]{5}$

(c) $\sqrt{3+\sqrt{17}}$
(d) $\sqrt[8]{5}$
(e) $\sqrt[10]{37}$
(f) $\sqrt[16]{37}$
(g) $\sqrt[3]{28}$
(h) $\sqrt[3]{27}$
(i) $\sqrt{13+\sqrt[3]{2}+\sqrt{11}}$
(j) $3+\sqrt[5]{4}$
(k) $\sqrt{3+\sqrt{19}+\sqrt{10}}$

(22) Is $\sqrt{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(23) Is $\sqrt{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(24) Is $\sqrt{9}$ a rational number? Is it a constructible number? Explain your reasoning.

(25) Is $\sqrt[3]{7}$ a rational number? Is it a constructible number? Explain your reasoning.

(26) Is $\sqrt[3]{8}$ a rational number? Is it a constructible number? Explain your reasoning.

(27) Is $\sqrt[3]{9}$ a rational number? Is it a constructible number? Explain your reasoning.

4.4 Impossibilities

Oddly enough, the importance of compass and straightedge constructions is not so much what we can construct, but what we cannot construct. It turns out that classifying what we cannot construct is an interesting question. There are three classic problems which are impossible to solve with a compass and straightedge alone:

(1) Doubling the cube.

(2) Squaring the circle.

(3) Trisecting the angle.

4.4.1 Doubling the Cube

The goal of this problem is to double the volume of a given cube. This boils down to trying to construct roots to the equation:

$$x^3 - 2 = 0$$

But we can see that the only root of the above equation is $\sqrt[3]{2}$ and we already know that this number is not constructible.

Question Why does doubling the cube boil down to constructing a solution to the equation $x^3 - 2 = 0$?

?

4.4.2 Squaring the Circle

Given a circle of radius r, we wish to construct a square that has the same area. Why would someone want to do such a thing? Well to answer this question you must ask yourself:

Question What is area?

?

So what is the deal with this problem? Well suppose you have a circle of radius 1. Its area is now π square units. How long should the edge of a square be if it has the same area? Well the square should have sides of length $\sqrt{\pi}$ units. In 1882, it was proved that π is not the root of any polynomial equation, and hence $\sqrt{\pi}$ is not constructible. Therefore, it is impossible to square the circle.

4.4.3 Trisecting the Angle

This might sound like the easiest to understand, but it's a bit subtle. Given any angle, the goal is to trisect that angle. It can be shown that this cannot be done using a compass and straightedge. In particular, it is impossible to trisect a 60 degree angle with compass and straightedge alone. However, we are not saying that you cannot trisect some angles with compass and straightedge alone, in fact there are *special* angles which can be trisected using a compass and straightedge. However the methods used to trisect those special angles will fail miserably in nearly all other cases.

Question Can you think of any angles that can be trisected using a compass and straightedge?

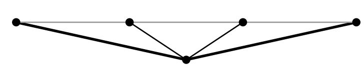
?

Just because it is impossible to trisect an arbitrary angle with compass and straightedge alone does not stop people from trying.

Question If you did not know that it was impossible to trisect an arbitrary angle with a compass and straightedge alone, how might you try to do it?

?

One common way that people try to trisect angles is to take an angle, make an isosceles triangle using the angle, and divide the line segment opposite the angle into three equal parts. While you can divide the opposite side into three equal parts, it in fact **never** trisects the angle. When you do this procedure to acute angles, it *seems* to work, though it doesn't really. You can see that it doesn't by looking at an obtuse angle:



Trisecting the line segment opposite the angle clearly leaves the middle angle much larger than the outer two angles. This happens regardless of the measure of the angle. This mistake is common among people who think that they can trisect an angle with compass and straightedge alone.

4.4.4 Origami's Time to Shine

We know that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathcal{C} = \mathcal{D}$$

Where does the set of origami numbers $\mathcal{O}$ fit into the parade? I'll tell you, if you promise not to tell anybody that I did... $\mathcal{O}$ is the leader of the pack! We already know that you can trisect angles using origami constructions. In fact you can even solve cubic equations! We'll show you how to do this.

Construction (Solving Cubic Equations) *We wish to solve equations of the form:*

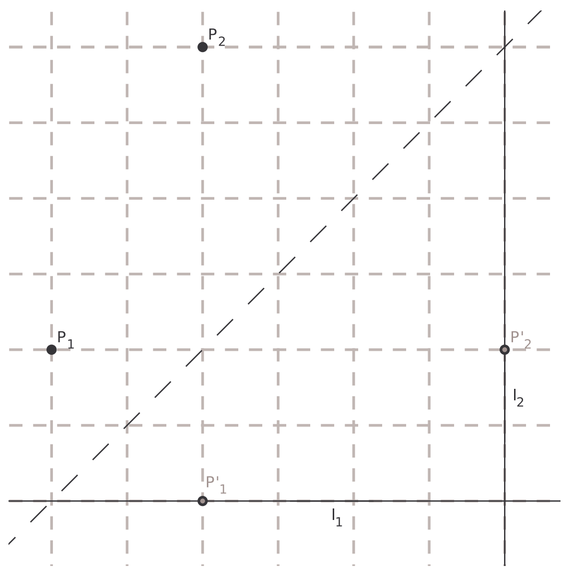
$$x^3 + ax^2 + bx + c = 0$$

(1) *Plot the points:* $P_1 = (a, 1)$ *and* $P_2 = (c, b)$.

(2) *Plot the lines:* $\ell_1 : y = -1$ *and* $\ell_2 : x = -c$.

(3) *With a single fold, place* P_1 *onto* ℓ_1 *and* P_2 *onto* ℓ_2.

(4) *The slope of the crease is a solution to* $x^3 + ax^2 + bx + c = 0$.



Question How do we get the "solution" from the slope?

?

Since origami constructions can duplicate every compass and straightedge construction and more, we have that $\mathcal{C} \subseteq \mathcal{O}$.

Problems for Section 4.4

(1) Explain the three classic problems that cannot be solved with a compass and straightedge alone.

(2) Use a compass and straightedge construction to trisect an angle of 90°. Explain the steps in your construction.

(3) Use a compass and straightedge construction to trisect an angle of 135°. Explain the steps in your construction.

(4) Use a compass and straightedge construction to trisect an angle of 45°. Explain the steps in your construction.

(5) Use a compass and straightedge construction to trisect an angle of 67.5°. Explain the steps in your construction.

(6) Use origami to construct an angle of 20°. Explain the steps in your construction.

(7) Use origami to construct an angle of 10°. Explain the steps in your construction.

(8) Is it possible to use compass and straightedge constructions to construct an angle of 10°? Why or why not?

(9) We have seen that:

$$\mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq \mathcal{C} \subseteq \mathcal{O}$$

Give explicit examples showing that the set inclusions above are strict—none of them are set equality. Explain your reasoning.

(10) Use origami to find a solution to the following cubic equations:

(a) $x^3 - x^2 - x + 1 = 0$
(b) $x^3 - 2x^2 - x + 2 = 0$
(c) $x^3 - 3x - 2 = 0$
(d) $x^3 - 4x^2 + 5x - 2 = 0$
(e) $x^3 - 2x^2 - 5x + 6 = 0$

Explain the steps in your constructions.

Chapter 5

City Geometry

> I always like a good math solution to any love problem.
>
> —Carrie Bradshaw

5.1 Welcome to the City

One day I was walking through the city—that's right, New York City. I had the most terrible feeling that I was lost. I had just passed a *Starbucks Coffee* on my left and a *Sbarro Pizza* on my right, when what did I see? Another *Starbucks Coffee* and *Sbarro Pizza*! Three options occurred to me:

(1) I was walking in circles.

(2) I was at the nexus of the universe.

(3) New York City had way too many *Starbucks* and *Sbarro Pizzas*!

Regardless, I was lost. My buddy Joe came to my rescue. He pointed out that the city is organized like a grid.

"Ah! city geometry!" I exclaimed. At this point all Joe could say was "Huh?"

Question What the heck was I talking about?

Let me tell you: *Euclidean geometry* is regular old plane (not plain!) geometry. It is the geometry that we've been exploring thus far in our journey. In *city geometry* we have points and lines, just like in Euclidean geometry. However,

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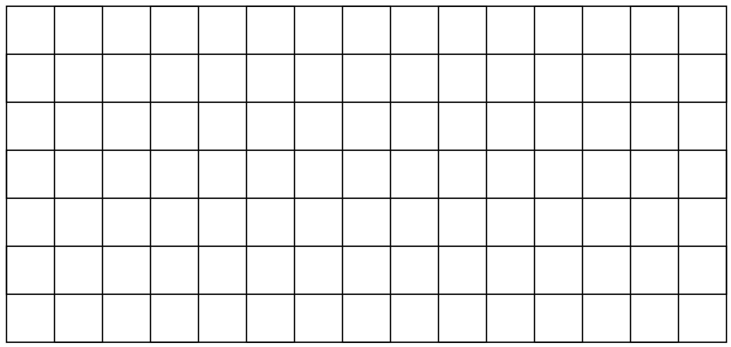
The approach taken in this section was adapted from [13].

most cities can be viewed as a grid of city blocks

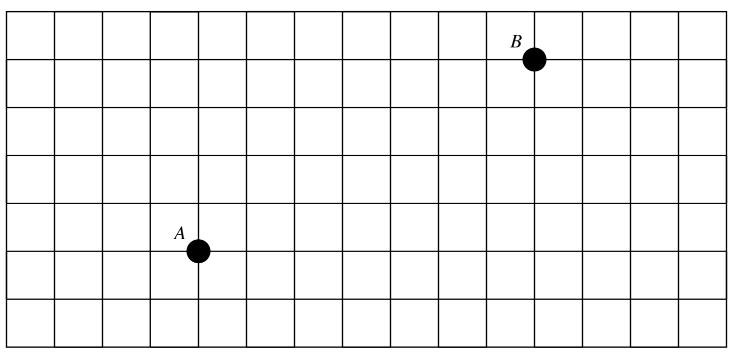


and when we travel in a city, we can only travel on the streets—we can't cut through the blocks. This means that we don't measure distance as the crow flies. Instead we use the *taxicab distance*:

Definition Given two points $A = (a_x, a_y)$ and $B = (b_x, b_y)$, we define the **taxicab distance** as:

$$d_T(A, B) = |a_x - b_x| + |a_y - b_y|$$

Example *Consider the following points:*



Let $A = (0, 0)$*. Now we see that* $B = (7, 4)$*. Hence*

$$\begin{aligned} d_T(A, B) &= |0 - 7| + |0 - 4| \\ &= 7 + 4 \\ &= 11. \end{aligned}$$

Of course in real life, you would want to add in the appropriate units to your final answer.

Question How do you compute the distance between A and B as the crow flies?

?

Definition The geometry were points and lines are those from Euclidean geometry but distance is measured via taxicab distance is called **city geometry**.

Question Compare and contrast the notion of a line in Euclidean geometry and in city geometry. In either geometry is a line the unique shortest path between any two points?

?

5.1.1 (Un)Common Structures

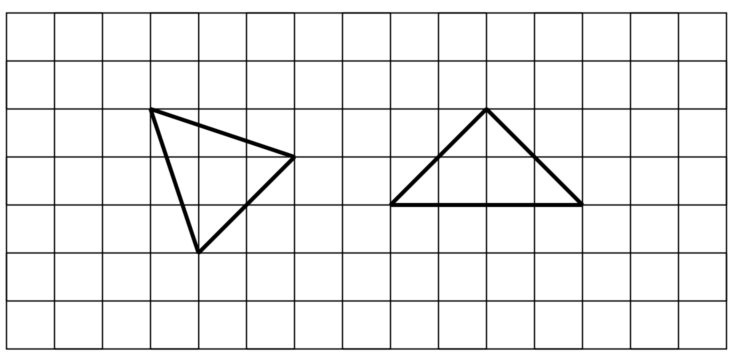
How different is life in city geometry from life in Euclidean geometry? Let's find out!

Triangles

If we think back to Euclidean geometry, we may recall some lengthy discussions on triangles. Yet so far, we have not really discussed triangles in city geometry.

Question What does a triangle look like in city geometry and how do you measure its angles?

I'll take this one. Triangles look the same in city geometry as they do in Euclidean geometry. Also, you measure angles in exactly the same way. However, there is one minor hiccup. Consider these two triangles in city geometry:



Question What are the lengths of the sides of each of these triangles? Why is this odd?

?

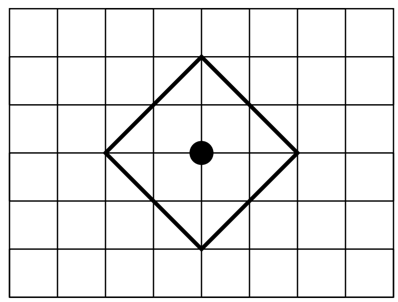
Hence we see that triangles are a bit funny in city geometry.

Circles

Circles are also discussed in many geometry courses and this course is no different. However, in city geometry the circles are a little less round. The first question we must answer is the following:

Question What is a circle?

Well, a circle is the collection of all points equidistant from a given point. So in city geometry, we must conclude that a circle of radius 2 would look like:



Question What sort of shape should a city geometry compass draw?

?

Question How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry?

?

Problems for Section 5.1

(1) Given two points A and B in city geometry, does $d_T(A, B) = d_T(B, A)$? Explain your reasoning.

(2) It was once believed that Euclid's five postulates

   (a) A line can be drawn from a point to any other point.
   (b) A finite line can be extended indefinitely.
   (c) A circle can be drawn, given a center and a radius.
   (d) All right angles are ninety degrees.
   (e) If a line intersects two other lines such that the sum of the interior angles on one side of the intersecting line is less than the sum of two right angles, then the lines meet on that side and not on the other side.

   were sufficient to completely describe plane geometry. Explain how city geometry shows that Euclid's five postulates are **not** enough to determine all of the familiar properties of the plane.

(3) In Euclidean geometry are all equilateral triangles congruent assuming they have the same side length? Is this true in city geometry? Explain your reasoning.

(4) How many points are there at the intersection of two circles in Euclidean geometry? How many points are there at the intersection of two circles in city geometry? Explain your reasoning.

(5) What sort of shape should a city geometry compass draw? Explain your reasoning.

(6) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for an equilateral triangle using a city geometry compass.

(7) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting a segment using a city geometry compass.

(8) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a perpendicular through a point using a city geometry compass.

(9) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for bisecting an angle using a city geometry compass.

(10) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for copying an angle using a city geometry compass.

(11) Give a detailed discussion of what happens if we attempt the compass and straightedge construction for a parallel through a point using a city geometry compass.

5.2 Anatomy of Figures and the City

When we study geometry, what do we seek? That's right—we wish to discover the points that can be obtained given a set of rules. With city geometry, the major rule involved is the taxicab distance. Let's answer these questions!

Question In regards to city geometry, what is a *point*?

?

Question In regards to city geometry, what is a *line*?

?

Question In regards to city geometry, what is a *circle*?

?

Now I'm going to quiz you about them (I know we've already gone over this *twice*, but it is fundamental so just smile and answer the questions):

Question Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

?

Question Place two lines randomly in the plane. How many points do you expect them to share?

?

Question Place three lines randomly in the plane. How many points do you expect all three lines to share?

?

Question Place three points randomly in the plane. Will you (almost!) always be able to draw a city geometry circle containing these points? If no, why not? If yes, how do you know?

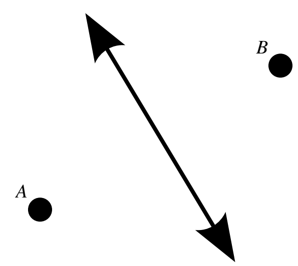
?

Midsets

Definition Given two points A and B, their **midset** is the set of points that are an equal distance away from both A and B.

Question How do we find the midset of two points in Euclidean geometry? How do we find the midset of two points in city geometry?

In Euclidean geometry, we just take the the following line:



If we had no idea what the midset should look like in Euclidean geometry, we could start as follows:

- Draw circles of radius r_1 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset. (Why?)

- Draw circles of radius r_2 centered at both A and B. If these circles intersect, then their points of intersection will be in our midset.

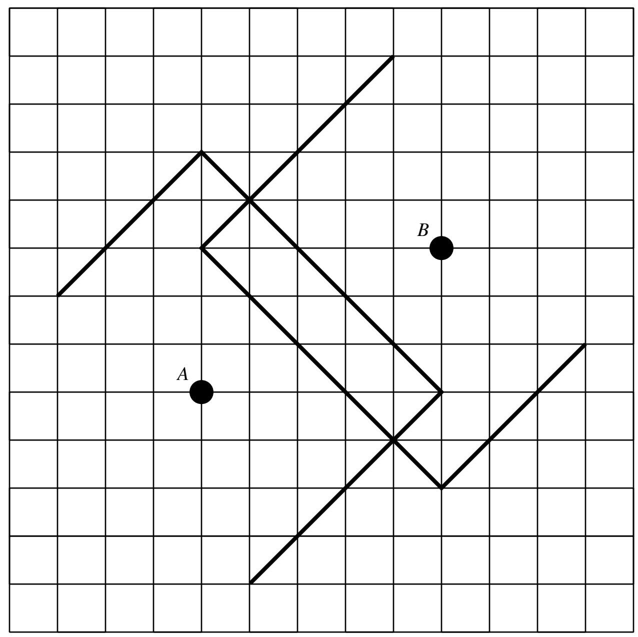
- We continue in this fashion until we have a clear idea of what the midset looks like. It is now easy to check that the line in our picture is indeed the midset.

How do we do it in city geometry? We do it basically the same way.

Example *Suppose you wished to find the midset of two points in city geometry.*

We start by fixing coordinate axes. Considering the diagram below, if $A = (0, 0)$, then $B = (5, 3)$. We now use the same idea as in Euclidean geometry. Drawing circles of radius 3 centered at A and B respectively, we see that there are no points 3 points away from both A and B. Since $d_T(A, B) = 8$, this is to be expected. We will need to draw larger taxicab circles before we will find points in the midset. Drawing taxicab circles of radius 5, we see that the points

$(1,4)$ and $(4,-1)$ are both in our midset.



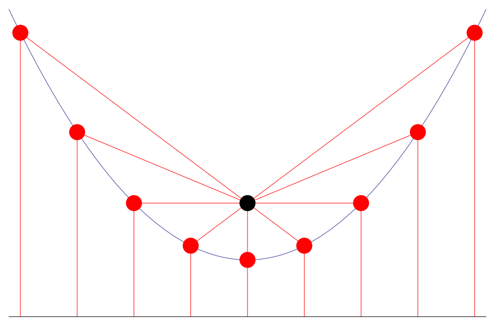
Now it is time to sing along. You draw circles of radius 6, to get two more points $(1,5)$ and $(4,-2)$. Drawing circles with larger radii yields more and more points "due north" of $(1,5)$ and "due south" of $(4,-2)$. However, if we draw circles of radius 4 centered at A and B respectively, their intersection is the line segment between $(1,3)$ and $(4,0)$. Unlike Euclidean circles, distinct city geometry circles can intersect in more than two points and city geometry midsets can be more complicated than their Euclidean counterparts.

Question How do you draw the city geometry midset of A and B? What could the midsets look like?

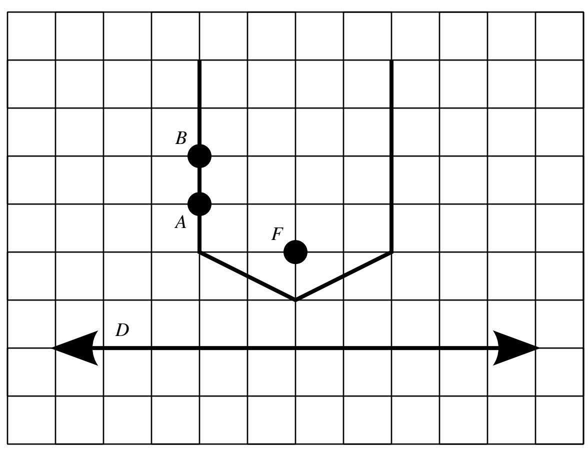
?

Parabolas

Recall that a parabola is a set of points such that each of those points is the same distance from a given point, F, as it is from a given line, D.



This definition still makes sense when we work with taxicab distance instead of Euclidean distance. To start, choose a value r and draw a line parallel to D at taxicab distance r away from D. Now draw a City circle of radius r centered at F. The points of intersection of this line and this circle will be r away from D and r away from F and so will be points on our City parabola. Repeat this process for different values of r.



Unlike the Euclidean case, the City parabola need not grow broader and broader as the distance from the line increases. In the picture above, as we go from A to B on the parabola, both the taxicab and Euclidean distances to the line D increase by 1. The taxicab distance from the point F also increases by 1 as we go from A to B but the Euclidean distance increases by less than 1. For the Euclidean distance from F to the parabola to keep increasing at the same rate as the distance to the line D, the Euclidean parabola has to keep spreading to the sides.

Question How do you draw city geometry parabolas? What do different parabolas look like?

?

A Paradox

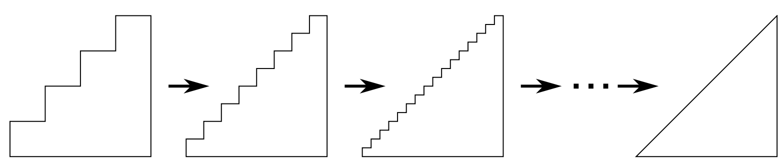
To be completely clear on what a paradox is, here is the definition we will be using:

Definition A **paradox** is a statement that seems to be contradictory. This means it seems both true and false at the same time.

There are many paradoxes in mathematics. By studying them we gain insight—and also practice tying our brain into knots! Here is a paradox:

Paradox $\sqrt{2} = 2$.

False-Proof Consider the following sequence of diagrams:



On the far right-hand side, we see a right-triangle. Suppose that the lengths of the legs of the right-triangle are one. Now by the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{1^2 + 1^2} = \sqrt{2}$.

However, we see that the triangles coming from the left converge to the triangle on the right. In every case on the left, the stair-step side has length 2. Hence when our sequence of stair-step triangles converges, we see that the hypotenuse of the right-triangle will have length 2. Thus $\sqrt{2} = 2$. ■

Question What is wrong with the proof above?

?

Problems for Section 5.2

(1) Suppose that you have two triangles $\triangle ABC$ and $\triangle DEF$ in city geometry such that

   (a) $d_T(A, B) = d_T(D, E)$.
   (b) $d_T(B, C) = d_T(E, F)$.
   (c) $d_T(C, A) = d_T(F, D)$.

   Is it necessarily true that $\triangle ABC \equiv \triangle DEF$? Explain your reasoning.

(2) In city geometry, if all the angles of $\triangle ABC$ are 60°, is $\triangle ABC$ necessarily an equilateral triangle? Explain your reasoning.

(3) In city geometry, if two right triangles have legs of the same length, is it true that their hypotenuses will be the same length? Explain your reasoning.

(4) Considering that π is the ratio of the circumference of a circle to its diameter, what is the value of π in city geometry? Explain your reasoning.

(5) Considering that the area of a circle of radius r is given by πr^2, what is the value of π in city geometry? Explain your reasoning.

(6) When is the Euclidean midset of two points equal to their city geometry midset? Explain your reasoning.

(7) Find the city geometry midset of $(-2, 2)$ and $(3, 2)$.

(8) Find the city geometry midset of $(-2, 2)$ and $(4, -1)$.

(9) Find the city geometry midset of $(-2, 2)$ and $(2, 2)$.

(10) Draw the city geometry parabola determined by the point $(0, 2)$ and the line $y = 0$.

(11) Draw the city geometry parabola determined by the point $(3, 0)$ and the line $x = 0$.

(12) Draw the city geometry parabola determined by the point $(2, 0)$ and the line $y = x$.

(13) Find the distance in city geometry from the point $(3, 4)$ to the line $y = -1/3x$. Explain your reasoning.

(14) Draw the city geometry parabola determined by the point $(0, 4)$ and the line $y = x/3$. Explain your reasoning.

(15) Draw the city geometry parabola determined by the point $(0, 6)$ and the line $y = x/2$. Explain your reasoning.

(16) Draw the city geometry parabola determined by the point $(1, 4)$ and the line $y = 2x/3$. Explain your reasoning.

(17) Draw the city geometry parabola determined by the point $(3, 3)$ and the line $y = x/2$. Explain your reasoning.

(18) Find all points P such that $d_T(P, A) + d_T(P, B) = 8$. Explain your work. (In Euclidean geometry, this condition determines an *ellipse*. The solution to this problem could be called the *city geometry ellipse*.)

(19) True/False: Three noncollinear points lie on a unique Euclidean circle. Explain your reasoning.

(20) True/False: Three noncollinear points lie on a unique city geometry circle. Explain your reasoning.

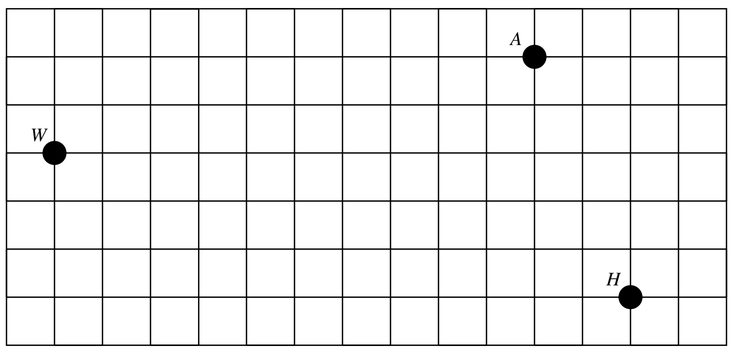
(21) Explain why no Euclidean circle can contain three collinear points. Can a city geometry circle contain three collinear points? Explain your conclusion.

(22) Can you find a false-proof showing that $\pi = 2$?

5.3 Getting Work Done

If you are interested in *real-world* types of problems, then maybe city geometry is the geometry for you. The concepts that arise in city geometry are directly applicable to everyday life.

Question Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night?



Solution We need to compute

$$d_T(A,H) + d_T(H,W)$$

Let's start by fixing a coordinate system and making A the origin. Then H is $(2,-5)$ and W is $(-10,-2)$. Then

$$\begin{aligned} d_T(A,H) &= |0-2| + |0-(-5)| \\ &= 2+5 \\ &= 7 \end{aligned}$$

and

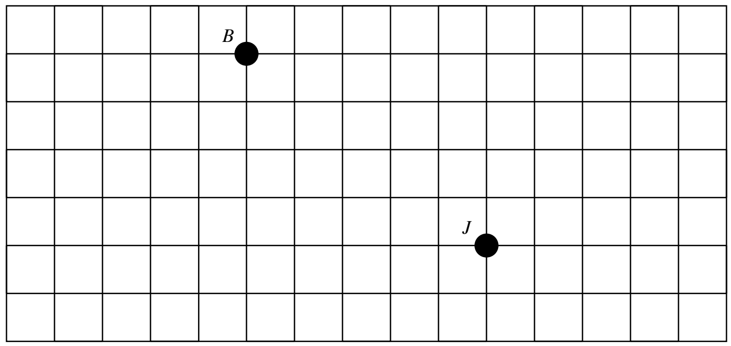
$$\begin{aligned} d_T(H,W) &= |2-(-10)| + |-5-(-2)| \\ &= 12+3 \\ &= 15. \end{aligned}$$

Will must trudge $7 + 15 = 22$ blocks in his gorilla suit. ■

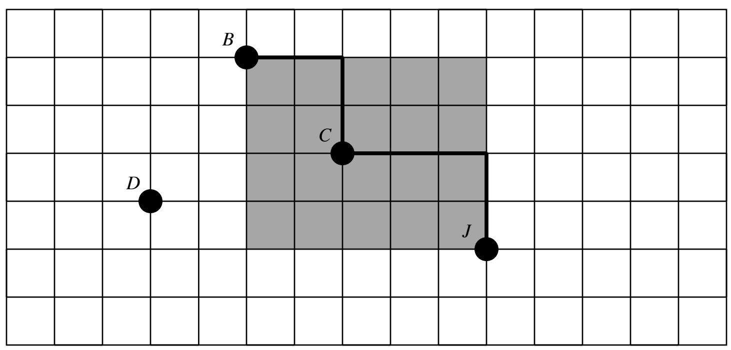
Okay, that's enough monkey business—I feel like pizza and a movie.

Question Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art*

Theater (B) to watch a movie. Where should they park to minimize the total distance walked by both?



Solution Again, let's set up a coordinate system so that we can say what points we are talking about. If J is $(0, 0)$, then B is $(-5, 4)$.



No matter where they park, Brad and Melissa's two paths joined together must make a path from B to J. This combined path has to be at least 9 blocks long since $d_T(B, J) = 9$. They should look for a parking spot in the rectangle formed by the points $(0, 0), (0, 4), (-5, 0)$, and $(-5, 4)$.

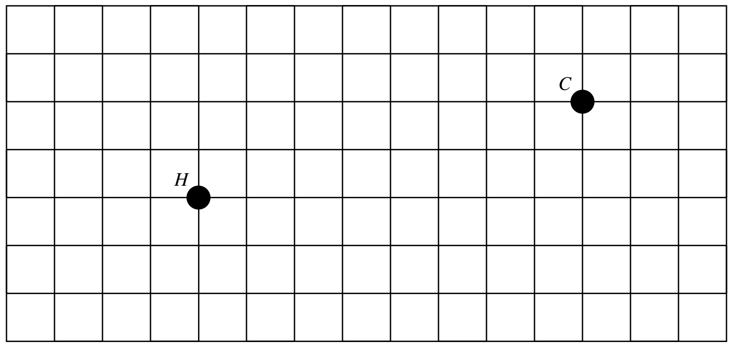
Suppose they park within this rectangle and call this point C. Melissa now walks 4 blocks from C to B and Brad walks 5 blocks from C to J. The two paths joined together form a path from B to J of length 9.

If they park outside the rectangle described above, for example at point D, then the corresponding path from B to J will be longer than 9 blocks. Any path from B to J going through D goes a block too far west and then has to backtrack a block to the east making it longer than 9 blocks. ■

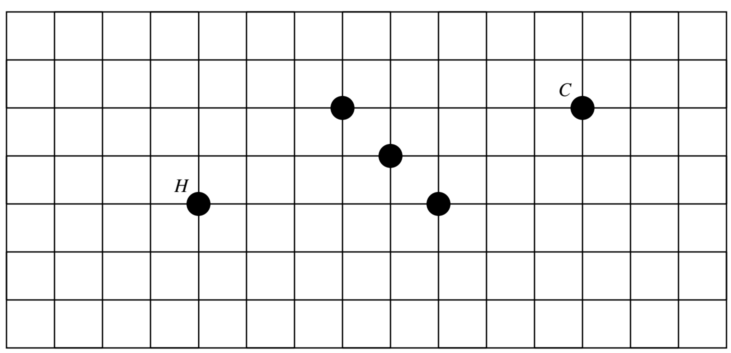
Question If we consider the same question in Euclidean geometry, what is the answer?

?

Question Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom live?



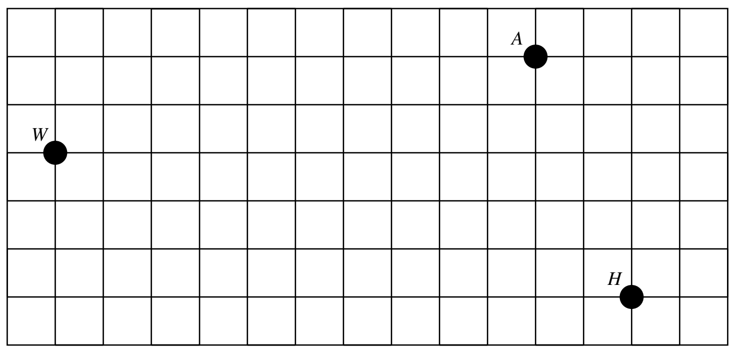
Solution If we fix a coordinate system with its origin at Altgeld Hall, H, then C is at $(8, 2)$. We see that $d_T(H, C) = 10$. If Tom wants to live as close as possible to both of these, he should look for an apartment, A, such that $d_T(A, H) = d_T(A, C) = 5$. He would then be living halfway along one of the shortest paths from Altgeld to the restaurant. Mark all the points 5 blocks away from H. Now mark all the points 5 blocks away from C.



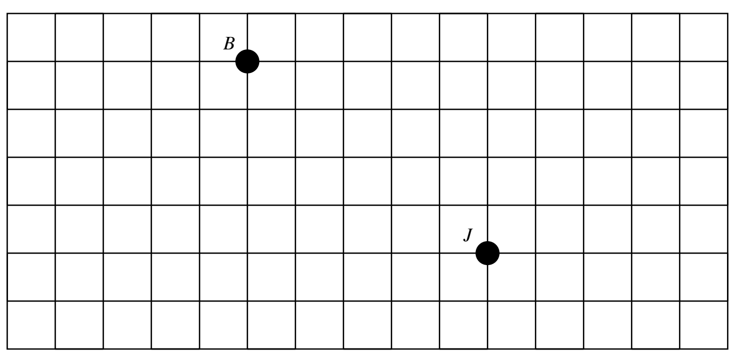
We now see that Tom should check out the apartments near $(5, 0), (4, 1)$, and $(3, 2)$. ∎

Problems for Section 5.3

(1) Will just bought himself a brand new gorilla suit. He wants to show it off at three parties this Saturday night. The parties are being held at his friends' houses: the Antidisestablishment (A), Hausdorff (H), and the Wookie Loveshack (W). If he travels from party A to party H to party W, how far does he travel this Saturday night? Explain your reasoning.

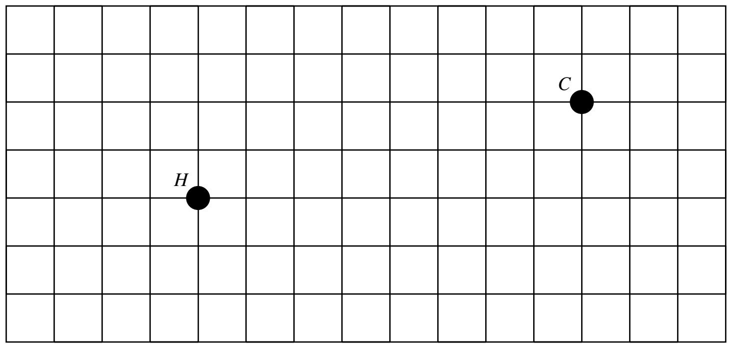


(2) Brad and Melissa are going to downtown Champaign, Illinois. Brad wants to go to *Jupiter's* for pizza (J) while Melissa goes to *Boardman's Art Theater* (B) to watch a movie. Where should they park to minimize the total distance walked by both? Explain your reasoning.

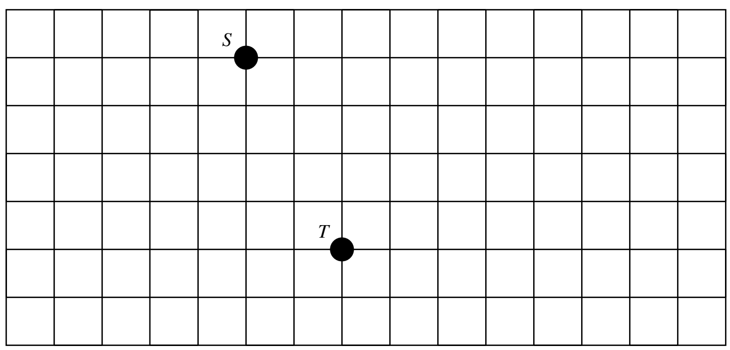


(3) Tom is looking for an apartment that is close to Altgeld Hall (H) but is also close to his favorite restaurant, *Crane Alley* (C). Where should Tom

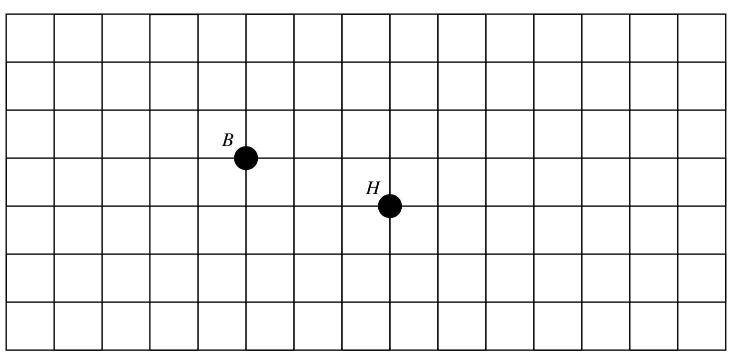
live? Explain your reasoning.



(4) Johann and Amber are going to German Village. Johann wants to go to *Schmidt's* (S) for a cream-puff while Amber goes to the *Thurman Cafe* (T) for some spicy wings. Where should they park to minimize the total distance walked by both if Amber insists that Johann should not have to walk a longer distance than her? Explain your reasoning.



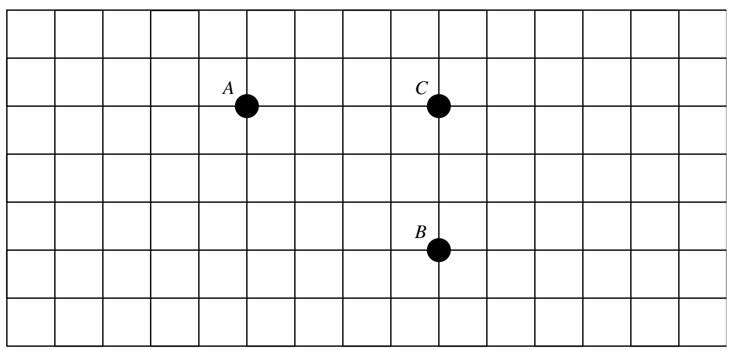
(5) Han and Tom are going to downtown Clintonville. Han wants to go to get a haircut (H) and Tom wants to look at the bookstore (B). Where should they park to keep the total distance walked by both less than 8 blocks? Explain your reasoning.



(6) The university is installing emergency phones across campus. Where should they place them so that their students are never more than a block away from an emergency phone? Explain your reasoning.

(7) Tom and Ben have devised a ingenious *Puzzle-Stroll.*[1] Here is one of the puzzles:

> To find what you seek, you must be one with the city—using it's distance, the treasure is 4 blocks from (A), 3 blocks from (B), and 2 blocks from (C).



Where's the treasure? Explain your reasoning.

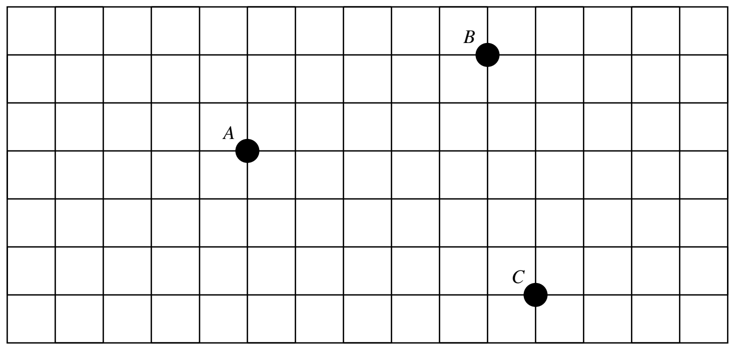
(8) Johann is starting up a new business, *Cafe Battle Royale*. He knows mathematicians drink a lot of coffee so he wants it to be near Altgeld Hall. Balancing this against how expensive rent is near campus, he decides the cafe should be 3 blocks from Altgeld Hall. Where should his cafe be located? Explain your reasoning.

(9) *Cafe Battle Royale, Inc.* is expanding. Johann wants his potential customers to always be within 4 blocks of one of his cafes. Where should his cafes be located? Explain your reasoning.

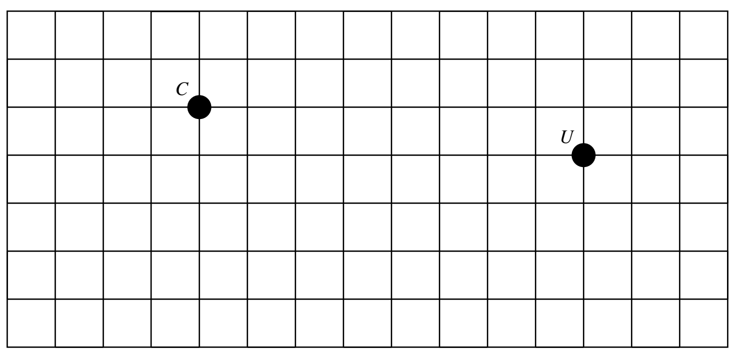
(10) There are hospitals located at A, B, and C. Ambulances should be sent to medical emergencies from whichever hospital is closest. Divide the city into regions in a way that will help the dispatcher decide which ambulance

[1]A.K.A. a *scavenger-hunt*.

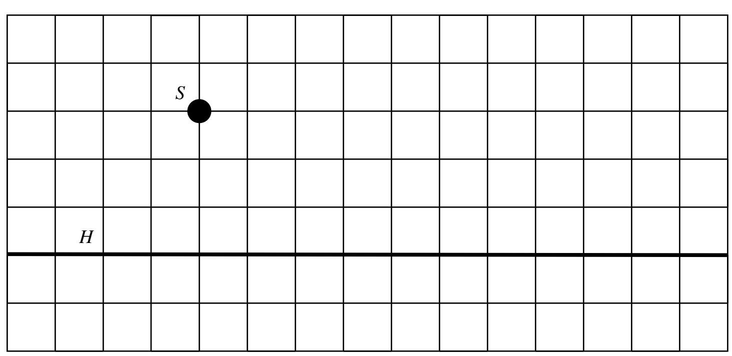
to send. Explain your reasoning.



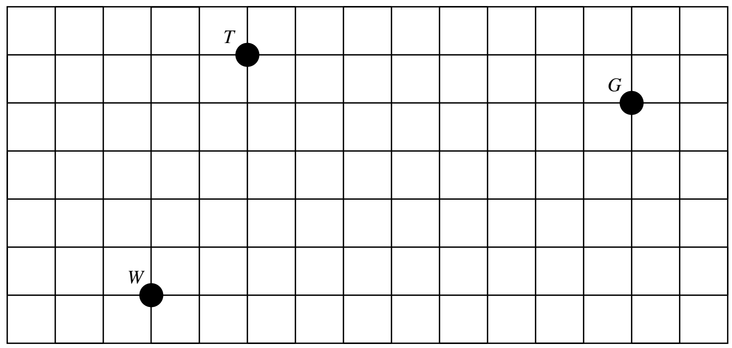
(11) Sylvia is going to open a new restaurant called *Grillvia's* where customers make their own food and then she grills it for them. She wants her restaurant to be equidistant from the heart of Champaign (C) and the heart of Urbana (U). Where should she put her restaurant? Explain your reasoning.



(12) Chris wants to live an equal distance from his favorite hangout *Studio 35* (S) and High Street (H) where he can catch the Number 2 bus. Where should he live? Explain your reasoning.



(13) Lisa just bought a 3-wheeled zebra-striped electric car and its range is limited. Suppose that each day Lisa likes to go to work (W), and then to the tea shop (T) **or** the garden shop (G) but not both, and then back home (H). Where should Lisa live? Give several options depending on how efficient her zebra-striped car is. Explain your reasoning.



Chapter 6

Isometries

> And since you know you cannot see yourself, so well as by reflection, I, your glass, will modestly discover to yourself, that of yourself which you yet know not of.
>
> —William Shakespeare

6.1 Matrices as Functions

We're going to discuss some basic functions in geometry. Specifically, we will talk about translations, reflections, and rotations. To start us off, we need a little background on matrices.

Question What is a *matrix*?

You might think of a matrix as just a jumble of large brackets and numbers. However, we are going to think of matrices as *functions*. Just as we write $f(x)$ for a function f acting on a number x, we'll write:

$$\mathsf{M}\mathbf{p} = \mathbf{q}$$

to represent a matrix M mapping point $\mathbf{p}$ to point $\mathbf{q}$. A point $\mathbf{p}$ is often represented as an ordered pair of coordinates, $\mathbf{p} = (x, y)$. However, to make things work out nicely, we need to write our points all straight and narrow, with a little buddy at the end:

$$(x, y) \rightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Throughout this chapter, we will abuse notation slightly, freely interchanging several notations for a point:

$$\mathbf{p} \leftrightsquigarrow (x, y) \leftrightsquigarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

With this in mind, our work will be done via matrices and points that look like this:

$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Now recall the nitty gritty details of *matrix multiplication*:

$$\begin{aligned} \mathsf{M}\mathbf{p} &= \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \cdot 1 \\ dx + ey + f \cdot 1 \\ 0 \cdot x + 0 \cdot y + 1 \cdot 1 \end{bmatrix} \\ &= \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix} \end{aligned}$$

Question Fine, but what does this have to do with geometry?

In this chapter we are going to study a special type of functions, called *isometries*. These are function that preserves distances. Let's see what we mean by this:

Definition An **isometry** is a function M that maps points in the plane to other points in the plane such that

$$d(\mathbf{p}, \mathbf{q}) = d(\mathsf{M}\mathbf{p}, \mathsf{M}\mathbf{q}),$$

where d is the distance function.

Question How do you compute the distance between two points again?

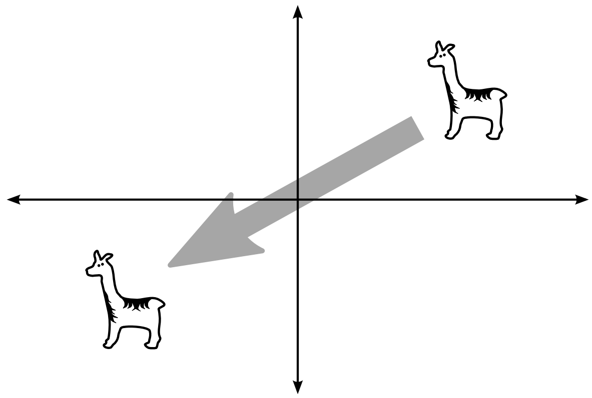
?

We're going to see that several ideas in geometry, specifically translations, reflections, and rotations which all seem very different, are actually all isometries. Hence, we will be thinking of these concepts as matrices.

6.1.1 Translations

Of all the isometries, *translations* are probably the easiest. With a translation, all we do is move our object in a straight line, that is, every point in the plane is moved the same distance and the same direction. Let's see what happens to

Louie Llama when he is translated:

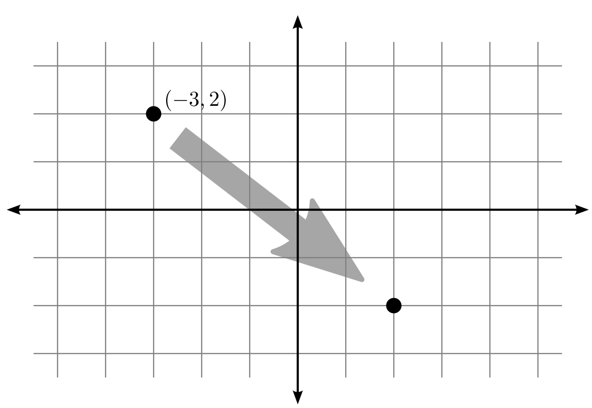


Pretty simple eh? We can give a more "mathematical" definition of a translation involving our newly-found knowledge of matrices! Check it:

Definition A **translation**, denoted by $\mathsf{T}_{(u,v)}$, is a function that moves every point a given distance u in the x-direction and a given distance v in the y-direction. We will use the following matrix to represent translations:

$$\mathsf{T}_{(u,v)} = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (-3, 2)$*. Use a matrix to translate* $\mathbf{p}$ 5 *units right and* 4 *units down.*



Here is how you do it:

$$\begin{aligned}
\mathsf{T}_{(5,-4)}\mathbf{p} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} -3+0+5 \\ 0+2-4 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence, we end up with the point $(2, -2)$*. But you knew that already, didn't you?*

Question Can you demonstrate with algebra why translations are isometries?

?

Question We know how to translate individual points. How do we move entire figures and other funky shapes?

?

6.1.2 Reflections

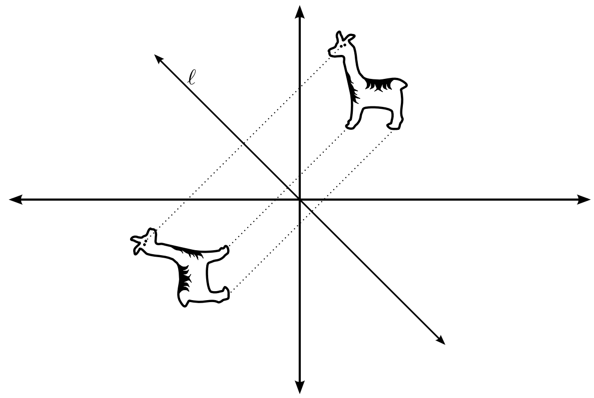
The act of reflection has fascinated humanity for millennia. It has a strong effect on our perception of beauty and has a defined place in art—not to mention how useful it is for the application of make-up. Here is our definition of a reflection:

Definition The **reflection** across a line ℓ, denoted by F_ℓ, is the function that maps a point $\mathbf{p}$ to a point $\mathsf{F}_\ell\mathbf{p}$ such that:

(1) If $\mathbf{p}$ is on ℓ, then $\mathsf{F}_\ell\mathbf{p} = \mathbf{p}$.

(2) If $\mathbf{p}$ is not on ℓ, then ℓ is the perpendicular bisector of the segment connecting $\mathbf{p}$ and $\mathsf{F}_\ell\mathbf{p}$.

You might be saying, "Huh?" It's not as hard as it looks. Check out this picture of the situation, again Louie Llama will help us out:



A Collection of Reflections

We are going to begin with a trio of reflections. We'll start with a **horizontal reflection** across the y-axis. Using our matrix notation, we write:

$$\mathsf{F}_{x=0} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

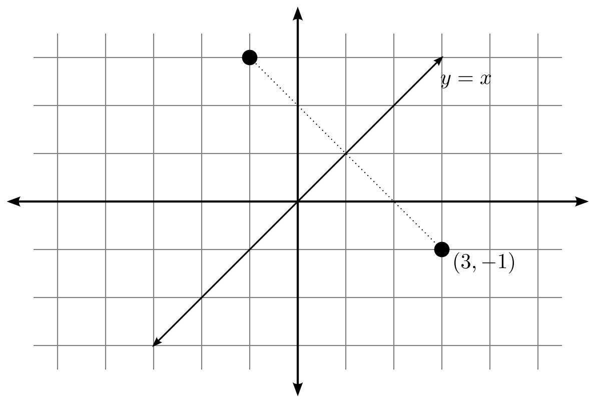
The next reflection in our collection is a **vertical reflection** across the x-axis. Using our matrix notation, we write:

$$\mathsf{F}_{y=0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The final reflection to add to our collection is a **diagonal reflection** across the line $y = x$. Using our matrix notation, we write:

$$\mathsf{F}_{y=x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (3, -1)$. *Use a matrix to reflect* $\mathbf{p}$ *across the* *line $y = x$.*



Here is how you do it:

$$\begin{aligned}
\mathsf{F}_{y=x}\mathbf{p} &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 - 1 + 0 \\ 3 + 0 + 0 \\ 0 + 0 + 1 \end{bmatrix} \\
&= \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence we end up with the point $(-1, 3)$.

Question Let $\mathbf{p}$ be some point in Quadrant I of the (x, y)-plane. What reflection will map this point to Quadrant II? What about Quadrant IV? What about Quadrant III?

?

Question Can you demonstrate with algebra why each of our reflections above are isometries?

?

6.1.3 Rotations

Imagine that you are on a swing set, going higher and higher until you are actually able to make a full circle[1]. At the point where you are directly above

[1] Face it, I think we all dreamed of doing that when we were little—or in my case, last week.

where you would be if the swing were at rest, where is your head, comparatively? Your feet? Your hands?

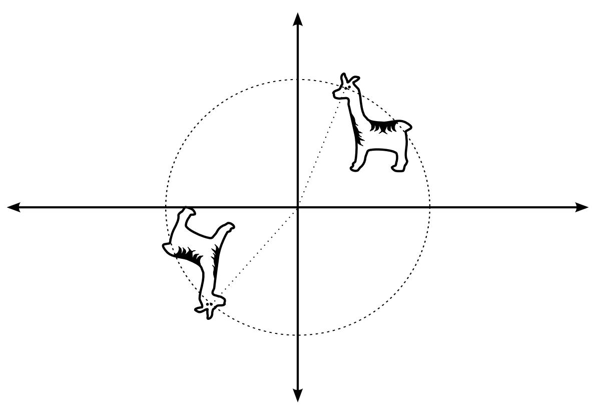
Rotations should bring circles to mind. This is not a coincidence. Check out our definition of a *rotation*:

Definition A **rotation** of θ degrees about the origin, denoted by R_θ, is a function that maps a point $\mathbf{p}$ to a point $\mathsf{R}_\theta\mathbf{p}$ such that:

(1) The points $\mathbf{p}$ and $\mathsf{R}_\theta\mathbf{p}$ are equidistant from the origin.

(2) An angle of θ degrees is formed by $\mathbf{p}$, the origin, and $\mathsf{R}_\theta\mathbf{p}$.

Louie Llama, can you do the honors?

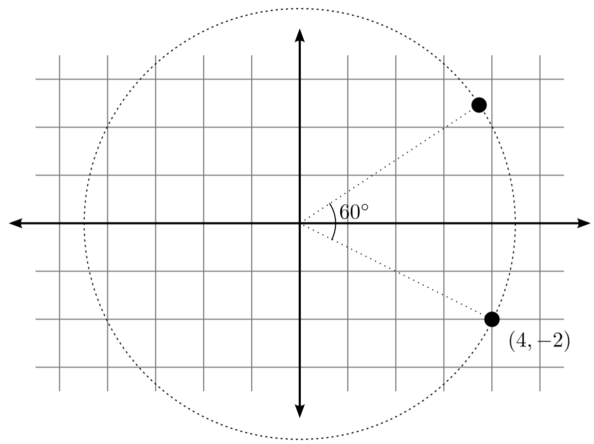


WARNING *Positive angles denote a counterclockwise rotation. Negative angles denote a clockwise rotation.*

Looking back on trigonometry, there were some angles that kept on coming up. Some of these were 90°, 60°, and 45°. We'll focus on these angles too.

$$\mathsf{R}_{90} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{R}_{60} = \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathsf{R}_{45} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example *Consider the point* $\mathbf{p} = (4, -2)$. *Use a matrix to rotate* $\mathbf{p}$ $60°$ *about* *the origin.*



Here is how you do it:

$$\begin{aligned}
\mathsf{R}_{60}\mathbf{p} &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2+\sqrt{3}+0 \\ 2\sqrt{3}-1+0 \\ 0+0+1 \end{bmatrix} \\
&= \begin{bmatrix} 2+\sqrt{3} \\ 2\sqrt{3}-1 \\ 1 \end{bmatrix}
\end{aligned}$$

Hence, we end up with the point $(2+\sqrt{3}, 2\sqrt{3}-1)$.

Question Do the numbers in the matrices above look familiar? If so, why?

?

Question How do you rotate a point 180 degrees?

?

Question Can you demonstrate with algebra why our rotations above are isometries?

?

Problems for Section 6.1

(1) How do you compute the distance between two points $\mathbf{p}$ and $\mathbf{q}$ in the plane?

(2) Use algebra to explain why:

$$d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p} - \mathbf{q}, \mathbf{o}) = d(\mathbf{o}, \mathbf{p} - \mathbf{q})$$

where $\mathbf{o} = (0, 0)$.

(3) What is an isometry?

(4) What is a translation?

(5) What is a rotation?

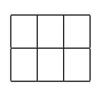
(6) What is a reflection?

(7) Reflecting back on this chapter, suppose I translate a point $\mathbf{p}$ to $\mathbf{p}'$. Does it make any difference if I move the point $\mathbf{p}$ along a wiggly path

or a straight path? Explain your reasoning.

(8) Reflecting back on this chapter, is a rotation the continuous *act* of *moving* a point through an angle around some fixed point, or is it just a final picture compared to the initial one? Explain your reasoning.

(9) In the vector illustrator *Inkscape* there is an option to transform an image via a "matrix." If you select this tool, you are presented with 6 boxes to fill in with numbers:



Use what you've learned in this chapter to make a guess as to how this tool works.

(10) In what direction does a positive rotation occur?

(11) Is a 270° rotation the same as a −90° rotation? Explain your reasoning.

(12) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(13) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 0 & 8 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(14) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(15) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 0 & 2 & 0 \\ -3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(16) Consider the following matrix:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Is M an isometry? Explain your reasoning.

(17) Use a matrix to translate the point $(-1, 6)$ three units right and two units up. Sketch this situation and explain your reasoning.

(18) The matrix $\mathsf{T}_{(-2,6)}$ was used to translate the point $\mathbf{p}$ to $(-1, -3)$. What is $\mathbf{p}$? Sketch this situation and explain your reasoning.

(19) Use a matrix to reflect the point $(5, 2)$ across the x-axis. Sketch this situation and explain your reasoning.

(20) Use a matrix to reflect the point $(-3, 4)$ across the y-axis. Sketch this situation and explain your reasoning.

(21) Use a matrix to reflect the point $(-1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(22) Use a matrix to reflect the point $(1, 1)$ across the line $y = x$. Sketch this situation and explain your reasoning.

(23) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point $\mathbf{p}$ to $(4, 3)$. What is $\mathbf{p}$? Explain your reasoning.

(24) The matrix $\mathsf{F}_{y=0}$ was used to reflect the point $\mathbf{p}$ to $(0, -8)$. What is $\mathbf{p}$? Explain your reasoning.

(25) The matrix $\mathsf{F}_{x=0}$ was used to reflect the point $\mathbf{p}$ to $(-5, -1)$. What is $\mathbf{p}$? Explain your reasoning.

(26) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point $\mathbf{p}$ to $(9, -2)$. What is $\mathbf{p}$? Explain your reasoning.

(27) The matrix $\mathsf{F}_{y=x}$ was used to reflect the point $\mathbf{p}$ to $(-3, -3)$. What is $\mathbf{p}$? Explain your reasoning.

(28) Considering the point $(3, 2)$, use a matrix to rotate this point $60°$ about the origin. Sketch this situation and explain your reasoning.

(29) Considering the point $(\sqrt{2}, -\sqrt{2})$, use a matrix to rotate this point $45°$ about the origin. Sketch this situation and explain your reasoning.

(30) Considering the point $(-7, 6)$, use a matrix to rotate this point $90°$ about the origin. Sketch this situation and explain your reasoning.

(31) Considering the point $(-1, 3)$, use a matrix to rotate this point $0°$ about the origin. Sketch this situation and explain your reasoning.

(32) Considering the point $(0, 0)$, use a matrix to rotate this point $120°$ about the origin. Sketch this situation and explain your reasoning.

(33) Considering the point $(1, 1)$, use a matrix to rotate this point $-90°$ about the origin. Sketch this situation and explain your reasoning.

(34) The matrix R_{90} was used to rotate the point $\mathbf{p}$ to $(2, -5)$. What is $\mathbf{p}$? Explain your reasoning.

(35) The matrix R_{60} was used to rotate the point $\mathbf{p}$ to $(0, 2)$. What is $\mathbf{p}$? Explain your reasoning.

(36) The matrix R_{45} was used to rotate the point $\mathbf{p}$ to $(-\frac{1}{2}, \frac{5}{2})$. What is $\mathbf{p}$? Explain your reasoning.

(37) The matrix R_{-90} was used to rotate the point $\mathbf{p}$ to $(4, 3)$. What is $\mathbf{p}$? Explain your reasoning.

(38) If someone wanted to plot the graph of $y = x^2$, they might start by filling in the following table:

x	x^2
0	
1	
-1	
2	
-2	
3	
-3	

Reflect each point you obtain from the table above about the line $y = x$. Give a plot of this situation. What curve do you obtain? What is this new curve's relationship to $y = x^2$? Explain your reasoning.

(39) Some translation T was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (1, 2)$ and $\mathbf{q} = (3, 4)$, find T and explain your reasoning.

(40) Some translation T was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (-2, 3)$ and $\mathbf{q} = (2, 3)$, find T and explain your reasoning.

(41) Some reflection F was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (1, 4)$ and $\mathbf{q} = (1, -4)$, find F and explain your reasoning.

(42) Some reflection F was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (5, 0)$ and $\mathbf{q} = (0, 5)$, find F and explain your reasoning.

(43) Some rotation R was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (3, 0)$ and $\mathbf{q} = (0, 3)$, find R and explain your reasoning.

(44) Some rotation R was used to map point $\mathbf{p}$ to point $\mathbf{q}$. Given $\mathbf{p} = (\sqrt{2}, \sqrt{2})$ and $\mathbf{q} = (0, 2)$, find R and explain your reasoning.

(45) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (0,0),\\ (1,0) &\mapsto (3,0),\\ (0,1) &\mapsto (0,5).\end{aligned}$$

Find M and explain your reasoning.

(46) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (-1,1),\\ (1,0) &\mapsto (3,0),\\ (0,1) &\mapsto (0,5).\end{aligned}$$

Find M and explain your reasoning.

(47) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (1,1),\\ (1,0) &\mapsto (2,1),\\ (0,1) &\mapsto (1,2).\end{aligned}$$

Find M and explain your reasoning.

(48) Some matrix M maps

$$\begin{aligned}(0,0) &\mapsto (2,2),\\ (1,1) &\mapsto (3,3),\\ (-1,1) &\mapsto (1,3).\end{aligned}$$

Find M and explain your reasoning.

(49) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (0,0), \\ (1,1) &\mapsto (0,3), \\ (-1,1) &\mapsto (5,0). \end{aligned}$$

Find M and explain your reasoning.

(50) Some matrix M maps

$$\begin{aligned} (0,0) &\mapsto (1,2), \\ (1,1) &\mapsto (-3,1), \\ (-1,1) &\mapsto (2,-3). \end{aligned}$$

Find M and explain your reasoning.

6.2 The Algebra of Matrices

6.2.1 Matrix Multiplication

We know how to multiply a matrix and a point. Multiplying two matrices is a similar procedure:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j & k & l \\ m & n & o \\ p & q & r \end{bmatrix} = \begin{bmatrix} aj+bm+cp & ak+bn+cq & al+bo+cr \\ dj+em+fp & dk+en+fq & dl+eo+fr \\ gj+hm+ip & gk+hn+iq & gl+ho+ir \end{bmatrix}$$

Variables are all good and well, but let's do this with actual numbers. Consider the following two matrices:

$$\mathsf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad \text{and} \qquad \mathsf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's multiply them together and see what we get:

$$\begin{aligned} \mathsf{MI} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1\cdot1+2\cdot0+3\cdot0 & 1\cdot0+2\cdot1+3\cdot0 & 1\cdot0+2\cdot0+3\cdot1 \\ 4\cdot1+5\cdot0+6\cdot0 & 4\cdot0+5\cdot1+6\cdot0 & 4\cdot0+5\cdot0+6\cdot1 \\ 7\cdot1+8\cdot0+9\cdot0 & 7\cdot0+8\cdot1+9\cdot0 & 7\cdot0+8\cdot0+9\cdot1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ &= \mathsf{M} \end{aligned}$$

Question What is IM equal to?

?

It turns out that we have a special name for I. We call it the **identity matrix**.

WARNING *Matrix multiplication is **not** generally commutative. Check it out:*

$$\mathsf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \textit{and} \qquad \mathsf{R} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

When we multiply these matrices, we get:

$$\begin{aligned}\mathsf{FR} &= \begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}1\cdot 0+0\cdot 1+0\cdot 0 & 1\cdot(-1)+0\cdot 0+0\cdot 0 & 1\cdot 0+0\cdot 0+0\cdot 1\\ 0\cdot 0+(-1)\cdot 1+0\cdot 0 & 0\cdot(-1)+(-1)\cdot 0+0\cdot 0 & 0\cdot 0+(-1)\cdot 0+0\cdot 1\\ 0\cdot 0+0\cdot 1+1\cdot 0 & 0\cdot(-1)+0\cdot 0+1\cdot 0 & 0\cdot 0+0\cdot 0+1\cdot 1\end{bmatrix}\\ &= \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

On the other hand, we get:

$$\mathsf{RF} = \begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Question Can you draw some nice pictures showing geometrically that matrix multiplication is not commutative?

?

Question Is it always the case that $(\mathsf{LM})\mathsf{N} = \mathsf{L}(\mathsf{MN})$?

?

6.2.2 Compositions of Matrices

It is often the case that we wish to apply several isometries successively to a point. Consider the following:

$$\mathsf{M} = \begin{bmatrix}a & b & c\\ d & e & f\\ 0 & 0 & 1\end{bmatrix} \qquad \mathsf{N} = \begin{bmatrix}g & h & i\\ j & k & l\\ 0 & 0 & 1\end{bmatrix} \qquad \text{and} \qquad \mathbf{p} = \begin{bmatrix}x\\ y\\ 1\end{bmatrix}$$

Now let's compute

$$\begin{aligned}\mathsf{M}(\mathsf{N}\mathbf{p}) &= \begin{bmatrix}a & b & c\\ d & e & f\\ 0 & 0 & 1\end{bmatrix}\left(\begin{bmatrix}g & h & i\\ j & k & l\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}\right)\\ &= \begin{bmatrix}a & b & c\\ d & e & f\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}gx+hy+i\\ jx+ky+l\\ 1\end{bmatrix}\\ &= \begin{bmatrix}agx+ahy+ai+bjx+bky+bl+c\\ dgx+dhy+di+ejx+eky+el+f\\ 1\end{bmatrix}\end{aligned}$$

Now *you* compute $(\mathsf{MN})\mathbf{p}$ and compare what *you* get to what we got above.

Compositions of Translations

A composition of translations occurs when two or more successive translations are applied to the same point. Check it out:

$$\begin{aligned} \mathsf{T}_{(5,-4)}\mathsf{T}_{(-3,2)} &= \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \mathsf{T}_{(5+(-3),(-4)+2)} \\ &= \mathsf{T}_{(2,-2)} \end{aligned}$$

Theorem 9 *The composition of two translations $\mathsf{T}_{(u,v)}$ and $\mathsf{T}_{(s,t)}$ is equal to the translation $\mathsf{T}_{(u+s,v+t)}$.*

Question How do you prove the theorem above?

?

Question Can you give a single translation that is equal to the following composition?

$$\mathsf{T}_{(-7,5)}\mathsf{T}_{(0,-6)}\mathsf{T}_{(2,8)}\mathsf{T}_{(5,-4)}$$

?

Question Are compositions of translations commutative? Are they associative?

?

Compositions of Reflections

A composition of reflections occurs when two or more successive reflections are applied to the same point. Check it out:

$$\begin{aligned} \mathsf{F}_{y=0}\mathsf{F}_{y=x} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Question Is the composition $\mathsf{F}_{y=0}\mathsf{F}_{y=x}$ still a reflection?

?

Question Are compositions of reflections commutative? Are they associative?

?

Compositions of Rotations

A composition of rotations occurs when two or more successive rotations are applied to the same point. Check it out:

$$\begin{aligned}\mathsf{R}_{60}\mathsf{R}_{60} &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Theorem 10 *The product of two rotations* R_θ *and* R_φ *is equal to the rotation* $\mathsf{R}_{\theta+\varphi}$.

From this we see that:

$$\mathsf{R}_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question What is the rotation matrix for a 360° rotation? What about a 405° rotation?

?

Question Are compositions of rotations commutative? Are they associative?

?

6.2.3 Mixing and Matching

Life gets interesting when we start composing translations, reflections, and rotations together. First we'll compose a reflection with a rotation:

$$\begin{aligned}\mathsf{F}_{y=0}\mathsf{R}_{60} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

Question Does this result look familiar?

?

Now how about a rotation composed with a translation:

$$\begin{aligned}\mathsf{R}_{90}\mathsf{T}_{(3,-4)} &= \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 3\\ 0 & 1 & -4\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}0 & -1 & 4\\ 1 & 0 & 3\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

Question Does $\mathsf{R}_{90}\mathsf{T}_{(3,-4)} = \mathsf{T}_{(3,-4)}\mathsf{R}_{90}$?

?

Question Find a matrix that represents the reflection $\mathsf{F}_{y=-x}$.

I'll take this one. Note that

$$\begin{aligned}\mathsf{F}_{y=-x} &= \mathsf{R}_{180}\mathsf{F}_{y=x}\\ &= \mathsf{R}_{90}\mathsf{R}_{90}\mathsf{F}_{y=x}\\ &= \begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\\ &= \begin{bmatrix}0 & -1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\end{aligned}$$

OK looks good, but you, the reader, are going to have to check the above computation yourself.

Question How do we deal with reflections that are not across the lines $y = 0$, $x = 0$, or $y = x$? How would you reflect points across the line $y = 1$?

?

Problems for Section 6.2

(1) Give a single translation that is equal to $\mathsf{T}_{(-3,2)}\mathsf{T}_{(5,-1)}$. Explain your reasoning.

(2) Consider the two translations $\mathsf{T}_{(-4,8)}$ and $\mathsf{T}_{(4,-8)}$. Do these commute? Explain your reasoning.

(3) Give a single reflection that is equal to $\mathsf{F}_{x=0}\mathsf{F}_{y=0}$. Sketch this situation and explain your reasoning.

(4) Given any point $\mathbf{p} = (x, y)$, express $\mathsf{T}_{(4,2)}\mathsf{T}_{(6,-5)}\mathbf{p}$ as $\mathsf{T}_{(u,v)}\mathbf{p}$ for some values of u and v. Sketch this situation and explain your reasoning.

(5) Give a matrix for R_{-45}. Explain your reasoning.

(6) Give a matrix for R_{-60}. Explain your reasoning.

(7) Sam suggests that:

$$\mathsf{R}_{-90} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Why does he suggest this? Is it even true? Explain your reasoning.

(8) Give a matrix for $\mathsf{F}_{y=-x}$. Explain your reasoning.

(9) Given the point $\mathbf{p} = (-4, 2)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{F}_{y=x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(10) Given the point $\mathbf{p} = (5, 0)$, use matrices to compute $\mathsf{F}_{y=x}\mathsf{F}_{y=-x}\mathbf{p}$. Sketch this situation and explain your reasoning.

(11) Give a single rotation that is equal to $\mathsf{R}_{45}\mathsf{R}_{60}$. Explain your reasoning.

(12) Given the point $\mathbf{p} = (1, 3)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{90}\mathbf{p}$. Sketch this situation and explain your reasoning.

(13) Given the point $\mathbf{p} = (-7, 2)$, use matrices to compute $\mathsf{R}_{45}\mathsf{R}_{-45}\mathbf{p}$. Sketch this situation and explain your reasoning.

(14) Given the point $\mathbf{p} = (-2, 5)$, use matrices to compute $\mathsf{R}_{90}\mathsf{R}_{-90}\mathsf{R}_{360}\mathbf{p}$. Sketch this situation and explain your reasoning.

(15) Given the point $\mathbf{p} = (5, 4)$, use matrices to compute $\mathsf{F}_{y=0}\mathsf{T}_{(2,-4)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(16) Given the point $\mathbf{p} = (-1, 6)$, use matrices to compute $\mathsf{R}_{45}\mathsf{T}_{(0,0)}\mathbf{p}$. Sketch this situation and explain your reasoning.

(17) Given the point $\mathbf{p} = (11, 13)$, use matrices to compute $\mathsf{T}_{(-6,-3)}\mathsf{R}_{135}\mathbf{p}$. Sketch this situation and explain your reasoning.

(18) Given the point $\mathbf{p} = (-7, -5)$, use matrices to compute $\mathsf{R}_{540}\mathsf{F}_{x=0}\mathbf{p}$. Sketch this situation and explain your reasoning.

(19) Give a composition of matrices that will take a point and reflect it across the x-axis and then rotate the result 90° around the origin. Sketch this situation and explain your reasoning.

(20) Give a composition of matrices that will take a point and translate it three units up and 2 units left and then rotate it 90° clockwise around the origin. Sketch this situation and explain your reasoning.

(21) Give a composition of matrices that will take a point and rotate it 270° around the origin, reflect it across the line $y = x$, and then translate the result down 5 units and 3 units to the right. Sketch this situation and explain your reasoning.

(22) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $x = 1$. Sketch this situation and explain your reasoning.

(23) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = -4$. Sketch this situation and explain your reasoning.

(24) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and reflect it across the line $y = x + 5$. Sketch this situation and explain your reasoning.

(25) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it 45° around the point $(2, 3)$. Sketch this situation and explain your reasoning.

(26) Give a composition of translations and any of the following matrices

$$\{\mathsf{F}_{x=0}, \mathsf{F}_{y=0}, \mathsf{F}_{y=x}, \mathsf{R}_{90}, \mathsf{R}_{60}, \mathsf{R}_{45}\}$$

that will take a point and rotate it 90° clockwise around the point $(-3, 4)$ Sketch this situation and explain your reasoning.

6.3 The Theory of Groups

One of the most fundamental notions in all of modern mathematics is that of a *group*. Sadly, many students never see a group in their education.

Definition A **group** is a set of elements (in our case matrices) which we will call $\mathcal{G}$ such that:

(1) There is an associative operation (in our case matrix multiplication).

(2) The set is closed under this operation (the product of any two matrices in the set is also in the set).

(3) There exists an identity $\mathsf{I} \in \mathcal{G}$ such that for all $\mathsf{M} \in \mathcal{G}$,

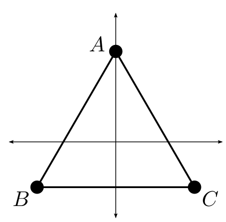
$$\mathsf{IM} = \mathsf{MI} = \mathsf{M}.$$

(4) For all $M \in \mathcal{G}$ there is an inverse $\mathsf{M}^{-1} \in \mathcal{G}$ such that

$$\mathsf{MM}^{-1} = \mathsf{M}^{-1}\mathsf{M} = \mathsf{I}.$$

6.3.1 Groups of Rotations

Let's see a group. Here we have an equilateral triangle centered at the origin of the (x, y)-plane:



Question The matrix R_{360} will rotate this triangle completely around the origin. What matrix will rotate this triangle one-third of a complete rotation?

As a gesture of friendship, I'll take this one. One-third of 360 is 120. So we see that R_{120} will rotate the triangle one-third of a full rotation. Do you remember this matrix? Here it is:

$$\mathsf{R}_{120} = \begin{bmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In spite of the fact that this matrix is messy and that matrix multiplication is somewhat tedious, you should realize that

$$\mathsf{R}_{120}^2 = \mathsf{R}_{240} \qquad \text{and} \qquad \mathsf{R}_{120}^3 = \mathsf{R}_{360}.$$

Let's put these facts (and a few more) together in what is called a *group table*. Remember multiplication tables from elementary school? Well, we're going to make something like a "multiplication table" of rotations. We'll start by listing the identity and powers of a one-third rotation along the top and left-hand sides. Setting $\mathsf{R} = \mathsf{R}_{120}$ we have:

$\circ$	I	R	R^2	R^3	$\cdots$
I					
R					
R^2					
R^3					
$\vdots$					

Since $\mathsf{R}^3 = \mathsf{I}$, we need only take our table to R^2. At this point we can write out the complete table:

$\circ$	I	R	R^2
I	I	R	R^2
R	R	R^2	I
R^2	R^2	I	R

Since matrix multiplication is associative, and we see from the table that every element has an inverse, we see that

$$\{\mathsf{I}, \mathsf{R}, \mathsf{R}^2\}$$

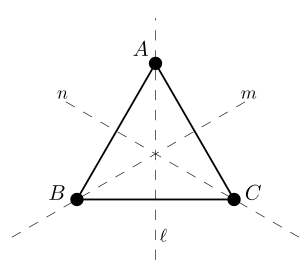
is a group.

Question What rotation matrices would we use when working with a square? A pentagon? A hexagon?

?

6.3.2 Groups of Reflections

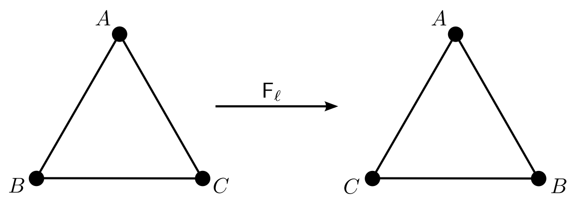
Let's see another group. Again consider an equilateral triangle. This time we are interested in the three lines of reflection that preserve this triangle:



Question Suppose that the triangle above is centered at the origin of the (x, y)-plane. What are equations for ℓ, m, and n?

?

The easiest of the reflections above is the reflection over F_ℓ.



We'll start our group table off with just two elements: I and $\mathsf{F} = \mathsf{F}_\ell$.

∘	I	F
I	I	F
F	F	I

Notice that when we apply F twice we're right back where we started. Hence, $\mathsf{FF} = \mathsf{I}$. Since matrix multiplication is associative, we see that

$$\{\mathsf{I}, \mathsf{F}\}$$

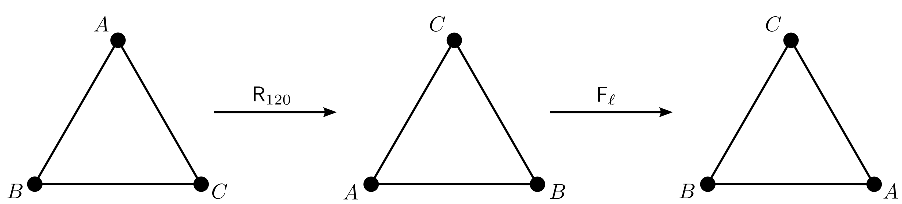
forms a group. Specifically this is a group of reflections of the triangle.

Question Above we used F_ℓ. What would happen if we used F_m or F_n? Also, what are the equations for the lines of symmetry of the square centered at the origin?

?

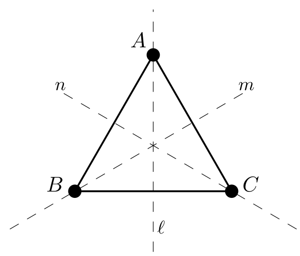
6.3.3 Symmetry Groups

Now let's mix our rotations and reflections. Consider our original triangle and apply $\mathsf{F}_\ell \mathsf{R}_{120}$:



What you may not immediately notice is that we obtain the same transformation by taking the original triangle and applying F_m.

As it turns out, every possible symmetry of the equilateral triangle can be represented using reflections and rotations. Each of these reflections and rotations can be expressed as a composition of a single reflection and a single rotation. The collection of all symmetries forms a group called the *symmetry group* of the equilateral triangle.



Let's see the group table, note we'll let $\mathsf{R} = \mathsf{R}_{120}$ and $\mathsf{F} = \mathsf{F}_\ell$:

$\circ$	I	R	R^2	F	FR	FR^2
I	I	R	R^2	F	FR	FR^2
R	R	R^2	I	FR^2	F	FR
R^2	R^2	I	R	FR	FR^2	F
F	F	FR	FR^2	I	R	R^2
FR	FR	FR^2	F	R^2	I	R
FR^2	FR^2	F	FR	R	R^2	I

This table shows every symmetry of the triangle, including the identify I. By comparing the rows and columns of the group table, you can see that every element has an inverse. This combined with the fact that the matrix multiplication is associative shows that the symmetries of the triangle,

$$\{\mathsf{I}, \mathsf{R}, \mathsf{R}^2, \mathsf{F}, \mathsf{FR}, \mathsf{FR}^2\}$$

form a group.

Question Can you express the symmetries of squares in terms of reflections and rotations? What does the group table look like for the symmetry group of the square?

?

Problems for Section 6.3

(1) State the definition of a group of matrices.

(2) How many lines of reflectional symmetry exist for a square? Provide a drawing to justify your answer.

(3) What are the equations for the lines of reflectional symmetry that exist for the square? Explain your answers.

(4) How many lines of reflectional symmetry exist for a regular hexagon? Provide a drawing to justify your answer.

(5) What are the equations for the lines of reflectional symmetry for a regular hexagon? Explain your answers.

(6) How many consecutive rotations are needed to return the vertexes of a square to their original position? Provide a drawing to justify your answer, labeling the vertexes.

(7) How many degrees are in one-fourth of a complete rotation of the square? Explain your answer.

(8) How many degrees are in one-sixth of a complete rotation of the regular hexagon? Explain your answer.

(9) In this section, we've focused on a 3-sided figure, a 4-sided figure, and a 6-sided figure. Why do we not include the rotation group for the pentagon in this section? If we did, how many degrees would be in one-fifth of a complete rotation?

(10) With notation used in this section, draw pictures representing the action of the following isometries F_ℓ, R, RF_ℓ and $\mathsf{F}_\ell\mathsf{R}$ on the equilateral triangle.

(11) Consider a square centered at the origin. Draw pictures representing the action of $\mathsf{F}_{y=0}$, R_{90}, $\mathsf{R}_{90}\mathsf{F}_{y=0}$, and $\mathsf{F}_{y=0}\mathsf{R}_{90}$ on this square.

(12) Consider a hexagon centered at the origin. Draw pictures representing the action of $\mathsf{F}_{x=0}$, R_{60}, $\mathsf{R}_{60}\mathsf{F}_{x=0}$, and $\mathsf{F}_{x=0}\mathsf{R}_{60}$ on this hexagon.

(13) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}. Explain and illustrate your answer.

(14) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is $\mathsf{F}_{x=0}$. Explain and illustrate your answer.

(15) Find two symmetries of the equilateral triangle, neither of which is the identity, such that their composition is R_{120}^2. Explain and illustrate your answer.

(16) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{180}. Explain and illustrate your answer.

(17) Find two symmetries of the square, neither of which is the identity, such that their composition is F_{ℓ}. Explain and illustrate your answer.

(18) Find two symmetries of the square, neither of which is the identity, such that their composition is R_{270}. Explain and illustrate your answer.

(19) Use a group table to help you write out the symmetries of the equilateral triangle. List all elements that commute with every other element in the table. Explain your reasoning.

(20) Use a group table to help you write out the symmetries of the square. List all elements that commute with every other element in the table. Explain your reasoning.

(21) Use a group table to help you write out the symmetries of the regular hexagon. List all elements that commute with every other element in the table. Explain your reasoning.

(22) Let M be a symmetry of the equilateral triangle. Define

$$C(\mathsf{M}) = \{\text{all symmetries that commute with } \mathsf{M}\}.$$

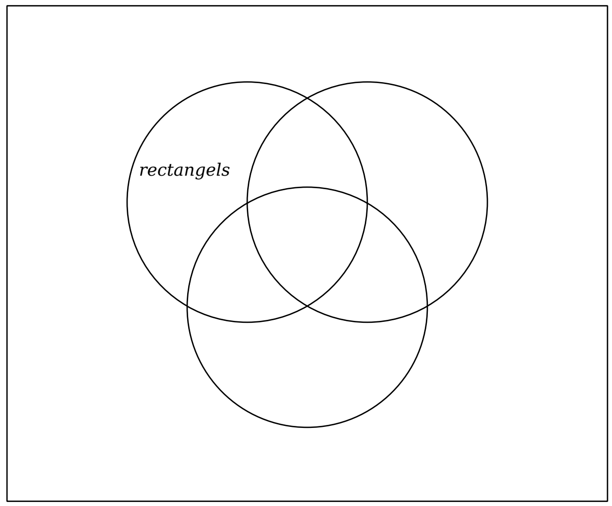
Write out $C(\mathsf{M})$ for every symmetry M of the equilateral triangle. Make some observations.

Appendix A

Activities

A.1 It's What the Book Says

Fifth graders were given the following task: Put the terms **square**, **rhombus**, **parallelogram**, in the Venn diagram below.



1) What are *squares*, *rhombuses*, and *parallelograms*?

2) Critique the question above based on mathematical content.

3) Create a Venn diagram showing the correct relationship between *rectangles*, *squares*, *rhombuses*, and *parallelograms*. Be ready to present and defend your diagram to your peers.

A.2 Forget Something?

1) Draw a Venn diagram with one set. List every possible relationship between an element and this set.

2) Draw a Venn diagram with two intersecting sets. List every possible relationship between an element and these sets.

3) Draw a Venn diagram with three intersecting sets. List every possible relationship between an element and these sets.

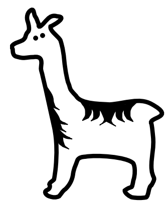
4) Describe and explain any patterns you see occurring.

5) Draw a Venn diagram with four intersecting sets. List every possible relationship between an element and these sets.

6) Are you **sure** that your diagram for Problem 5 is correct? If so explain why. If not, draw a correct Venn diagram.

A.3 Louie Llama and the Triangle

We are going to investigate why the interior angles of a triangle sum to 180°. We won't be alone on this journey, we'll have help. Meet Louie Llama:



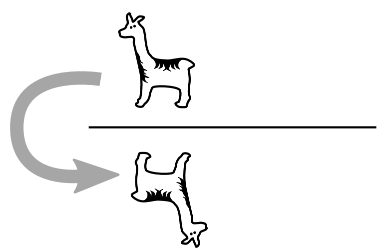
Louie Llama is rather radical for a llama and doesn't mind being rotated at all.

1) Draw a picture of Louie Llama rotated 90° counterclockwise.

2) Draw a picture of Louie Llama rotated 180° counterclockwise.

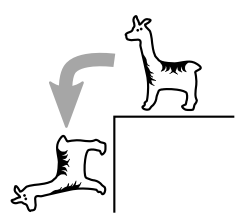
3) Draw a picture of Louie Llama rotated 360° counterclockwise.

4) Sometimes Louie Llama likes to walk around lines he finds:



Through what angle did Louie Llama just rotate?

Now we're going to watch Louie Llama go for a walk. Draw yourself any triangle, draw a crazy scalene triangle—those are the kind that Louie Llama likes best. Louie Llama is going to proudly parade around this triangle. When Louie Llama walks around corners he rotates. Check it out:



Take your triangle and denote the measure of its angles as a, b, and c. Start Louie Llama out along a side adjacent to the angle of measure a. He should be on the outside of the triangle, his feet should be pointing toward the triangle, and his face should be pointing toward the angle of measure b.

5) Sketch Louie Llama walking to the angle of measure b. Walk him around the angle. As he goes around the angle his feet should always be pointing toward the triangle. Through what angle did Louie Llama just rotate?

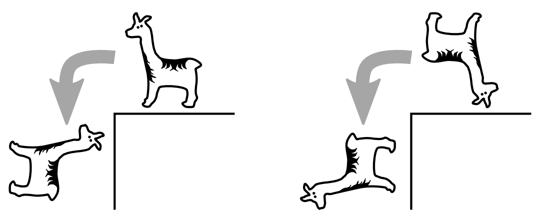
6) Sketch Louie Llama walking to the angle of measure c. Walk him around the angle. Through what angle did Louie Llama just rotate?

7) Finally sketch Louie Llama walking back to the angle of measure a. Walk him around the angle. He should be back at his starting point. Through what angle did Louie Llama just rotate?

8) All in all, how many degrees did Louie Llama rotate in his walk?

9) Write an equation where the right-hand side is Louie Llama's total rotation and the left-hand side is the sum of each rotation around the angle. Can you solve for $a + b + c$?

As you may have guessed, Louie Llama isn't your typical llama, for one thing he likes to walk backwards and on his head! He also like to do somersaults. Louie Llama can somersault around corners in two different ways:



10) What does Louie Llama's somersault have to do with the angle of the corner? Can you precisely explain how Louie Llama rotates when he somersaults around corners?

11) Can you walk Louie Llama around your original triangle allowing him to walk backwards (or even on his head!), letting him do somersaults as he pleases around corners, and **directly** arrive at the equation

$$a + b + c = 180°?$$

12) Can you rephrase what we did above in terms of *exterior angles* and *interior angles*?

13) Can you walk Louie Llama around other shapes and figure out what the sum of their interior angles are?

A.4 Louie Llama and Regular Polygons

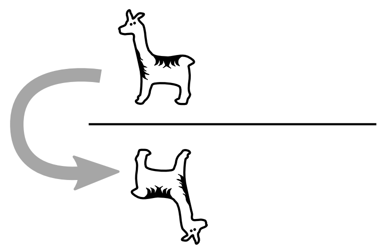
Louie Llama is a very curious llama. He knows that each angle of a regular 3-gon is 60°. He also knows that each angle of a regular 4-gon is 90°. But what he really wants to know, are the measure of each angle of a regular n-gon. In this activity we'll see if we can answer this question.

1) Draw a picture of Louie Llama rotated 90° counterclockwise.

2) Draw a picture of Louie Llama rotated 180° counterclockwise.

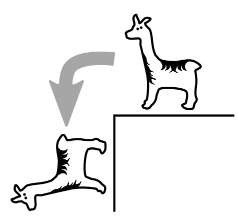
3) Draw a picture of Louie Llama rotated 360° counterclockwise.

4) Sometimes Louie Llama likes to walk around lines he finds:



Through what angle did Louie Llama just rotate?

Again, we're going to watch Louie Llama go for a walk. Draw yourself any a regular 3-gon. When Louie Llama walks around corners he rotates. Check it out:



Since your 3-gon is regular, each of its angles has measure θ.

5) Sketch Louie Llama walking around our 3-gon. As he goes around a corner, through what angle does Louie Llama rotate?

6) Find the measure of each angle of a 3-gon. Explain your reasoning.

7) Sketch a regular 4-gon and find the measure of each angle of a 4-gon. Explain your reasoning.

8) Sketch a regular 5-gon and find the measure of each angle of a 5-gon. Explain your reasoning.

9) Sketch a regular 6-gon and find the measure of each angle of a 6-gon. Explain your reasoning.

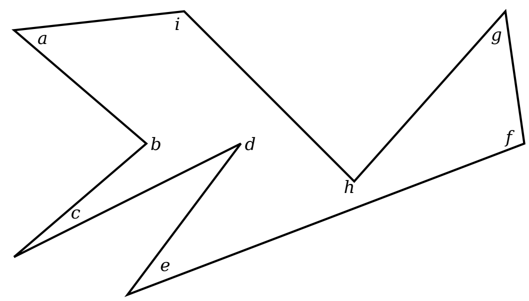
Now it is time to generalize!

10) Sketch a regular n-gon and find the measure of each angle of a n-gon. Explain your reasoning. Note, your answer should be a formula.

A.5 Angles in a Funky Shape

We are going to investigate the sum of the interior angles of a funky shape.

1) Using a protractor, measure the interior angles of the crazy shape below:



Use this table to record your findings:

a	b	c	d	e	f	g	h	i

2) Find the sum of the interior angles of the polygon above.

3) What should the sum be? Explain your reasoning.

A.6 Centers, Circles, and Lines Oh My!

In this activity, we are going to explore the basic centers, circles, and lines related to triangles. To make things more pleasant, I suggest you let *GeoGebra* help you out.

1) Draw yourself a triangle. Now construct the perpendicular bisectors of the sides—notice anything? Does this work for every triangle?

2) Now bisect the angles—notice anything? Does this work for every triangle?

3) Now construct the lines containing the altitudes—notice anything? Does this work for every triangle?

4) Now construct the medians—notice anything? Does this work for every triangle?

5) Now construct the circumcircle using the "three-point-circle" tool. Construct the circumcenter using the "midpoint-or-center" tool. Notice anything in connection to the lines drawn above? Does this work for every triangle?

6) The incenter is found via the intersection of the angle bisectors. Construct the incircle.

7) Fill in the following handy chart summarizing the information you found above.

	what are they? (draw pictures)	associated point?	always inside the triangle?	meaning?
perpendicular bisectors				
angle bisectors				
lines containing altitudes				
lines containing the medians				

Be sure to put this in a safe place like in a safe, or under your bed.

A.7 The Euler Line and Nine-Point Circle

1) Here are some easy questions to get the brain-juices flowing!

(a) Place two points randomly in the plane. Do you expect to be able to draw a single line that connects them?

(b) Place three points randomly in the plane. Do you expect to be able to draw a single line that connects them?

(c) Place two lines randomly in the plane. How many points do you expect them to share?

(d) Place three lines randomly in the plane. How many points do you expect all three lines to share?

(e) Place three points randomly in the plane. Will you (almost!) always be able to draw a circle containing these points? If no, why not? If yes, how do you know?

(f) Place four points randomly in the plane. Do you expect to be able to draw a circle containing all four at once? Explain your reasoning.

Round one!

2) Use *GeoGebra* to construct the circumcenter of a triangle. Hide all extraneous lines and points. Label this point C.

3) Use *GeoGebra* to construct the centroid of the same triangle. Hide all extraneous lines and points. Label this point N.

4) Use *GeoGebra* to construct the orthocenter of the same triangle. Hide all extraneous lines and points. Label this point O.

5) Connect C and O with a segment. Did a miracle happen—or has *GeoGebra* been a naughty monkey?

Round two—let's talk about circles baby!

6) Keeping the same triangle as used in the previous problems, use *GeoGebra* to mark the midpoint of the segment that connects C and O. Label this point M.

7) On the same triangle use *GeoGebra* to mark the midpoints of each side. Note, you should just be able to "unhide" them as they are already there.

8) On the same triangle use *GeoGebra* to mark where the altitudes meet the lines containing the sides of the triangle. Hide all extraneous lines and points.

9) On the same triangle use *GeoGebra* to mark the midpoints of the segments joining the orthocenter and the vertexes. Hide all extraneous lines and points.

10) On the same triangle use *GeoGebra* to draw a circle centered at M that goes through one of the midpoints of the triangle.

11) Did a miracle happen—or has *GeoGebra* morphed into *Geogezilla*?

A.8 Verifying our Constructions

When we do our compass and straightedge constructions, we should take care to verify that they actually work as advertised. We'll walk you through this process. To start, remember what a circle is:

Definition A **circle** is the set of points that are a fixed distance from a given point.

1) Is the center of a circle part of the circle?

2) Construct an equilateral triangle. Why does this construction work?

Now recall the SSS Theorem:

Theorem 11 (SSS) *Specifying three sides uniquely determines a triangle.*

3) Now we'll analyze the construction for copying angles.

(a) Use a compass and straightedge construction to duplicate an angle. Explain how you are really just "measuring" the sides of some triangle.

(b) In light of the SSS Theorem, can you explain why the construction used to duplicate an angle works?

4) Now we'll analyze the construction for bisecting angles.

(a) Use compass and straightedge construction to bisect an angle. Explain how you are really just constructing an equilateral triangle. Draw this equilateral triangle in your figure.

(b) Find two triangles on either side of your angle bisector where you may use the SSS Theorem to argue that they have equal side lengths and equal angle measures.

(c) Can you explain why the construction used to bisect angles works?

Recall the SAS Theorem:

Theorem 12 (SAS) *Specifying two sides and the angle between them uniquely determines a triangle.*

5) Now we'll analyze the construction for bisecting segments.

(a) Use a compass and straightedge construction to bisect a segment. Explain how you are really just constructing two equilateral triangles.

(b) Note that the bisector divides each of the above equilateral triangles in half. Find two triangles on either side of your bisector where you may use the SAS Theorem to argue that they have equal side lengths and angle measures.

(c) Can you explain why the construction used to bisect segments works?

6) Now we'll analyze the construction for copying angles.

(a) Use a compass and straightedge construction to construct a perpendicular through a point. Explain how you are really just constructing an isosceles triangle.

(b) Find two triangles in your construction where you may use the SAS Theorem to argue that they have equal side lengths and angle measures.

(c) Can you explain why the construction used to construct a perpendicular through a point works?

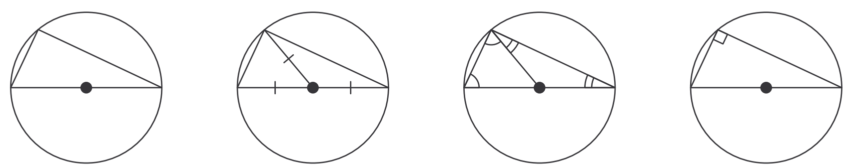
A.9 Of Angles and Circles

In this handout we are going to look at pictures and see if we can explain how they "prove" theorems.

Theorem 13 *Any triangle inscribed in a circle having the diameter as a leg is a right triangle.*

1) Can you tell me in English what this theorem says? Provide some examples of this theorem in action.

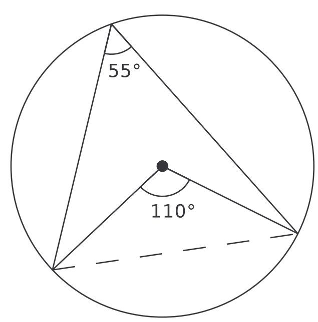
2) Here is a series of pictures, designed to be read from left to right.



Explain how these pictures "prove" the above theorem. In the process of your explanation, you may need to label parts of the pictures and do some algebra.

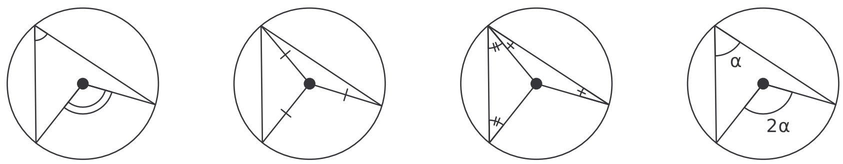
Theorem 14 *Given a chord of a circle, the central angle defined by this chord is twice any inscribed angle defined by this chord.*

I'll play nice here and give you a picture of this theorem in action:



3) Can you tell me in English what this theorem says? For one thing, what do the fancy words mean? Specifically, what is meant by *chord*, *central angle*, and *inscribed angle*?

4) Here is a series of pictures, designed to be read from left to right.



Explain how these pictures "prove" the above theorem. In the process of your explanation, you may need to label parts of the pictures and do some algebra.

Corollary *Given a chord of a circle, all inscribed angles defined by this chord are equal.*

5) Firstly—what the heck is a corollary? Secondly—what is it saying? Thirdly—why is it true?

A.10 I'm Into Triangles

In this activity, we're going to see if we can discover a simple method for breaking *every* polygon into triangles.

1) Draw yourself a polygon with at least 8 sides. Show how to break this polygon into triangles.

2) See if you can figure out exactly **what** your method was for breaking the polygon into triangles. Write it down.

3) Find a casual acquaintance and declare "I challenge you to present me with a polygon that cannot be broken into triangles." Can you use your method to break their polygon into triangles?

4) Draw a polygon that would be really difficult to break into triangles.

5) Come up with a simple method that will **always** work for breaking a polygon into triangles. As a hint, draw a stick person in your polygon, and try to imagine what they see...

A.11 Morley's Miracle

Here is a construction that wasn't discovered until 1899. To make life easier, I'm going to allow you to use the following (somewhat imprecise) method for trisecting angles:

(a) Fold the paper so that the crease leads up to the angle, with the edge of the flap being folded-over bisecting the new angle of the crease and the edge that was not moved.

(b) Now fold the edge that was not moved on top of the flap that was just made. It should fit perfectly near the angle. If done correctly, the steps above should trisect the angle.

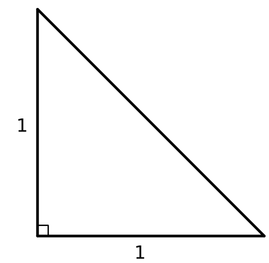
1) What does that say above? I know, I know, it sounds complicated. See if you can figure it out anyhow.

So now get your tracing paper out, make a big scalene triangle, and trisect all three angles.

2) Connect adjacent trisectors. Do you see a miracle happening? (I know, I know, if *anybody* can follow these directions than we truly will have a miracle on our hands!)

A.12 Please be Rational

Let's see if we can give yet another proof that the square root of two is not rational. Consider the following isosceles right triangle:

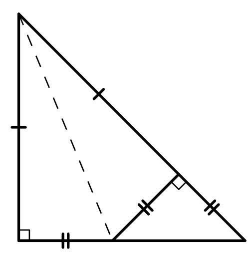


1) Using the most famous theorem of all, how long is the unmarked side?

2) Suppose that the unmarked side has a rational length. In that case how could we express it?

3) Explain why there would then be a *smallest* isosceles right triangle with integer sides. Considering the problem above, how long would the sides be? Draw and label a picture.

4) Now fold your smallest isosceles right triangle with integer sides along the dotted line like so:



Explain why the segments we have marked above as "equal" are in fact equal.

5) Explain how we have now found an isosceles right triangle with integer sides that is now smaller than the smallest isosceles right triangle with integer sides. Is this possible? What must we now conclude?

A.13 Reading Information From a Graph

On the next page is the graph of a function called $h(t)$, which represents the distance (in miles) and direction (east = positive, west = negative) Johnny is from home t hours after noon. It does not have a simple formula, so don't try to find one. Answer the following questions about h, briefly explaining how you obtained your answer(s):

1) On the given graph of h, what are the smallest and greatest values of t? What are the greatest and smallest values of $h(t)$? What do these answers say about Johnny?

2) Evaluate the following expressions: $h(0)$, $h(3)$, and $h(-3)$. What do each of these say about Johnny?

3) For each of the following, solve for t (i.e., find all the values of t that make the statement true). Describe what you did with the graph to determine the solutions. Where possible, interpret the statement and its solutions in terms of Johnny.

(a) $h(t) = 0$

(b) $h(t) = 3$

(c) $h(t) \leq 3$

(d) $h(t) = h(4.5)$

(e) $h(t) = t$

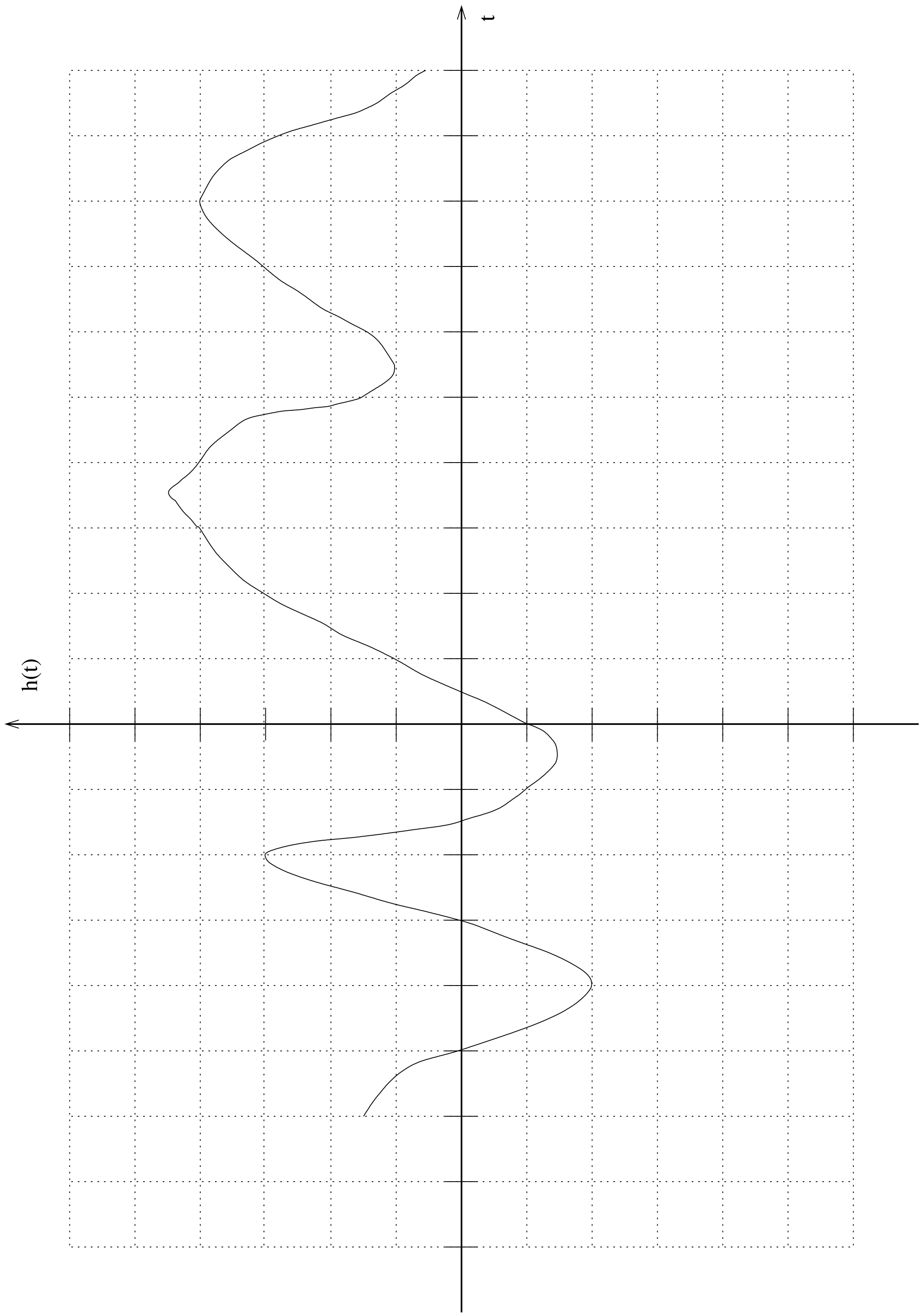
(f) $h(t) = -t$

(g) $h(t) = h(-t)$

(h) $h(t) = -h(-t)$

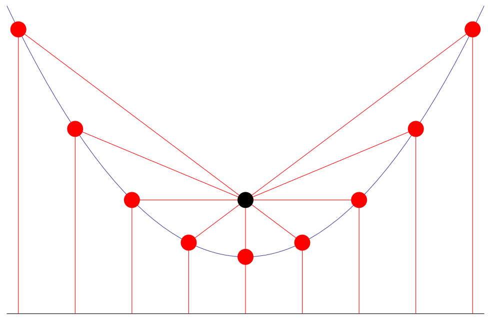
(i) $h(t + 1) = h(t)$

(j) $h(t) + 1 = h(t)$


t
h(t)

A.14 Bola, Para Bola

We've mentioned several times that a parabola is the set of points that are equidistant from a given point (the focus) and a given line (the directrix):



In this activity we are going to try an reconcile the definition given above with the equation that you know and love (admit it!):

$$y = ax^2 + bx + c$$

1) How do we compute the distance between two points? Be explicit!

2) Let's see if we can derive the formula for a parabola with its focus at $(0, 1)$ and its directrix being the line $y = 0$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

3) Let's see if we can derive the formula for a parabola with its focus at $(0, 1)$ and its directrix being the line $y = -1$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

4) Let's see if we can derive the formula for a parabola with its focus at $(1, 1)$ and its directrix being the line $y = -2$.

(a) Given a point (x, y), write an expression for the distance from this point to the focus.

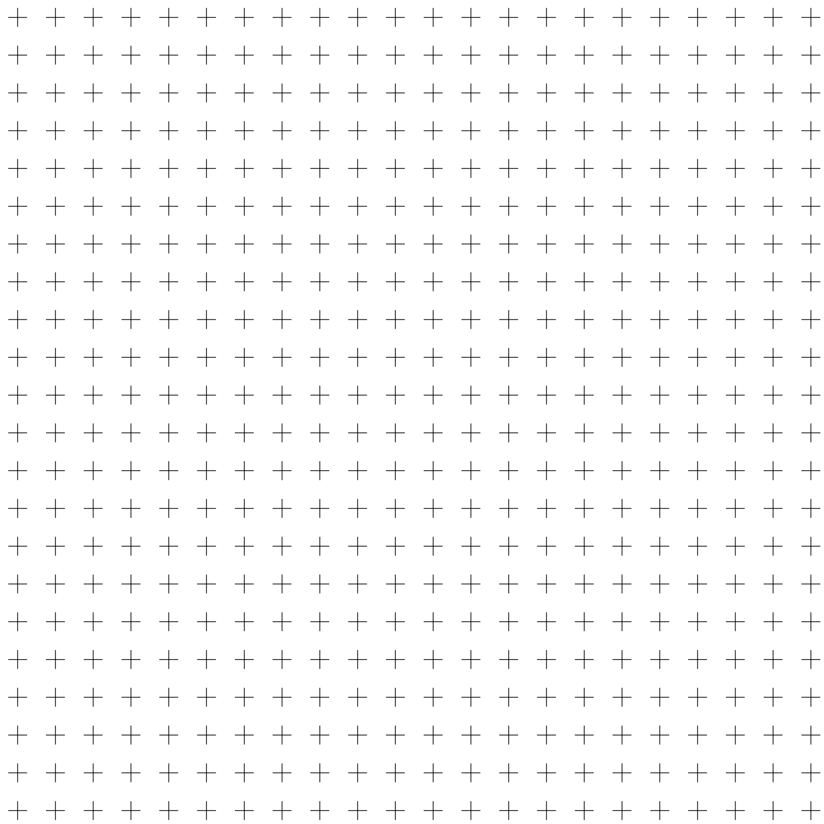
(b) Write an expression for the distance from (x, y) to the directrix.

(c) Use these two expressions and some algebra to find the formula for the parabola.

A.15 The Path Not Taken

In Euclidean geometry, there is a unique shortest path between two points. Not so in city geometry, here you have many different choices. Let's investigate this further.

1) Place two points 5 units apart on the grid below. How many paths are there that follow the grid lines? Note, if your answer is 1, then maybe you should pick another point!



Be sure to demand that your results are shared with the rest of the class.

2) Do the first problem again, except for points that are 4 units apart and then for points that are 6 units apart. What do you notice? Can you explain this?

3) Construct a chart showing your findings from your work above, and other findings that may be relevant.

4) Suppose you know how many paths there are to all points of distance n away from a given point. Can you easily figure out how many paths there are to all points of distance $n + 1$ away? Try to explain this in the context of paths in city geometry.

A.16 Midsets Abound

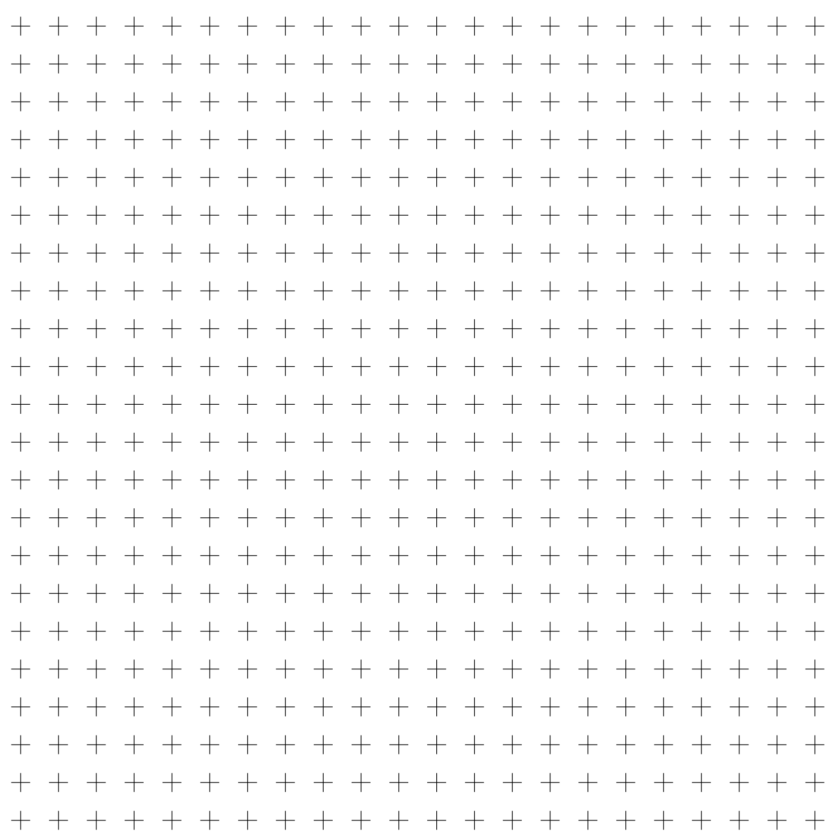
In this activity we are going to investigate *midsets*.

Definition Given two points A and B, their **midset** is the set of points that are an equal distance away from both A and B.

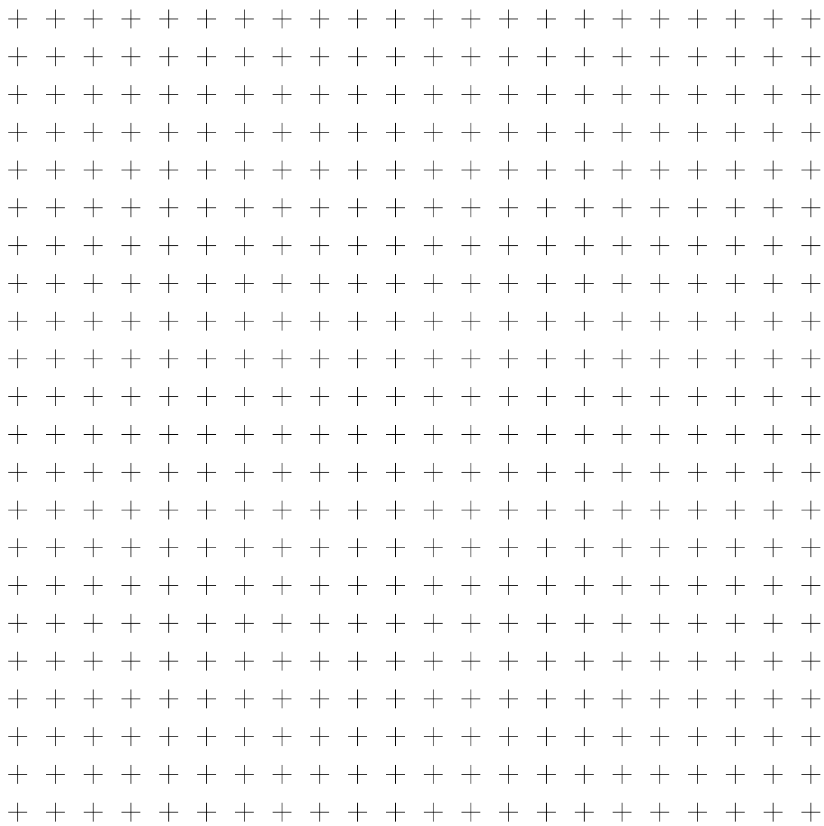
1) Draw two points in the plane A and B. See if you can sketch the Euclidean midset of these two points.

2) See if you can use coordinate constructions to find the equation of the midset of two points A and B. If necessary, set $A = (2, 3)$ and $B = (5, 7)$.

3) Now working in city geometry, place two points and see if you can find their midset.



4) Let's try to classify the various midsets in city geometry:



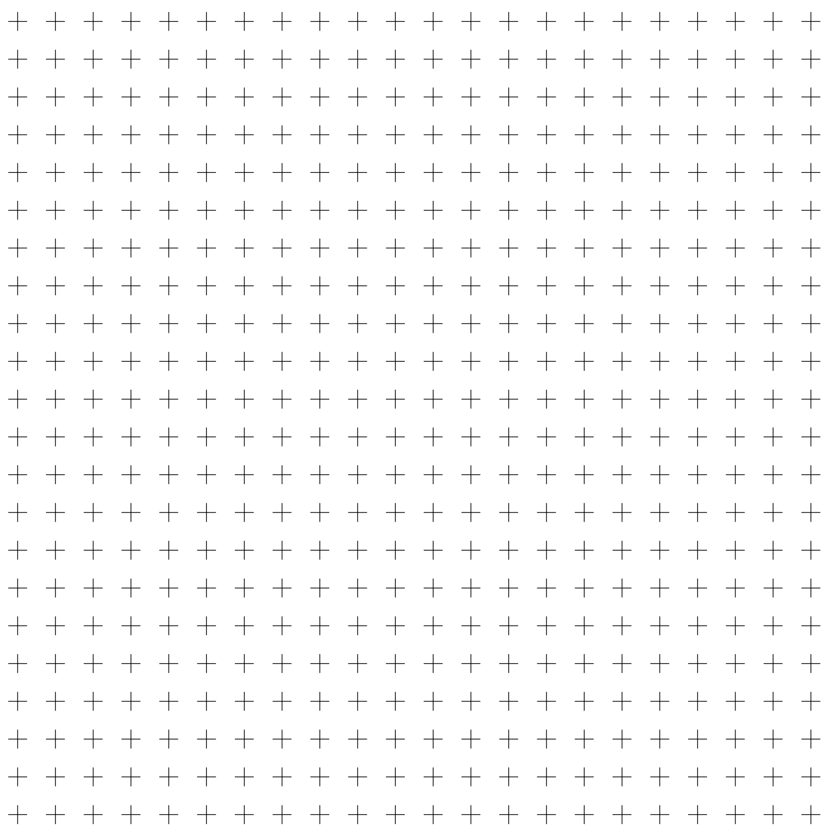
A.17 Tenacity Paracity

In this activity we are going to investigate city geometry parabolas.

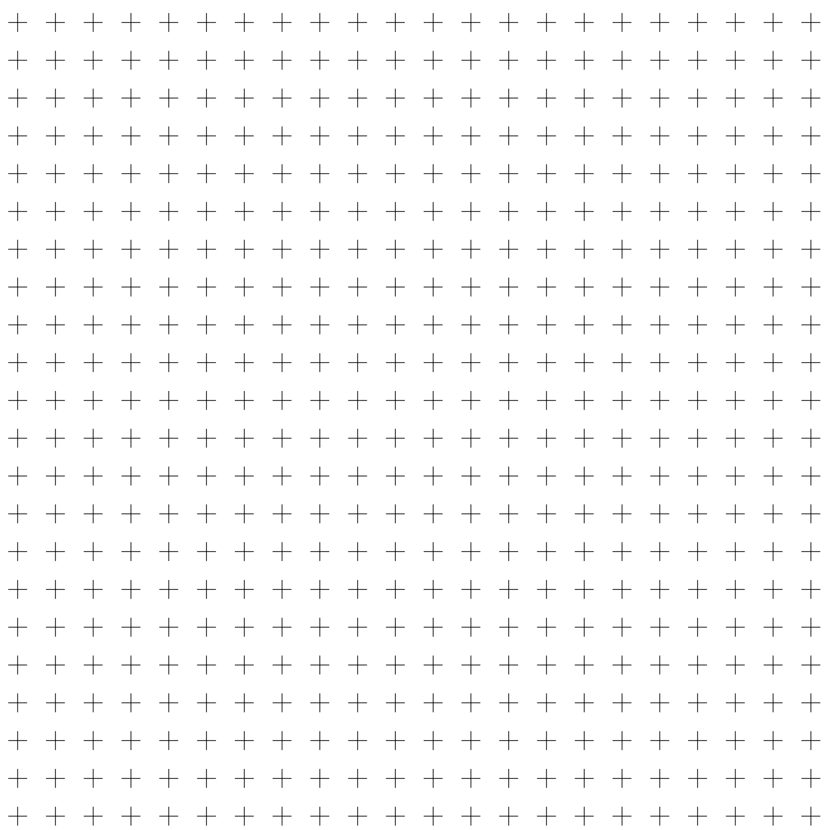
1) Remind me again, what is the definition of a *parabola*?

2) Use coordinate constructions to find the equation of the parabola with its focus at $(1, 2)$ and its directrix being the line $y = -3$.

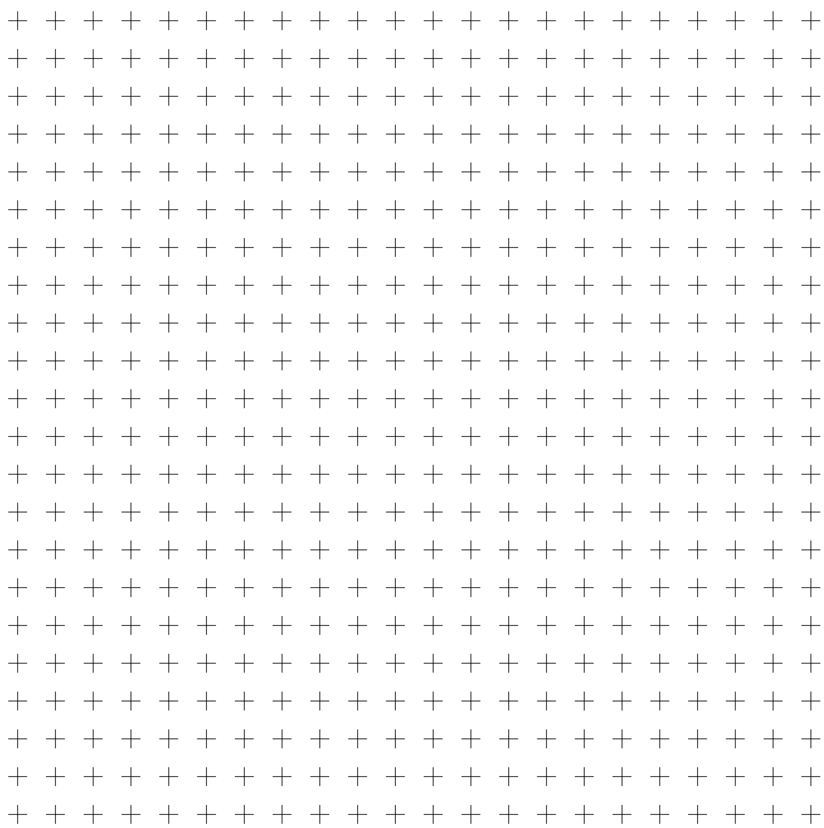
3) Sketch the city geometry parabola when the focus is the point $(0, 2)$ and the directrix is $y = 0$



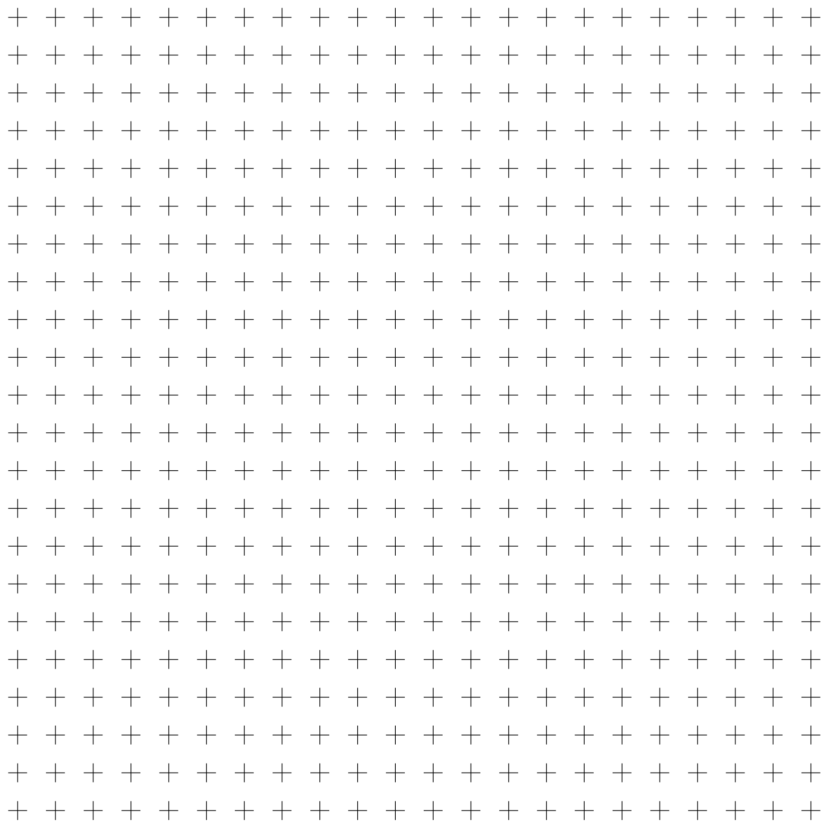
4) Sketch the city geometry parabola when the focus is the point $(4, 4)$ and the directrix is $y = -x$



5) Sketch the city geometry parabola when the focus is the point $(0, 4)$ and the directrix is $y = x/3$



6) Sketch the city geometry parabola when the focus is the point $(4, 1)$ and the directrix is $y = 3x/2$



7) Explain how to find the distance between a point and a line in city geometry.

8) Give instructions for sketching city geometry parabolas.

A.18 Who Mapped the What Where?

Let M represent some mysterious matrix that maps the plane to itself. So M is of the form:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

1) If I tell you that $\mathbf{p} = (3, 4)$ and

$$\mathsf{M}\mathbf{p} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix},$$

give 3 possible matrices for M.

2) Now suppose that in addition to the fact above, I tell that

$$\mathsf{M}\mathbf{o} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

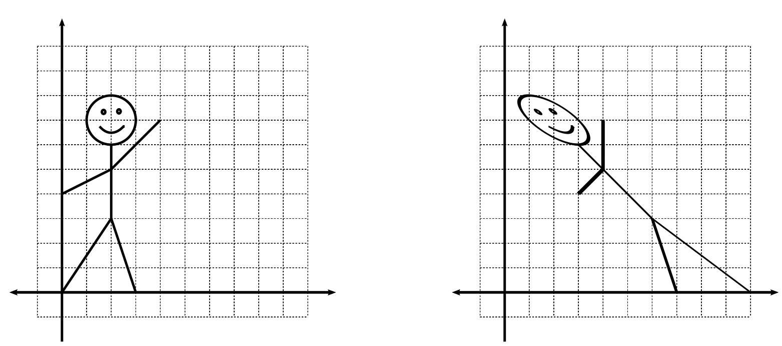
where $\mathbf{o}$ is the origin. Give 3 possible matrices for M.

3) Now suppose that in addition to the two facts above, I tell you that

$$\mathsf{M}\mathbf{q} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

where $\mathbf{q} = (1, 1)$. How many possibilities do you have for M now? What are they?

4) Here is a picture of my buddy *Sticky*:



As you can see, he's been dancing with some matrix M. Can you tell me which matrix it was? What are good points to pay attention to?

A.19 How Strange Could It Be?

In this activity, we are going to investigate just how strange a map given by

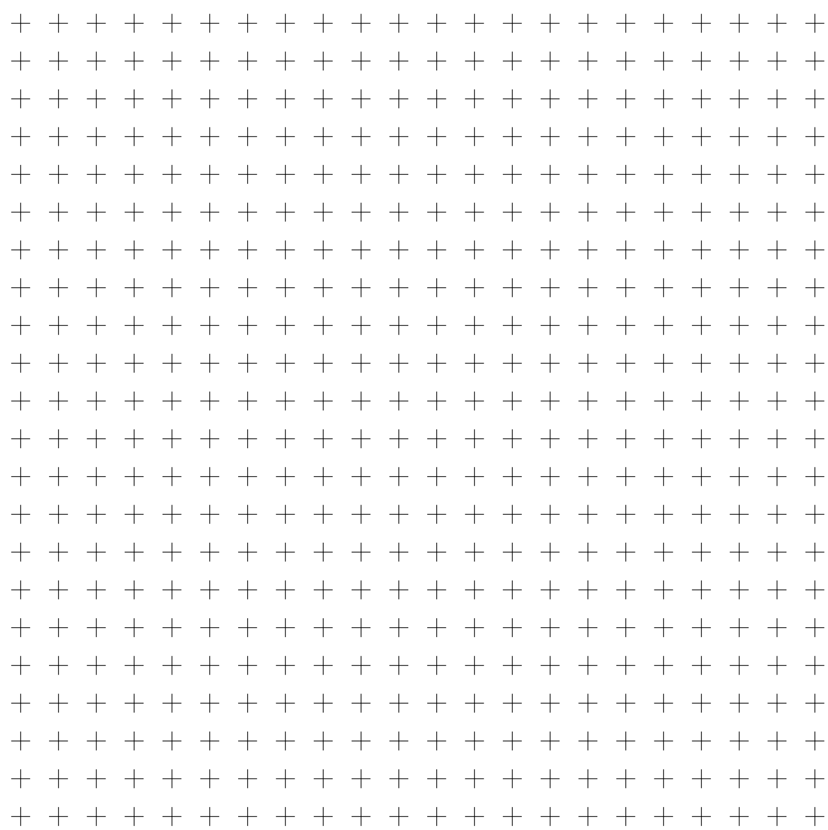
$$\mathsf{M} = \begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix}$$

could possibly be.

1) Let α, β, γ, and δ be real numbers, and let x be a variable. Consider the point:

$$(\alpha x + \beta, \gamma x + \delta)$$

Choose values for the Greek letters and plot this for varying values of x. What sort of curve do you get?



2) Now consider the line $y = mx + p$. Express its coordinates *without* using y.

3) Apply M to the coordinates you found above. What do you get? What does this tell you about what happens to lines after you apply a matrix to them?

4) Tell me some things about the line $y = mx + q$.

5) Apply M to the coordinates associated to $y = mx + q$. What does this tell you about what happens to parallel lines after you apply a matrix to them?

6) What's going to happen to a parallelogram after you apply a matrix to it?

A.20 Subs at Sea

Take your favorite n-gon and consider the group of symmetries, call it G. Write out the group table for G. Got it? Good.

1) Let M be an element of your group. Define

$$C(\mathsf{M}) = \{\text{all elements of } G \text{ that commute with } \mathsf{M}\}.$$

For every element $\mathsf{M} \in G$, write out $C(\mathsf{M})$. Make some observations.

2) Exactly which elements of G commute with everything?

3) Repeat the exercises above for a different group.

4) Are the sets $C(\mathsf{M})$ groups themselves? Explain why or why not.

5) For any given M, what do you notice about the number of elements found in $C(\mathsf{M})$?

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