Math 2167

Calculus for Middle Grades Teachers

Contents

1 Functions Everywhere

In this activity, we will describe some basic relationships arising from real-world contexts.

1) You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table.

(a) Sketch a graph of the temperature of the water as a function of the elapsed time, call it $T(t)$, and explain your reasoning.

(b) Recall a function is **one-to-one** if for every element in the domain, there is exactly one in the range; and for every element in the range, there is exactly one element in the domain. Is the function one-to-one? Explain your reasoning.

(c) If $T(t)$ is one-to-one, sketch a graph of its inverse. If $T(t)$ is not one-to-one, give a restricted domain on which it is one-to-one and sketch a graph of its inverse on this domain.

2) Consider the number of hours of daylight as a function of the time of year.

(a) Sketch a graph of the temperature of the water as a function of the elapsed time, call it $N(t)$, and explain your reasoning.

(b) Is the function one-to-one? Explain your reasoning.

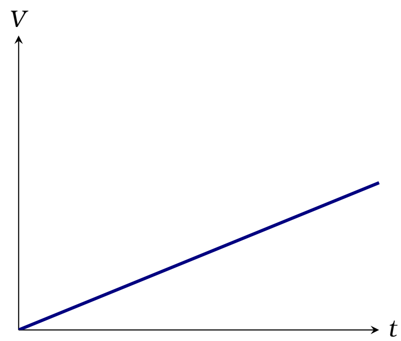
(c) If $N(t)$ is one-to-one, sketch a graph of its inverse. If $N(t)$ is not one-to-one, give a restricted domain on which it is one-to-one and sketch a graph of its inverse on this domain.

3) Louise swims to the island in the middle of a big lake. When she starts out, she has lots of energy. However, she tires soon and her speed decreases until she reaches the island.

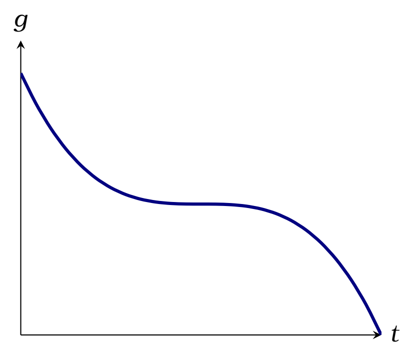
(a) Sketch a graph of Louise's distance from the island as a function of time and explain your reasoning.

(b) Sketch a graph of Louise's distance traveled as a function of time and explain your reasoning.

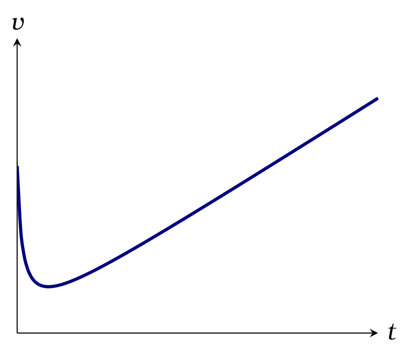
4) For each of the plots below, describe in words the situation that the graph depicts. Use both the language of **amounts** as well as **rates**.



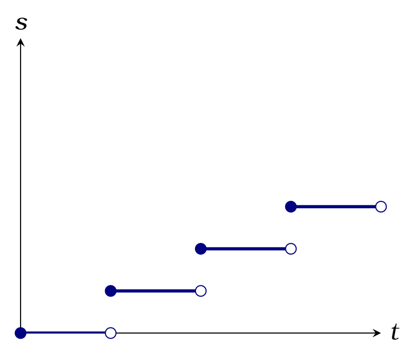
Volume of water in a bathtub.



Amount of gas left in a car.



Value of a vintage car.



Savings for a new car.

2 Language of Functions

We use functions as a special case of relationships between two or more variables, only two in this course, to predict the value of one variable given the value of the other. For this activity, our function will first be given to us in “formula” form.

1) Suppose you are standing on a bridge that is 60 meters above sea-level. You toss a ball up into the air with an initial velocity of 30 meters per second. If t is the time (in seconds) after we toss the ball, then the height at time t is approximately $f(t) = -5t^2 + 30t + 60$. Find the following values and explain what it means in the “real-life” problem:

(a) $f(2)$

(b) $f(17)$

(c) $f(-4)$

(d) Find t if $f(t) = 100$.

(e) Find t if $f(t) = -20$.

(f) Find t if $f(t) = 150$.

2) With the same context as above, if h is a small (positive) value what is the meaning of $f(t+h)$? How does this compare this to the meaning of $f(t)+h$? Explain your reasoning.

Now we’re going to get slightly more abstract.

3) Consider $f(x) = x^2$. What is meant by $f^{-1}(x)$, how does this compare to $f(x)^{-1}$? Explain your reasoning. Are either of these functions? Explain your reasoning.

4) What is the difference (if any!) between the values of x that satisfy $x^2 = 4$ and $\sqrt{4}$? Explain your reasoning.

5) What is the relationship between $\left(\sqrt{x}\right)^2$ and $\sqrt{x^2}$?

3 Taking it to the Limit

Calculus courses usually start with a section on *limits*. We don't want to rock the boat too much, so we'll talk about them too.

1) Limits are a way of examining a function where it may not be defined. Intuitively, $\lim_{x\to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. As an example, consider the function defined by the formula

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

Plot $f(x)$ by making a table of values for $f(x)$. What do you notice? What is $\lim_{x\to 2} f(x)$?

2) Consider the *greatest integer function*. This is the function that maps any real number x to the greatest integer less than or equal to x. It is denoted by

$$f(x) = \lfloor x \rfloor.$$

Sketch a plot of $f(x)$.

3) Again let $f(x) = \lfloor x \rfloor$. Compute the following:

(a) $\lim_{x\to 1.5} f(x)$

(b) $\lim_{x\to 0.\bar{3}} f(x)$

(c) $\lim_{x\to \pi} f(x)$

(d) $\lim_{x\to e} f(x)$

(e) $\lim_{x\to 0} f(x)$

(f) $\lim_{x\to 1} f(x)$

4) A function is *continuous* if whenever inputs are nearby each other, the associated outputs are nearby each other. Give some examples of continuous functions.

5) It's a fact that if a function is continuous at $x = a$, then

$$\lim_{x\to a} f(x) = f(a).$$

What is this saying?

6) Compare/contrast the following limits:

(a) $\lim_{x\to 2} \dfrac{x^2 - 3x + 2}{x - 2}$

(b) $\lim_{x\to 2} \dfrac{x^2 + 3x + 2}{x - 2}$

(c) $\lim_{x\to 2} \dfrac{x^2 - 3x + 2}{x + 2}$

4 How Does it Rate?

A major topic in calculus is *rates*. Let's dive right in and look at some rates.

1) Water is being poured into a large tub in such a way that the amount of water in the tub t minutes after the pouring started is:

$$f(t) = \frac{2t^2 + 3t + 1}{t + 1} \text{ gallons.}$$

We want to find out how fast water is being poured in at time $t = 3$ minutes.

(a) How much water is in the tub when we starting pouring water in?

(b) How much water is in the tub at 3 minutes?

(c) Find the rate of pouring at 3 minutes. Give evidence that your answer is correct.

2) Now water is being poured into a large tub in such a way that the amount of water in the tub t minutes after the pouring started is:

$$g(t) = \frac{t^2}{2} + 3 \text{ gallons.}$$

We want to find out how fast water is being poured in at time $t = 3$ minutes—can you use the same procedure you used above? Explain why or why not.

3) List some strategies for figuring out the rate that water is poured into the tub at $t = 3$ minutes. Whatever strategy you have, be prepared to justify **why** it works.

5 What Do Limits Have To Do With It?

As we previously stated, calculus courses usually start with a section on *limits*. Let's see if we can make it clear why this is the case.

1) Remind me again, what is *slope*? I want at least **four** different interpretations.

2) Now, assume you have a function $f(x)$. Consider:

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

what does this have to do with your answers to the question above? **Draw a picture**.

3) Consider the function $g(x) = \dfrac{x^2}{2} + 3$. Write

$$\frac{g(3+h)-g(3)}{(3+h)-3}.$$

What is this computing? Can you simply set h to zero? Why would you even want to do this? Explain what is going on here.

Recall, limits are a way of examining a function where it may not be defined. This is exactly the sort of situation we are in! We'll call the rate of a function at a point the **derivative** and it is defined as follows:

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

4) Compute:

(a) $\dfrac{d}{dx}x^2$

(b) $\dfrac{d}{dx}\left(x^2 - 3x\right)$

(c) $\dfrac{d}{dx}\left(x^2 + 2\right)$

(d) $\dfrac{d}{dx}(3x+2)$

(e) $\dfrac{d}{dx}16$

(f) $\dfrac{d}{dx}\sqrt{x}$

(g) $\dfrac{d}{dx}\sqrt{x+2}$

(h) $\dfrac{d}{dx}x^3$

6 Rated PG (for Language)

In this activity, we will work on understanding the language of derivatives. Keep in mind that

$$\frac{d}{dx}f(x) = f'(x) = \text{the slope of a tangent line to } f(x).$$

1) Consider the following statements:

> The population of Columbus in the year 2000 was 715971. The population is increasing at a rate of 7851 people per year in the year 2000.

Write an equivalent statment using functions and derivatives of functions.

2) Consider the following statements:

> Judy bikes along a flat country road at 300 meters per minute for 30 minutes. Next Judy comes to a hill and over the next 3 minutes, her speed drops at a constant rate of 40 meters per minute each minute.

Write an equivalent statment using $v(t)$ and $v'(t)$ where $v(t)$ is Judy's velocity at time t.

3) Consider the following statements:

> Judy bikes along a flat country road at 300 meters per minute for 30 minutes. Next Judy comes to a hill and over the next 3 minutes, her speed drops at a constant rate of 40 meters per minute each minute.

Write an equivalent statment using $p(t)$ and $p'(t)$ where $p(t)$ is Judy's (east or west) position from home at time t.

4) Let $H(t)$ stand for the height of an airplane (in feet above the ground) as a function of time (in minutes). Translate the following into English:

(a) $H(0) = 10000$

(b) $H'(0) = 0$

(c) $H'(5) = -520$

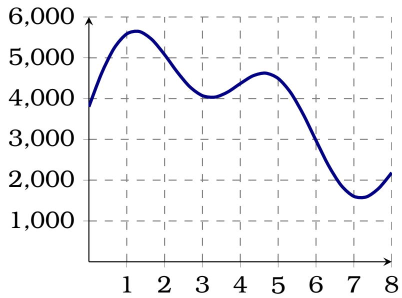
(d) $H(5) - H(0) = -2010$

5) Let $f(t)$ be a function such that $f(0) = 5$ and $0 < f'(t) < 12$ for all t. What can be said about the value of $f(3)$? What about $f(-4)$?

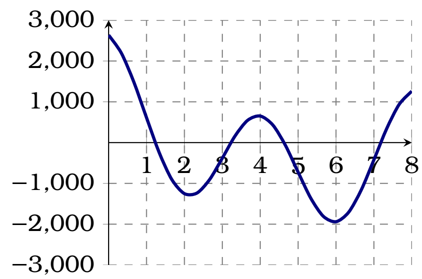
7 Where am I? How Fast am I Going?

If a function gives the position of an object, then that function's derivative tells you the velocity. Let's explore functions and their derivatives first in a real-world context and then in the abstract.

1) In the gondola of a hot air balloon, two instruments monitor the balloon's course over the course of an 8 minute period. An altimeter shows the balloon's height and a rate-of-climb meter shows how fast the balloon rises or falls. Here are the resultant graphs—heights change a lot due to updrafts and downdrafts.



Altitude (ft) vs. time (min)



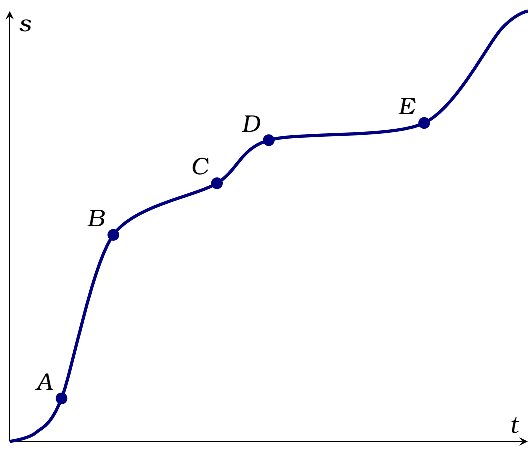
Vertical velocity (ft/min) vs. time (min)

(a) Find when the balloon is going up (height increasing) and going down (height decreasing). Look at each graph separately to answer the question. Explain your reasoning.

(b) At what time is the velocity of the balloon zero? What is happening to the balloon at those times? Look at each graph separately to answer the question. Explain your reasoning.

(c) Use the second graph to answer when the balloon is rising the fastest and when it is falling the fastest. What does the first graph do at those times? Explain your reasoning.

2) The graph below shows the position function of a car as it goes to work. Answer the following questions:



(a) What was the initial velocity of the car?

(b) Was the car going faster at B or at C?

(c) Was the car slowing down or speeding up at A, B, and C?

(d) What happened between D and E?

(e) At what times did the car turn back for home?

3) For the following, suppose an object can move only along the positive x-axis. Sketch the graph of the object's position vs. time graph and its velocity vs. time graph.

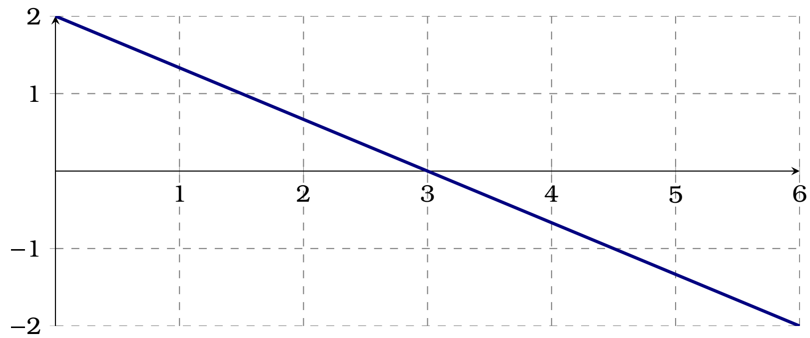
(a) The object is standing still.

(b) The object is moving away from the origin at a constant velocity.

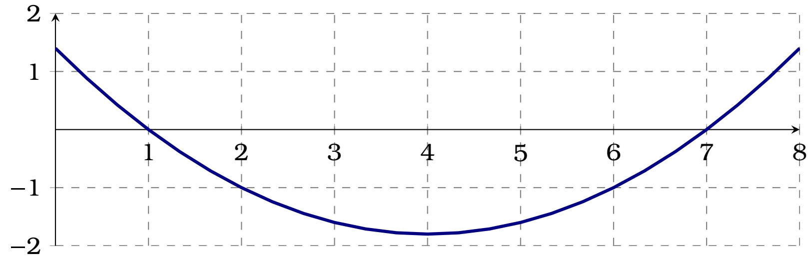
(c) The object is moving toward the origin at a steadily increasing speed.

4) For the following, given the graph of the function, graph its derivative.

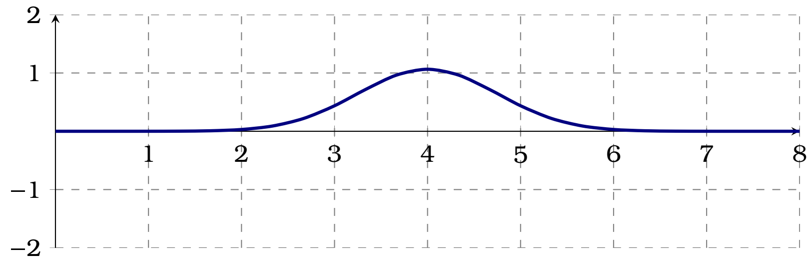
(a)



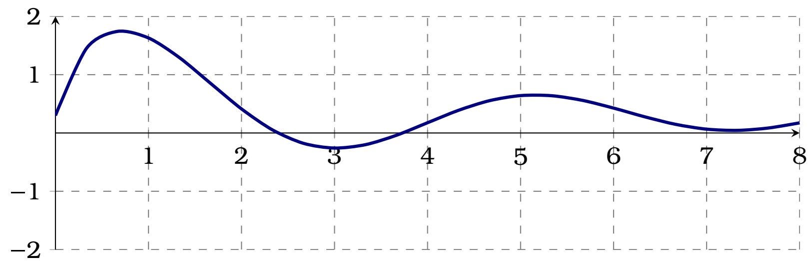
(b)



(c)



(d)



8 The Power Rule, Sum Rule, and Constant Multiple Rule

If you learn nothing else in a calculus class, then maybe you should know the *power rule*.

1) Compute $\dfrac{d}{dx}x$. Explain your reasoning.

2) Compute $\dfrac{d}{dx}x^2$. Explain your reasoning.

3) Compute $\dfrac{d}{dx}x^3$. Explain your reasoning.

4) Compute $\dfrac{d}{dx}x^4$. Explain your reasoning.

5) Compute $\dfrac{d}{dx}x^5$. Explain your reasoning.

6) Compute $\dfrac{d}{dx}x^n$ where n is a positive integer. Explain your reasoning.

7) State the power rule. Give (and compute) examples where the exponent is: a positive integer, a fraction with unit numerator, a negative integer, a rational number, a real number.

8) Compute:

$$\frac{d}{dx}\sqrt{x} \qquad \frac{d}{dx}\sqrt[5]{x} \qquad \frac{d}{dx}\frac{1}{x^7} \qquad \frac{d}{dx}\frac{1}{\sqrt[3]{x^4}}$$

9) Explain why

$$\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x).$$

This is the sum rule.

10) Explain why if a is a real number,

$$\frac{d}{dx}af(x) = af'(x).$$

This is the constant multiple rule

11) Putting the power rule, sum rule, and constant multiple rule together, what can you now differentiate with ease? Give a variety of examples.

12) Calculus Cal says that

$$\frac{d}{dx}e^x = xe^{x-1}.$$

Is he right? Either prove him right or prove him wrong.

9 Lines for Curves

In this activity we will study *linear approximations*.

1) Smart Sally says that the line

$$\ell(x) = \frac{1}{4}(x-4)+2$$

is a good approximation for $f(x) = \sqrt{x}$ when x is close to 4.

(a) Plot $\ell(x)$ and $f(x)$. Explain how this shows that she is correct.

(b) Use concepts of calculus to explain why Sally is correct.

2) Consider $f(x) = \sqrt[3]{x}$. Find the equation of the line tangent to $f(x)$ when $x = 27$. Use this to approximate $\sqrt[3]{28}$. Explain your reasoning and sketch a plot of this situation.

3) Consider $f(x) = \sqrt[5]{x}$. Find the equation of the line tangent to $f(x)$ when $x = 243$. Use this to approximate $\sqrt[5]{250}$. Explain your reasoning and sketch a plot of this situation.

4) In the problems above, we've been working with *linear approximations*. Explain what this means and how linear approximations work.

5) Suppose you want to know $\sqrt[3]{10}$. Explain how to use a linear approximation to estimate this value. Sketch a plot to clarify why your method works.

6) Calculus Cal attempts to use Smart Sally's method of computing linear approximations to compute $\sqrt[3]{2}$ with

$$\ell(t) = \frac{1}{27}(x-27)+3.$$

Smart Sally laughs at him and says, "That's never going to work!" Why is she sure this won't work? Explain what Cal did right and what Cal did wrong. Sketch a plot of this situation to enhance your explanation.

7) Calculus Cal attempts to use Smart Sally's method of computing linear approximations to compute 11^5 with

$$\ell(t) = 5 \cdot 10^4(x-10)+100000.$$

Smart Sally laughs at him and says, "That's never going to work!" Why is she sure this won't work? Explain what Cal did right and what Cal did wrong. Sketch a plot of this situation to enhance your explanation.

8) Explain the general procedure for finding linear approximations, and what the potential pit-falls are for such a procedure.

10 What's so Special About e?

I remember the number e,

$$e = 2.718281828459045\ldots,$$

being a very mysterious number. Calculus might be able to help explain why someone might be interested in this number.

1) Find a function $f(x)$ such that

$$f(0) = 1.$$

How many such functions can you find? What's the simplest function you can find?

2) Find a function $f(x)$ such that

$$f(0) = 1 \qquad \textit{and} \qquad f'(0) = 1$$

How many such functions can you find? What's the simplest function you can find?

3) Find a function $f(x)$ such that

$$f(0) = 1, \qquad f'(0) = 1, \qquad \textit{and} \qquad f''(0) = 1$$

How many such functions can you find? What's the simplest function you can find?

4) Find a function $f(x)$ such that

$$f(0) = 1, \qquad f'(0) = 1, \qquad f''(0) = 1, \qquad \textit{and} \qquad f'''(0) = 1$$

How many such functions can you find? What's the simplest function you can find?

5) Find a function $f(x)$ such that

$$f(0) = 1, \qquad f'(0) = 1, \qquad f''(0) = 1, \qquad f'''(0) = 1, \qquad \textit{and} \qquad f''''(0) = 1$$

How many such functions can you find? What's the simplest function you can find?

6) Use your work above to help you find a function $f(x)$ such that $f(0) = 1$ and every derivative of $f(x)$ is equal to 1 when evaluated at 0.

7) The function $f(x) = e^x$ is a function where

$$f'(x) = f(x).$$

Can you find some others with this property?

8) Can you use your work above to help you compute an approximation for e?

9) Find a function $f(x)$ such that

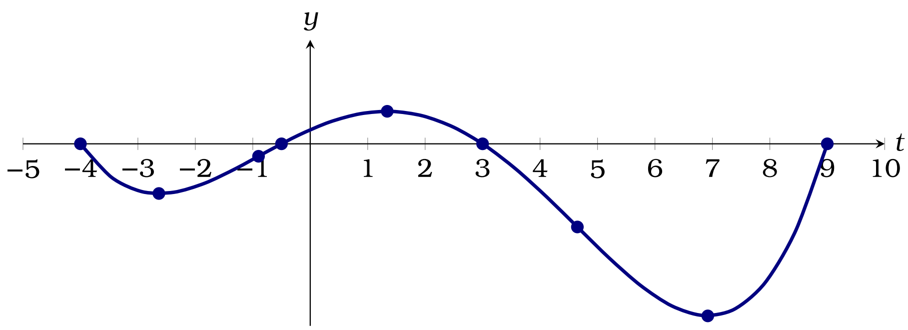
$$f'(x) = f(x).$$

and $f(0) = -17$.

10) Give a story problem explaining what e^x is in terms of mile-markers.

11 Graphing All the Way!

Here is a graph of Harry's position (in miles) from home (positive = east, negative = west) as he travels along an east-west road throughout the day (i.e., as time increases). Assume noon is when $t = 0$ hours. We can call Harry's position function $f(t)$. Although we have no formula for $f(t)$, everything we will do here will correspond to algebraic methods we would do if we had a formula for his position. Don't worry—you'll do these things with a formula soon enough!



1) For what times is Harry's position east of home? West of home? What did you look for on the graph to answer those questions? What special thing happens at the demarcation times? If we had a formula for $f(t)$, what would you do with the formula to determine those demarcation times? If you did not have the graph of $f(t)$ and only had the answers to the initial questions above, what would you know and not know about the graph?

2) Now let's examine Harry's *velocity*, not just where he is. For what times is Harry moving eastward? Moving westward? What did you look for on the graph to answer those questions? What special thing happens at the demarcation times? If we had a formula for $f(t)$, what would you do with the formula to determine those demarcation times? If you did not have the graph of $f(t)$ and only had the answers to the initial questions to 1 and 2, what would you now know and not know about the graph?

3) Now let's precisely examine Harry's *acceleration*. For what times is his eastward speed increasing? For what times is his eastward speed decreasing? What did you look for on the graph to answer those questions? What special thing happens at the demarcation times? If we had a formula for $f(t)$, what would you do with the formula to determine those demarcation times? What happens if we examine the westward speed?

4) Fill in pictures in the table below:

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	Here $f'(x) < 0$ and $f''(x) > 0$. This means that $f(x)$ slopes down and is getting *less steep*. In this case the curve is **concave up**.	Here $f'(x) > 0$ and $f''(x) > 0$. This means that $f(x)$ slopes up and is getting *steeper*. In this case the curve is **concave up**.
$f''(x) < 0$	Here $f'(x) < 0$ and $f''(x) < 0$. This means that $f(x)$ slopes down and is getting *steeper*. In this case the curve is **concave down**.	Here $f'(x) > 0$ and $f''(x) < 0$. This means that $f(x)$ slopes up and is getting less *steep*. In this case the curve is **concave down**.

12 Sketch THIS!

Let's apply what we've learned to sketch curves using the tools of calculus. For each problem, follow the guide below:

(a) Find the y-intercept.

(b) Compute $f'(x)$ and $f''(x)$.

(c) Find when $f'(x)$ is zero.

(d) Find when $f(x)$ is increasing and it is decreasing.

(e) Find local maxima and minima for $f(x)$.

(f) Find when $f''(x)$ is zero.

(g) Find when $f(x)$ is concave up and when it is concave down.

(h) If possible find the roots of $f(x)$.

1) Let's sketch $f(x) = x^5 - x$.

2) Let's sketch $f(x) = 2x^3 - 3x^2 - 12x$.

3) Let's sketch $f(x) = x^5 - 5x^4 + 5x^3$.

4) Let's sketch $f(x) = x + 1/x$.

5) Let's sketch $f(x) = x^2 + 1/x$.

13 The Product Rule and Quotient Rule

In this activity, we're going to investigate the derivatives of products of functions.

1) Give examples that prove that the derivative of the product of two given functions is not equal to the product of the derivatives of those functions, that is, give example of $f(x)$ and $g(x)$ where

$$\frac{d}{dx}f(x)g(x) \neq f'(x)\cdot g'(x).$$

2) Give two explanations, one geometrical and one algebraic, of why when h is small,

$$f'(x)\cdot h \approx f(x+h) - f(x) \qquad \text{and} \qquad g'(x)\cdot h \approx g(x+h) - g(x).$$

3) Now suppose you have two functions $f(x)$ and $g(x)$. Their product can be interpreted as the area of a $f(x)\times g(x)$ rectangle.

(a) Draw the $f(x)\times g(x)$ rectangle.

(b) Now suppose x changes (increasing is easiest!) to $x+h$, where we assume that h is small. Show this change on your drawing.

(c) Label your drawing with $f'(x)\cdot h$ and $g'(x)\cdot h$ in the appropriate places.

(d) Use your drawing to explain why

$$\frac{d}{dx}f(x)\cdot g(x) = f'(x)g(x) + f(x)g'(x).$$

This is known as the **product rule**.

4) Compute:

(a) $\dfrac{d}{dx}e^x\sqrt{x}$

(b) $\dfrac{d}{dx}e^{2x}$

(c) $\dfrac{d}{dx}e^{3x}$

(d) $\dfrac{d}{dx}f(x)g(x)h(x)$

5) Give examples that prove that the derivative of the quotient of two given functions is not equal to the quotient of the derivatives of those functions, that is, give example of $f(x)$ and $g(x)$ where

$$\frac{d}{dx}\frac{f(x)}{g(x)} \neq \frac{f'(x)}{g'(x)}.$$

6) Now suppose you have two functions $f(x)$ and $g(x)$. Let

$$Q(x) = \frac{f(x)}{g(x)}.$$

The quotient of $f(x)$ and $g(x)$ can be represented by $Q(x)$ on a $g(x) \times Q(x)$ rectangle with area $f(x)$.

(a) Draw the $g(x) \times Q(x)$ rectangle.

(b) Now suppose x changes (increasing is easiest!) to $x+h$, where we assume that h is small. Show this change on your drawing.

(c) Label your drawing with $f'(x) \cdot h$, $g'(x) \cdot h$, and $Q'(x) \cdot h$ in the appropriate places.

(d) Use your drawing to explain why

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

This is known as the **quotient rule**.

7) Compute:

(a) $\frac{d}{dx}\frac{x^2}{e^x}$

(b) $\frac{d}{dx}\frac{3x^4 - 12x^5 + 23}{x+1}$

(c) $\frac{d}{dx}\frac{e^x}{\sqrt{x}}$

(d) $\frac{d}{dx}e^{-2x}$

(e) $\frac{d}{dx}\frac{4x}{e^{3x}}$

(f) $\frac{d}{dx}\frac{f(x)g(x)}{h(x)j(x)}$

8) Suppose that an unfortunate calculus student forgot which of the following formulas was the quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} \stackrel{?}{=} \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

or

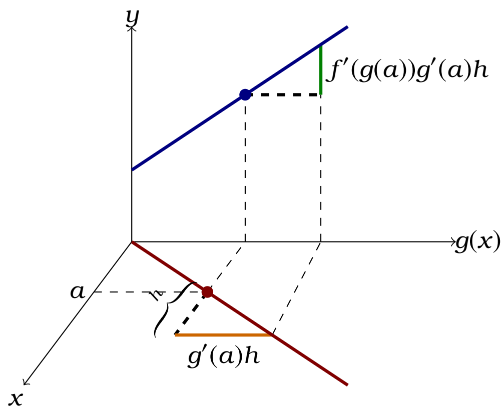
$$\frac{d}{dx}\frac{f(x)}{g(x)} \stackrel{?}{=} \frac{f(x)g'(x) - g(x)f'(x)}{g(x)^2}$$

Explain how one could use reasoning to figure this out.

14 My Rule, Your Rule, Our Rule, Chain Rule!

In this activity, we discuss the *chain rule*, my favorite rule.

1) Consider the following picture:



Explain how the picture above strongly suggests that

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

2) Explain how to compute

$$\frac{d}{dx}e^{5x}$$

two different ways, one way using the chain rule, and another using the product rule.

3) Compute:

(a) $\dfrac{d}{dx}\sqrt{x^3 + 7x^2 - 1}$

(b) $\dfrac{d}{dx}e^{\sqrt{x}}$

(c) $\dfrac{d}{dx}(x^5 + 14x)^{99}$

(d) $\dfrac{d}{dx}f(g(h(x)))$

4) Use the chain rule and the power rule to derive the quotient rule.

5) The volume of the balloon is dependent on the radius

$$V(r) = \frac{4}{3}\pi r^3$$

However, while you're blowing up the balloon, the radius in turn depends on how long you've been blowing it up. Let's say

$$r(t) = t^5 + 4t.$$

Thus, ultimately, volume of the balloon is dependent on the time you've been blowing it up.

(a) Compute $\dfrac{dV}{dr}$ and explain what it means in this context.

(b) Compute $\dfrac{dr}{dt}$ and explain what it means in this context.

(c) Compute $\dfrac{dV}{dt}$ and explain what it means in this context.

6) Let's see if we can find
$$\frac{d}{dx}6^x.$$

(a) Use the relation betweeen e^x and $\ln(x)$ to explain why
$$6^x = e^{\ln(6^x)}$$

(b) Explain how to use the chain rule to compute $\dfrac{d}{dx}6^x$.

7) Let's see if we can find
$$\frac{d}{dx}\ln(x).$$

(a) Use the relation betweeen e^x and $\ln(x)$ to explain why provided $x \neq 0$
$$e^{\ln(x)} = x$$
and hence $\dfrac{d}{dx}e^{\ln(x)} = 1$.

(b) Explain how to use the chain rule to deduce that
$$\frac{d}{dx}e^{\ln(x)} = e^{\ln(x)} \cdot (\ln(x))'$$

(c) Solve for $(\ln(x))'$.

8) Compute:

(a) $\dfrac{d}{dx}\ln(x^4 + x^3 + 1)$

(b) $\dfrac{d}{dx}\pi^{4x^2+x}$

(c) $\dfrac{d}{dx}\ln(f(x))$

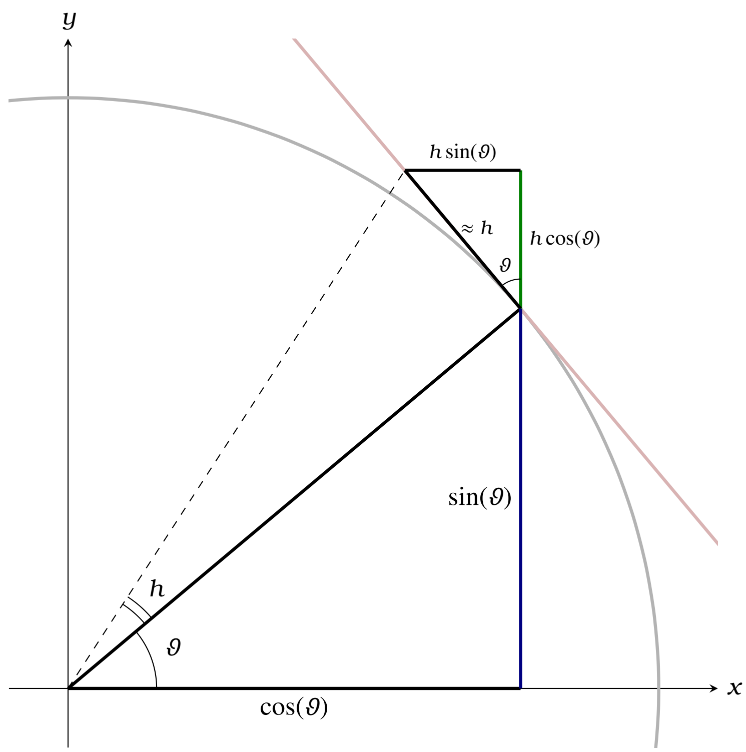
(d) $\dfrac{d}{dx}a^{f(x)}$

15 Sine and Cosine

Let's say "hi" to our old *frienemies* sine and cosine.

1) Explain how the following diagram strongly suggests that

$$\frac{d}{d\vartheta}\sin(\vartheta) = \cos(\vartheta).$$



2) Now use the same picture as before to explain why

$$\frac{d}{d\vartheta}\cos(\vartheta) = -\sin(\vartheta).$$

3) In calculus we use **radians**. Someone claims that

$$\frac{d}{d\vartheta}\sin(\vartheta^\circ) = \cos(\vartheta^\circ).$$

Prove them right or prove them wrong. Big hint: Sketch a plot and label your axes.

4) What is a *radian* and where did we use the fact that we were "thinking in radians" in the first two problems?

5) Compute
$$\frac{d}{d\vartheta}\tan(\vartheta).$$

6) Let $f(x) = \sin(x)$. Compute $f'(x), f''(x), f'''(x), f''''(x)$. What do you notice?

7) Give an example of a polynomial where
$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0.$$

8) Give an example of a polynomial where
$$f(0) = 0, \qquad f'(0) = 1, \qquad f''(0) = 0,$$
$$f^{(3)}(0) = -1, \qquad f^{(4)}(0) = 0, \qquad f^{(5)}(0) = 1.$$

9) Recall
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots.$$
Can you give a similar formula for $\sin(x)$? What about $\cos(x)$?

10) Compute:

(a) $\frac{d}{dx}\sin(3x)$

(b) $\frac{d}{dx}\cos(x^2)$

(c) $\frac{d}{dx}(\sin(x) - x\cos(x))$

(d) $\frac{d}{dx} - \ln(\cos(x))$

(e) $\frac{d}{dx}\frac{1}{\sin(x)}$

(f) $\frac{d}{dx}\sin^2(x)$

16 Derivative Rules

Let's get all of our derivative rules together in one place.

1) Write the constant rule and give an example.

2) Write the power rule and give an example.

3) Write the constant multiple rule and give an example.

4) Write the derivative rule for e^x and give an example.

5) Write the product rule and give an example.

6) Write the quotient rule and give an example.

7) Write the chain rule and give an example.

8) Write the derivative rule for a^x and give an example.

9) Write the derivative rule for $\ln(x)$ and give an example.

10) Write the derivative rules for sine and cosine and give examples.

11) Suppose that you could only test someone's knowledge of derivative rules with five questions. What questions would you ask?

17 Implicit Differentiation

We've been thinking about *explicit* functions. Now let's think about *implicit* functions.

1) Once, many years ago, I was bored in my math class and I wanted to plot a circle using my fancy graphing calculator. I looked up the formula for a circle and found

$$x^2 + y^2 = r^2.$$

What does this mean? Once I figured out the answer to that question, I realized that my fancy graphing calculator could only plot functions that started off as "$y =$." I eventually plotted a circle, how did I do it?

With implicit differentiation, we assume y is a function of x. This might be an impossible assumption but, as one finds with more advanced mathematics, the conditions for when this is a valid assumption are not too strict, so we can usually safely make this assumption.

2) If y is a function of x, tell me what is

$$\frac{d}{dx}y^2$$

Hint: Don't forget the chain rule!

3) If y is a function of x, tell me what is

$$\frac{d}{dx}e^y$$

Hint: Don't forget the chain rule!

4) If y is a function of x, tell me what is

$$\frac{d}{dx}\cos(y)$$

Hint: Don't forget the chain rule!

5) If y is a function of x, tell me what is

$$\frac{d}{dx}\ln(y)$$

Hint: Don't forget the chain rule!

6) If y is a function of x, differentiate both sides of

$$3x + 2y = 8$$

and solve for y'. Can you prove that your answer is correct?

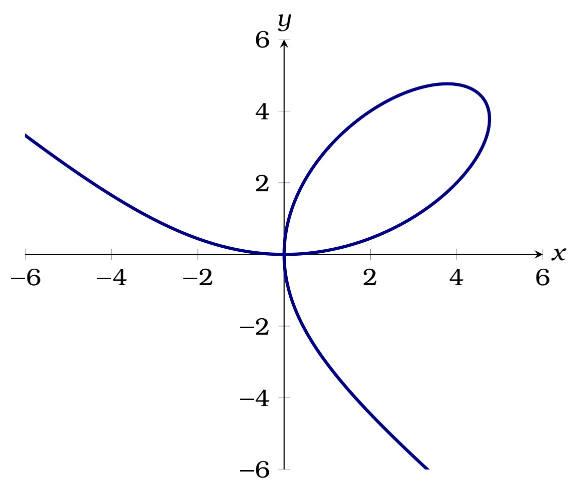
7) Use implicit differentiation to compute $\frac{dy}{dx}$ when

$$x^2 + y^2 = 1$$

Can you give evidence that your answer is correct?

8) Consider the curve defined by:

$$x^3 + y^3 = 9xy$$



(a) Compute $\frac{dy}{dx}$.

(b) Find the slope of the tangent line at $(4, 2)$.

18 Related Rates

Related rates problems are some of the hardest problems in a calculus class. One of the reasons that these questions are so hard is that they incorporate a lot of "middle grades" mathematics. Let's try our hand at some.

1) A 5 foot long ladder is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of 1 foot per second. How fast is the top of the ladder falling when the bottom of the ladder is 3 feet from the wall?

(a) Draw a picture of this situation.

(b) Write an equation.

(c) Differentiate the equation.

(d) Evaluate the equation at desired values and solve for the unknown.

Can you find examples of "middle grades" mathematics in this problem?

2) *Calculus Cal* has the following solution to the problem above:

> When we start off, we have a triangle with an $5'$ hypotenuse and two legs of $3'$ and $4'$. After the first second, we have a new triangle with a $5'$ hypotenuse and two legs of $4'$ and $3'$ because the horizontal leg became 1 foot longer in 1 second. Therefore the vertical became $1'$ shorter and hence was moving at $1'$ per second.

Is Cal correct? Why or why not?

3) A 10 foot long ladder is leaning against a wall. The bottom of the ladder is sliding away from the wall at a rate of 2 feet per second. How fast is the angle between the wall and the top of the ladder changing when the angle is $\pi/3$ radians?

(a) Draw a picture of this situation.

(b) Write an equation.

(c) Differentiate the equation.

(d) Evaluate the equation at desired values and solve for the unknown.

Can you find examples of "middle grades" mathematics in this problem?

4) A light is on the top of a 12 ft tall pole and a 6 ft tall person is walking away from the pole at a rate of 2 ft/sec. At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

(a) Draw a picture of this situation.

(b) Write an equation.

(c) Differentiate the equation.

(d) Evaluate the equation at desired values and solve for the unknown.

Can you find examples of "middle grades" mathematics in this problem?

5) Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep? Hint: The formula for the volume of a cone is $\frac{\pi}{3}r^2h$, where r is the radius of the cone and h is the height.

(a) Draw a picture of this situation.

(b) Write an equation.

(c) Differentiate the equation.

(d) Evaluate the equation at desired values and solve for the unknown.

Can you find examples of "middle grades" mathematics in this problem?

19 Maximums and Minimums

In this activity we are going to explore some maximums and minimums found in geometry using calculus.

1) What are the dimensions of the rectangle that has the largest area with a perimeter of 100 cm?

2) What are the dimensions of the rectangle that has the smallest perimeter with an area of 100 cm^2?

3) Explain why the following questions are silly:

(a) What are the dimensions of the rectangle that has the smallest area with a perimeter of 100 cm?

(b) What are the dimensions of the rectangle that has the largest perimeter with an area of 100 cm^2?

Now for some real problem solving.

The Main Question For a fixed perimeter what shape has the largest area? Give a complete explanation of why what you say is true is true.

4) When a question is too hard, answer some other question. Write down as many related questions as you can and try to solve them.

20 The Apothem

The *apothem* is something like a *radius* for regular polygons. Let's see if it can help us out.

1) The **apothem** of a regular n-gon is the line segment from the center of the polygon to the midpoint of one of its sides. Explain why

$$A = \frac{Pa}{2}$$

where A is the area of the regular n-gon, P is the perimeter of the regular n-gon, and a is the length of the apothem.

2) Explain why given a regular n-gon of perimeter 1,

$$a = \frac{1}{2n\tan(\pi/n)}.$$

3) Can you answer the main question?

> For a fixed perimeter what shape has the largest area? Give a complete explanation of why what you say is true is true.

If so do it. If not, what else needs to be explained?

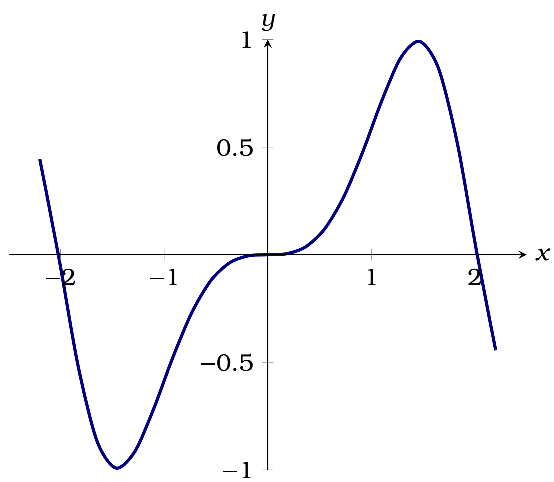
21 Integrals Compute Signed Area

The definite integral

$$\int_a^b f(x)\,dx$$

computes the signed area in the region $a \le x \le b$ between $f(x)$ and the x-axis. If the region is above the x-axis, then the area has positive sign. If the region is below the x-axis, then the area has negative sign. In this activity, use what you know about area to compute integrals.

1) Consider the plot of $f(x)$ below:



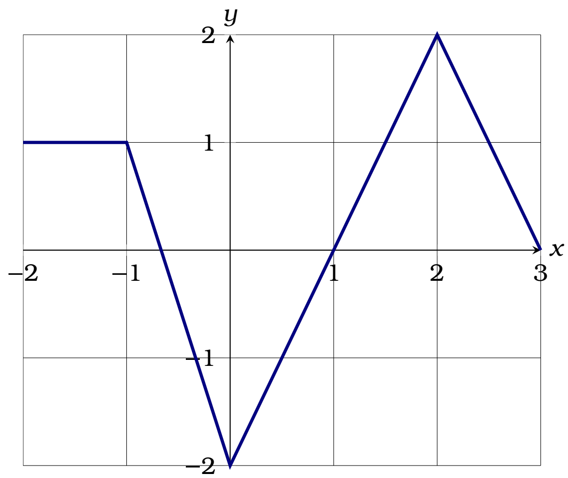
(a) Is $\displaystyle\int_1^2 f(x)\,dx$ positive, negative, or zero?

(b) Is $\displaystyle\int_{-1}^0 f(x)\,dx$ positive, negative, or zero?

(c) Is $\displaystyle\int_{-1}^1 f(x)\,dx$ positive, negative, or zero?

(d) Is $\displaystyle\int_{-1}^2 f(x)\,dx$ positive, negative, or zero?

2) Consider the plot of $g(x)$ shown below:



Compute:

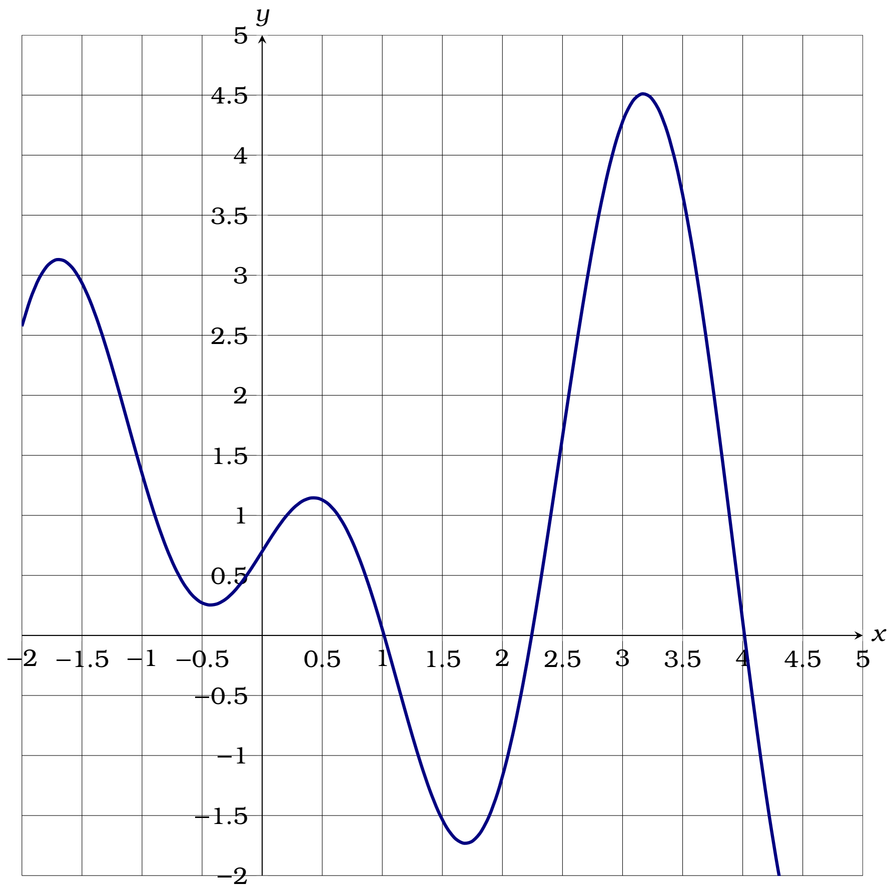
(a) $\displaystyle\int_{-2}^{-1} g(x)\,dx$

(b) $\displaystyle\int_{1}^{3} g(x)\,dx$

(c) $\displaystyle\int_{0}^{3} g(x)\,dx$

(d) $\displaystyle\int_{-1}^{3} g(x)\,dx$

3) Consider the following plot of $f(x)$:



Estimate

$$\int_{-1}^{4} f(x)\,dx$$

as accurately as possible.

22 Integrated Language

Let's get some practice using the notation of the integral, and the language of the integral. For the first three problems below, write an equivalent statement using function, derivative, and/or integral notation.

1) If $P(t)$ is the population of Everytown:

(a) The population of Everytown was 5.4 million in 2005.

(b) The population of Everytown was increasing at a rate of 110000 per year in 2005.

(c) The population of Everytown grew by 1000000 people between 2005 and 2008.

2) It's a beautiful day outside and Judy decides to take a bike trip. Let $s(t)$ be the position Judy is from home, east being negative and west being positive, after t minutes.

(a) At $t = 25$, Judy is biking along at 300 meters per minute.

(b) She comes to a hill. At $t = 27$, her velocity is dropping at a rate of 40 meters per minute each minute.

(c) At 31 minutes, Judy was 1400 meters ahead of where she was at 25 minutes. Side note: This may or may not mean the same as Judy traveled 1400 meters over that period of time. . . why?

3) If $T(t)$ = the temperature at OSU at time t hours after noon,

(a) The temperature at 11AM is 67 degrees.

(b) At 2PM the temperature was 75 degrees and rising.

(c) Between 5PM and 6PM, the temperature rose less than it did between 2PM and 5PM.

For the next two questions, translate the math into English.

4) Let $H(t)$ stand for the height of an airplane (in feet above the ground) as a function of time (in minutes).

(a) $H(0) = 10000$.

(b) $H'(0) = 0$.

(c) $H'(5) = -250$.

(d) $H(5) - H(0) = -2010$. Is there another way we can write this now?

5) Let $R(t)$ be Rachel's height at t years of age and $J(t)$ be Joe's height at t years of age.

(a) $R(16) < J(16)$.

(b) $R'(14) < J'(14)$.

(c) $\displaystyle\int_{12}^{15} R'(t)\,dt > \int_{12}^{15} J'(t)\,dt$.

6) Debbie the Dieter goes on every fad diet that she sees advertised on TV, thus wreaking havoc on her system. She is constantly losing and gaining weight. While her TV was being repaired one day, Deb, using her many charts of her weight through the years, figured out that her weight (in pounds) at time t years (her age) is given by the function $W(t)$. What would $\displaystyle\int_{30.5}^{37} W'(t)\,dt = 0$ mean in terms of her age and weight?

23 Winter Storm Warning

Let's dive a little deeper into working with integrals. Snow is starting to fall with a rate of $f'(t) = 1.5t - .25t^2 + .3$ inches per hour from noon until 4PM. There were already 5 inches on the ground when the storm started.

1) A natural question would be to ask, "how much snow fell during the storm?" What is the notation for this?

Because the rate is always changing, this is a difficult question to answer (Yet, we will eventually answer it!). Let's take what we know about constant rates and amounts and use that to help us answer our question.

2) Assume the rate stays the same as it was at the start of the storm. How much snow fell? Is this a realistic estimate?

3) Now assume the rate is the same as it is at the start for the first two hours, then changes to the rate at 2PM for the final two hours. How much snow fell? Is this a realistic estimate? Is it likely to be better or worse than the first estimate?

4) Now assume the rate stays constant by the hour (i.e., it only changes on the hour to its rate at those times of noon, 1PM, 2PM, and 3PM). How much snow fell?

5) Now do the same, but it changes on the half-hour.

6) What would we need to do to find the exact amount that fell?

7) Graph $f'(t)$ and interpret what you did in terms of the curve (i.e., geometrically) for all the parts.

8) What would you need to do geometrically to find the exact amount that fell?

9) If you knew the exact amount that fell, what would you need to do to determine how much snow is *on the ground*? Also, write the notation for the amount of snow on the ground.

24 The Accumulation Function

In this activity, we will study the accumulation function to help us "unlock" the code of the integral.

1) The accumulation function for a given function $f(x)$ is given by

$$F(x) = \int_a^x f(t)\,dt$$

where a is some fixed constant.

(a) Draw some pictures showing what $F(x)$ is doing.

(b) What is going on with the ts in the expression?

2) Let

$$F(x) = \int_a^x f(t)\,dt$$

where a is some fixed constant and draw a picture of $f(x)$ so that $F(x)$ is constant as x increases. Draw several more pictures. What must be true about $f(x)$ to ensure that $F(x)$ is constant? Explain your reasoning.

3) Let

$$F(x) = \int_a^x f(t)\,dt$$

where a is some fixed constant and draw a picture of $f(x)$ so that $F(x)$ is increasing as x increases. Draw several more pictures. What must be true about $f(x)$ to ensure that $F(x)$ is increasing? Explain your reasoning.

4) Let

$$F(x) = \int_a^x f(t)\,dt$$

where a is some fixed constant and draw a picture of $f(x)$ so that $F(x)$ is decreasing as x increases. Draw several more pictures. What must be true about $f(x)$ to ensure that $F(x)$ is decreasing? Explain your reasoning.

5) Summarize your work from above. What does it suggest?

25 The FUNdamental Theorem of Calculus!

You've made it! Here it is: Given a continuous function, if

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

where $F'(x) = f(x)$. In essence, the fundamental theorem gives a connection between area and (anti)derivatives. Moreover, it gives a recipe for computing areas.

1) To see if you get the idea of the computation, try your hand at these integrals:

(a) $\displaystyle\int_1^2 x^2\,dx$

(b) $\displaystyle\int_0^1 \cos(x)\,dx$

(c) $\displaystyle\int_4^5 \left(x^3 + e^x + \frac{1}{x^2}\right) dx$

2) To see if you get the concept, try your hand at these problems:

(a) Express the area of a 3×4 rectangle as an integral.

(b) Express the area of an isosceles right triangle with a short leg of length 4 as an integral.

3) Let $p(t)$ be the position of an object with respect to time. Let $v(t)$ be the velocity of an object with respect to time. Explain why

$$p(t) = p(a) + \int_a^t v(x)\,dx.$$

4) Let $p(t)$ be the position of an object with respect to time. Consider two points $(a, p(a))$ and $(b, p(b))$. Compare/contrast the point slope formula for a line

$$\ell(t) = m(t - a) + p(a)$$

with the fundamental theorem of calculus. Big hint: How is m related to $p'(t)$?

5) Try your hand at these integrals:

(a) Find the derivative of $F(x) = \displaystyle\int_1^x \left(t^2 - 3t\right) dt$. Explain your reasoning.

(b) Find the derivative of $F(x) = \displaystyle\int_{-4}^x e^{(t^2)}\,dt$. Explain your reasoning.

(c) Find the derivative of $F(x) = \displaystyle\int_0^x \tan(t^2)\,dt$. Explain your reasoning.

(d) Find the derivative of $F(x) = \displaystyle\int_1^{x^2} \left(t^2 - 3t\right) dt$. Explain your reasoning.

(e) Find the derivative of $F(x) = \displaystyle\int_{-4}^{\sin(x)} e^{\left(t^2\right)} dt$. Explain your reasoning.

(f) Find the derivative of $F(x) = \displaystyle\int_0^{e^x} \tan(t^2)\, dt$. Explain your reasoning.

6) Compute

$$\int_{-1}^{1} \frac{1}{x^2}\, dx.$$

Explain your reasoning. Hint: This is a trick question—the more naively you approach this question, the more likely that you will “solve” it correctly.

26 Templates for Computing Integrals

Unfortunately, we cannot tell you how to compute every integral. We advise that the mathematician view integrals as a sort of *puzzle*. A robust and simple way to compute integrals is guess-and-check.

How to Guess Integrals

(a) Make a guess for the antiderivative.

(b) Take the derivative of your guess.

(c) Note how the above derivative is different from the function whose antiderivative you want to find.

(d) Change your original guess by **multiplying** by constants or by **adding** in new functions.

1) Tell us how to integrate polynomials.

2) If the indefinite integral looks *something* like

$$\int \text{stuff}' \cdot (\text{stuff})^n \, dx \qquad \text{then guess} \qquad \text{stuff}^{n+1}$$

where $n \neq -1$. Try your hand at these integrals:

(a) $\displaystyle\int 2x(x^2+4)^5 \, dx$

(b) $\displaystyle\int \frac{(\ln(x))^4}{x} \, dx$

(c) $\displaystyle\int \frac{1}{\sqrt{2x+1}} \, dx$

(d) $\displaystyle\int \frac{x}{\sqrt{x^2+1}} \, dx$

(e) $\displaystyle\int x\sqrt{4-x^2} \, dx$

(f) $\displaystyle\int \frac{\sqrt{\ln(x)}}{x} \, dx$

3) If the indefinite integral looks *something* like

$$\int \text{junk} \cdot e^{\text{stuff}} \, dx \qquad \text{then guess} \qquad e^{\text{stuff}} \textbf{ or } \text{junk} \cdot e^{\text{stuff}}.$$

Try your hand at these integrals:

(a) $\displaystyle\int 3x^2 e^{x^3-1} \, dx$

(b) $\displaystyle\int x e^{3(x^2)} \, dx$

(c) $\displaystyle\int 2x e^{-(x^2)} \, dx$

(d) $\displaystyle\int \frac{8x}{e^{(x^2)}} \, dx$

(e) $\displaystyle\int x e^{5x} \, dx$

(f) $\displaystyle\int x e^{-x/2} \, dx$

4) If the indefinite integral looks *something* like

$$\int \frac{\text{stuff}'}{\text{stuff}}\,dx \qquad \text{then guess} \qquad \ln(\text{stuff}).$$

Try your hand at these integrals:

(a) $\displaystyle\int \frac{1}{2x}\,dx$

(b) $\displaystyle\int \frac{x^4}{x^5+1}\,dx$

(c) $\displaystyle\int \frac{x^2}{3-x^3}\,dx$

(d) $\displaystyle\int \frac{1}{x\ln(x)}\,dx$

(e) $\displaystyle\int \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}\,dx$

(f) $\displaystyle\int \frac{1}{x\ln(x^2)}\,dx$

5) If the indefinite integral looks *something* like

$$\int \text{junk}\cdot\sin(\text{stuff})\,dx \qquad \text{then guess} \qquad \cos(\text{stuff})\ \textbf{or}\ \text{junk}\cdot\cos(\text{stuff}),$$

likewise if you have

$$\int \text{junk}\cdot\cos(\text{stuff})\,dx \qquad \text{then guess} \qquad \sin(\text{stuff})\ \textbf{or}\ \text{junk}\cdot\sin(\text{stuff}),$$

Try your hand at these integrals:

(a) $\displaystyle\int 5x^4\sin(x^5+3)\,dx$

(b) $\displaystyle\int x\cos(-2x^2)\,dx$

(c) $\displaystyle\int x\sin(5x^2)\,dx$

(d) $\displaystyle\int 8x\cos(x^2)\,dx$

(e) $\displaystyle\int 6e^{3x}\sin(e^{3x})\,dx$

(f) $\displaystyle\int \frac{\cos(\ln(x))}{x}\,dx$

27 Computing Volumes

In this activity we will compute volumes of objects.

1) Remind me how to do these basic, but essential problems:

(a) Express the area of a 3×4 rectangle as an integral.

(b) Express the area of an isosceles right triangle with a short leg of length 4 as an integral.

2) Express the area of a circle as an integral.

3) What's the formula for the volume of a cylinder? Can you express this as an integral?

4) What's the formula for the volume of a box? Can you express this as an integral?

5) What's the formula for the volume of a sphere? Can you express this as an integral?

6) What's the formula for the volume of a cone? Can you express this as an integral?

7) What's the formula for the volume of a pyramid with a square base? Can you express this as an integral?