Unofficial
Math 501 Problems

The following document consists of problems collected from previous classes and comprehensive exams given at the University of Illinois at Urbana-Champaign. In the exercises below, all rings contain identity and all modules are unitary unless otherwise stated. Please report typos, suggestions, gripes, complaints, and criticisms to: `snapp@math.uiuc.edu`

1) Let R be a commutative ring with identity and let I and J be ideals of R. If the R-modules R/I and R/J are isomorphic, prove $I = J$. Note, we really do mean *equals*!

2) Let R and S be rings where we assume that R has an identity element. If M is any (R, S)-bimodule, prove that

$$\mathrm{Hom}_R({}_RR, M) \overset{S}{\simeq} M.$$

3) Let $0 \to A \to B \to C \to 0$ be an exact sequence of R-modules where R is a ring with identity. If A and C are projective, show that B is projective.

4) Let R be a ring with identity.

(a) Prove that every finitely generated right R-module is noetherian if and only if R is a right noetherian ring.

(b) Give an example of a finitely generated module which is not noetherian.

5) Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a short exact sequence of R-modules where R is any ring. Assume there exists an R-module homomorphism $\delta : B \to A$ such that $\delta\alpha = \mathrm{id}_A$. Prove that there exists an R-module homomorphism $\gamma : C \to B$ such that $\beta\gamma = \mathrm{id}_C$.

6) If R is any ring and ${}_RM$ is an R-module, prove that the functor $- \otimes_R M$ is right exact.

7) Let S and R be rings, with S a subring of R containing 1_R.

(a) Use tensor products to construct a covariant functor

$$\mathscr{F} : \mathbf{Mod}_S \to \mathbf{Mod}_R\,.$$

(b) If M_S is a flat module, prove that M_S is S-isomorphic with a submodule of $\mathscr{F}(M_S)$.

8) Let R be any ring and let M_R, N_R be flat R-modules. Prove that $M \oplus N$ is also flat.

9) Let R be a domain and let M_R be a flat R-module. Prove that M is R-torsion-free.

10) Give an example of a commutative ring R with identity and an R-module M such that the R-torsion elements of M do not form a submodule of M.

11) Let $(C, \mu_\lambda)_{\lambda \in \Lambda}$ and $(C', \mu'_\lambda)_{\lambda \in \Lambda}$ be coproducts of set of objects $\{A_\lambda : \lambda \in \Lambda\}$ in some category $\mathbf{C}$. Prove that there is an equivalence $\beta \in \mathbf{C}(C, C')$ such that $\mu'_\lambda = \beta\mu_\lambda$ for all λ in Λ.

12) Prove that the direct sum $\left(\bigoplus_{\lambda \in \Lambda} M_\lambda, \mu_\lambda\right)$ is the coproduct in the category ${}_R\mathbf{Mod}$ where $\mu_\lambda : M_\lambda \to \bigoplus_{\lambda \in \Lambda} M_\lambda$ via

$$(\mu_\lambda(a))_\nu = \begin{cases} a & \text{if } \nu = \lambda \\ 0_{M_\nu} & \text{if } \nu \neq \lambda. \end{cases}$$

13) Prove that $\mathrm{Hom}_R(-, M)$ is a contravariant functor from ${}_R\mathbf{Mod}$ to $\mathbf{Ab}$.

14) Let R and S be rings and $\alpha : R \to S$ a ring homomorphism. Show that there is a corresponding covariant functor $\mathscr{F}_\alpha : {}_S\mathbf{Mod} \to {}_R\mathbf{Mod}$.

15) Let L, M, N be submodules of an R-module and assume that $N \subseteq M$. Prove that $L + M/L + N$ is R-isomorphic with a quotient of M/N and $L \cap M/L \cap N$ is R-isomorphic with a submodule of M/N.

16) Let (F, μ) and (F', μ') be free on a set X in ${}_R\mathbf{Mod}$. Prove that $F \overset{R}{\simeq} F'$.

17) In the following diagram the squares are commutative, the rows are exact, and λ and ν are isomorphisms of modules. All maps are module homomorphisms.

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\nu} & & \\
0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & &
\end{array}$$

Prove:

(a) α is injective.

(b) β' is surjective.

(c) μ is an isomorphism.

18) Let R be a ring with identity. Prove that every R-module is projective if and only if R is a semisimple ring.

19) Let R be an arbitrary ring and let M be an R-module with submodules $M_1, M_2, \ldots, M_k$ such that each M/M_i is a simple module and

$$M_1 \cap M_2 \cap \cdots \cap M_k = \{0\}.$$

Prove that M is a semisimple module.

20) Let G be a group and F a field. Define a map $\theta : FG \to F$ by

$$\theta\left(\sum_{g \in G} f_g g\right) = \sum_{g \in G} f_g.$$

(a) Prove that θ is a ring homomorphism.

(b) Prove that $\mathrm{Ker}(\theta) = I_G$ where I_G is generated as an F-vector space by the set $\{g - 1 : g \in G\}$.

(c) Assume that G has finite order not divisible by $\mathrm{char}(F)$. Prove

$$FG = J_0 \oplus J_1$$

where J_0, J_1 are ideals of FG and $I_G J_0 = 0$, $I_G J_1 = J_1$.

(d) Show that $\dim_F(J_0) = 1$ and find a generator for J_0.

21) Let G be a nontrivial finite p-group and let F be a field of characteristic p, a prime. Let I_G be as in the above problem. Prove I_G is not a direct summand of the module ${}_{FG}(FG)$. What does this tell you about the ring FG?

22) Let R be a PID and denote by F the field of fractions of R. Regard F as an R-module in the obvious way.

(a) If X is a finitely generated R-submodule of F, show that $X = Rf$ for some $f \in F$.

(b) If F is a finitely generated R-module, prove $R = F$.

23) Let M be a finitely generated module over a PID R.

(a) If M is a torsion module prove that M is artinian.

(b) Assuming that R is not a field, show that M has a composition series if and only if it is a torsion module.

24) Describe all isomorphism types of abelian groups of order 2250, and give the invariant factors for each type.

25) Let M and N be finite abelian p-groups and let their invariant factors be

$$\underbrace{p,\ldots,p}_{m_1},\underbrace{p^2,\ldots,p^2}_{m_2},\ldots,\underbrace{p^r,\ldots,p^r}_{m_r},$$

and

$$\underbrace{p,\ldots,p}_{n_1},\underbrace{p^2,\ldots,p^2}_{n_2},\ldots,\underbrace{p^r,\ldots,p^r}_{n_r},$$

respectively, where $m_i, n_i \geqslant 0$. What are the invariant factors of $M \otimes_{\mathbb{Z}} N$? Justify your answer.

26) Prove that an artinian integral domain is field. [Use the DCC directly, no major theorems necessary.]

27) Let R be a ring and M a left R-module. If $r \in R$ and $a \in M$, define $a \cdot r$ to be $r \cdot a$.

(a) Prove that the assignment $(a, r) \mapsto a \cdot r$ makes M into a right module over R^{Opp}.

(b) Define a covariant functor $\mathrm{Opp} : {}_R\mathbf{Mod} \to \mathbf{Mod}_{R^{\mathrm{Opp}}}$.

28) Give an example of a nonzero abelian group A such that $A \otimes_{\mathbb{Z}} A = 0$.

29) Let M_λ be a right R-module for $\lambda \in \Lambda$ and let N be a left R-module, where R is any ring. Prove that

$$\left(\bigoplus_{\lambda\in\Lambda} M_\lambda\right) \otimes_R N \simeq \bigoplus_{\lambda\in\Lambda} (M_\lambda \otimes_R N)\,.$$

30) Let $A = \mathbb{Z}\oplus\mathbb{Z}_2\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3$. Let $A^{\otimes n}$ denote the nth tensor power, $A\otimes_{\mathbb{Z}}\cdots\otimes_{\mathbb{Z}}A$, ($n$ factors). Prove that

$$A^{\otimes n} \simeq \mathbb{Z} \oplus \mathbb{Z}_2^{2^n-1} \oplus \mathbb{Z}_3^{3^n-1}, \quad (n > 0),$$

where $\mathbb{Z}_k^m = \mathbb{Z}_k \oplus \cdots \oplus \mathbb{Z}_k$, ($m$ factors).

31) Let R be a commutative ring and let M be an R-module. The **exterior square** $M \wedge_R M$ is an abelian group generated by elements $a \wedge b$, $(a, b \in M)$, with all the properties of the tensor product *plus* $a \wedge a = 0$, for all $a \in M$.

(a) Explain how to define $M \wedge_R M$.

(b) Prove that $a \wedge b = -(b \wedge a)$, where $a, b \in M$.

(c) A mapping $\alpha : M \times M \to A$, where A is an abelian group, is called **alternating middle linear** if it is middle linear and also $\alpha((a,a)) = 0$, for all $a \in M$. Construct a "canonical" alternating middle linear mapping

$$\theta : M \times M \to M \wedge_R M.$$

(d) Use alternating middle linear maps to characterize $M \wedge_R M$ up to isomorphism by a mapping property.

32) Let R be an arbitrary ring with identity.

(a) Prove that a left R-module M is simple if and only if $M \simeq R/I$ for some maximal left ideal I.

(b) Use Zorn's Lemma to prove that simple left R-modules exist.

33) Let

$$A = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

be a matrix with entries in a field k of characteristic $p \geqslant 5$.

(a) Prove that A is similar to a Jordan block J.

(b) Find the order of A in the group $GL_4(k)$.

34) Let R be a commutative ring and let ${}_R\mathbf{Mod}$ be the category of all R-modules.

(a) If M is an R-module, prove that $R \otimes_R M$ is also an R-module.

(b) Prove that $M \simeq R \otimes_R M$.

(c) Prove that $\mathscr{I}$, the identity functor on ${}_R\mathbf{Mod}$, is naturally equivalent to the functor $R \otimes_R -$.

35) Define semisimple ring, and prove that a commutative ring R is semisimple if and only if R is isomorphic to a finite direct product of fields.

36) Let $S^2 = \{(a,b,c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that

$$I = \{f(x,y,z) \in \mathbb{R}[x,y,z] : f(a,b,c) = 0 \text{ for all } (a,b,c) \in S^2\}$$

is a finitely generated ideal in $\mathbb{R}[x,y,z]$.

37) Prove that a UFD satisfies the ascending chain condition for principal ideals, but that the ascending chain condition on all ideals need not hold.

38) Let R be a finitely generated commutative ring with identity. Prove that R is isomorphic with some quotient ring of a polynomial ring $\mathbb{Z}[t_1, \ldots, t_n]$. Explain why R is noetherian.

39) Compute $(\mathbb{Z}/14\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/15\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/12\mathbb{Z})$.

40) A module is called **semisimple** if it is a direct sum of simple modules.

(a) If a module is generated by simple submodules, prove that it is semisimple.

(b) Let N be a submodule of a semisimple module M. Prove that N is a direct summand of M.

(c) Prove that a submodule of a semisimple module is semisimple.

41) Prove that a noetherian module is a direct sum of finitely many (directly) indecomposable modules.

42) Let M be a module with a submodule N. If N and M/N are artinian modules, show that M is artinian.

43) Let R be a commutative ring with identity.

(a) Define what it means to be a projective R-module.

(b) Give three more characterizations of a projective R-modules.

(c) Let P and Q be projective modules over a commutative ring R with identity. Prove that $P \otimes_R Q$ is a projective R-module.

44) Let M be a module and let θ be a module endomorphism of M. Assume that r and s are positive integers which are minimal subject to $\mathrm{Im}(\theta^r) = \mathrm{Im}(\theta^{r+1})$ and $\mathrm{Ker}(\theta^s) = \mathrm{Ker}(\theta^{s+1})$. Prove that $r = s$ and that $M = \mathrm{Im}(\theta^r) \oplus \mathrm{Ker}(\theta^r)$.

45) Let G be a finite abelian group and let F be a finite field of order q where q is relatively prime to $|G|$.

(a) Prove that the group algebra FG is isomorphic as an F-algebra to a direct sum $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ where $F_i = GF(q^{e_i})$ and $e_1 + e_2 + \cdots + e_k = |G|$.

(b) If $|G| = 3$ and $|F| = 5$, find the correct values of the e_i's.

46) Prove that a finitely generated torsion module over a PID R is artinian. [Hint: Show that R/p^iR is artinian where $p \in R$ is irreducible.]

47) If P is an R-module, define

$$\operatorname{tr}(P) = \sum_{\alpha} \alpha(P)$$

where $\alpha \in \operatorname{Hom}_R(P, R)$.

(a) Show that if P is free, then $\operatorname{tr}(P) \simeq R$.

(b) Show that if P is projective, then $\operatorname{tr}(P)$ is a 2-sided ideal of R.

48) Let $A \leq B$ be R-modules. Call A **large** in B if the only submodule S of B such that $A \cap S = 0$ is $S = 0$.

(a) Show that given $A \leq B$ there is an intermediate submodule S that is maximal among the submodules that contain A as a large submodule.

(b) In the case where R is an integral domain, $A = R$, and B is the field of fractions of R, what is this intermediate S?

49) Show that if R is a commutative noetherian ring, then so are $R[x]$ and $S^{-1}R$, where S is a multiplicative subset of R. Do not use Hilbert's Basis Theorem.

50) Let A be a finitely generated module over a PID R. Show if $A \otimes_R A = 0$, then $A = 0$. Show that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$.

51) Let A be an R-module. A is called **divisible** if given $a \in A$ and $r \in R$ with r not a zero divisor in R, there exists $a' \in A$ such that $ra' = a$. Show:

(a) Every injective module is divisible.

(b) Every divisible module over a PID is injective.

52) Let A be an $n \times n$ matrix over an algebraically closed field k. Prove there exist matrices D and N such that:

(a) $A = D + N$.

(b) $DN = ND$.

(c) D is similar to a diagonal matrix.

(d) $N^n = 0$.

53) Let G be a finite group and let F be an algebraically closed field whose characteristic does not divide $|G|$. Prove:

(a) $FG \simeq \operatorname{Mat}_{n_1}(F) \times \cdots \times \operatorname{Mat}_{n_t}(F)$ where $\operatorname{Mat}_n(F)$ is the ring of all $n \times n$ matrices over F.

(b) $\dim_{\mathbb{C}}(Z(\mathbb{C}G)) = t$, where t is the number of matrix summands in the decomposition of $\mathbb{C}G$. $Z(R)$ denotes the center of the ring R.

54) Let M be a module over a ring with identity and assume that M satisfies both the ascending and descending chain condition on submodules.

(a) If α is a module endomorphism of M, prove that $M = \mathrm{Ker}(\alpha^i) \oplus \mathrm{Im}(\alpha^i)$.

(b) Show that there are submodules L and N such that $M = L \oplus N$ where $\alpha(L) = L$ and $\alpha|_L$ is an automorphism, while $\alpha(N) \subseteq N$ and $\alpha|_N$ is nilpotent.

55) Let M and N be nonzero R-modules, where R is a commutative ring with identity.

(a) A mapping $\varphi : \mathrm{Hom}_R(M, R) \otimes_R N \to \mathrm{Hom}_R(M, N)$ is defined by

$$\varphi(f \otimes y) : x \mapsto f(x)y,$$

where $f \in \mathrm{Hom}_R(M, R)$, $y \in N$, and $x \in M$. Show that φ is a well-defined R-module homomorphism.

(b) Given that M is a finitely generated free R-module, prove that φ is an isomorphism.

(c) Deduce from (b) that φ will also be an isomorphism if M is a finitely generated projective module.

56) Let M be a module over R, a commutative ring with identity. Then M is called ***R*-flat** if, for every ideal I of R, the natural map $I \otimes_R M \to R \otimes_R M = M$ is injective. Also M is **faithfully flat** if it is flat and if $N \otimes_R M = 0$ always implies that $N = 0$.

(a) Show that $\mathbb{Q}$ is $\mathbb{Z}$-flat.

(b) Is $\mathbb{Q}$ faithfully flat?

(c) Prove that a finitely generated free R-module is faithfully R-flat.

57) Let A be an abelian group.

(a) Suppose A is generated by n elements subject to r defining relations, where $n > r$. Prove that there are at least $n - r$ infinite summands in a direct sum decomposition of A with cyclic summands.

(b) Let A be generated by x_1, x_2, x_3, x_4 subject to the following defining relations:

$$\left\{ \begin{array}{rcrcrcrcl} 3x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 0 \\ x_1 & & & + & 2x_3 & + & x_4 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & = & 0 \end{array} \right.$$

Determine the structure of the group A.

58) Let A be an $n \times n$ matrix over $\mathbb{Q}$. Describe in detail, but without proofs, the rational canonical form of A.

59) Let A be an $n \times n$ matrix over $\mathbb{C}$. Describe in detail, but without proofs, the Jordon canonical form of A.

60) Let A be an $n \times n$ matrix over $\mathbb{Q}[x]$. Describe in detail, but without proofs, the Smith normal form of A.

61) Suppose that A is a 6×6 matrix with rational entries satisfying

$$(A^3 + A + 1)^2 = 0.$$

Find all possible rational canonical forms of A and indicate the invariant factors.

62) In the following diagram of R-modules and R-module homomorphisms the rows are exact and the squares are commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ \downarrow \lambda & & \downarrow \mu & & \downarrow \nu & & \\ A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0 \end{array}$$

(a) If λ and ν are surjective, prove that μ is also surjective.

(b) Let M be an R-module. Form a corresponding diagram for $\mathrm{Hom}_R(A, M)$, $\mathrm{Hom}_R(B, M)$ etc., and prove that its squares commute.

63) You are given an abelian group A with three generators subject to the relations:

$$7a + 5b + 2c = 0$$
$$3a + 3b = 0$$
$$13a + 11b + 2c = 0$$

By the structure theorem, A is a direct sum of cyclic groups. Find such a decomposition.

64) Prove the following statements, using only the definitions of projectivity and injectivity:

(a) $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

(b) $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module.

65) Let I be a principal ideal of a domain R. Prove that $I \otimes_R I \overset{R}{\simeq} I^2$ and deduce that $I \otimes_R I$ is R-torsion free.

66) Let $R = \mathbb{Z}[t]$ and $I = (2, t)$.

(a) Use the mapping property to construct an R-module homomorphism $\theta : I \otimes_R I \to \mathbb{Z}/2\mathbb{Z}$ such that $\theta(p \otimes q) = \frac{1}{2}p(0)q'(0) + 2\mathbb{Z}$. Note that $p(0)$ must be even and q' is the usual derivative. Also $\mathbb{Z}/2\mathbb{Z}$ is an R-module via $R \to \mathbb{Z}$.

(b) Prove that $u = t \otimes 2 - 2 \otimes t$ in $I \otimes_R I$ is nonzero by using (a).

(c) Prove that u in (b) is an R-torsion element.

67) Let R be a commutative noetherian ring with identity, M a finitely generated R-module, and let $f : M \to M$ be a surjective R-linear map. Show that f is an isomorphism.

68) Let $T : V \to V$ be a linear transformation of a vector space V over a the field F.

(a) Given T, explain how V becomes a $F[X]$-module, where X is an indeterminate.

(b) Suppose the dimension of V is finite. Define the order of $v \in V$ by

$$O(v) = \{g \in F[X] : g(T)v = 0\}.$$

Prove that $O(v)$ is a nonzero ideal of $F[X]$.

(c) Define $T : V \to V$ on a basis $e_1, \ldots, e_n$, $(n \geqslant 2)$ of V by $T(e_1) = ae_1$ and $T(e_{i+1} + e_i) = ae_{i+1} + e_i$ for $a \in F$ and $i = 1, \ldots, n-1$. Determine explicitly with proof the ideals $O(e_i)$, $i = 1, \ldots, n$.

69) Let R be a commutative ring with identity. Consider an exact sequence of finitely generated free R-modules

$$0 \to F \xrightarrow{\alpha} G \xrightarrow{\beta} H \to 0.$$

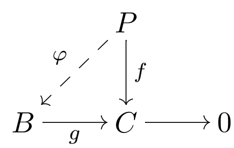
(a) Show that there exists an R-linear map $\gamma : H \to G$ such that $\beta\gamma = \mathrm{id}_H$.

(b) Show that $F \oplus H \simeq \alpha(F) \oplus \gamma(H) = G$.

70) Let $\alpha : M \to M$ be a surjective homomorphism of modules. Show that α need not be injective if M is artinian.

71) Let R be a commutative ring with identity. For R-modules A, B, C, F, P, Q, prove that the following are equivalent:

(a) Given any right-exact sequence $B \to C \to 0$ and a homomorphism $f : P \to C$, there exists $\varphi : P \to B$ such that the diagram below commutes.



(b) Every short exact sequence $0 \to A \to B \to P \to 0$ is split exact.

(c) There is a free module F such that $F \simeq P \oplus Q$ for some R-module Q.

72) Let F be the free abelian group with basis x_1, x_2, x_3, x_4. Define elements

$$\begin{aligned} y_1 &= 2x_1 + 2x_2 + 4x_3 + 6x_4, \\ y_2 &= 2x_1 + x_2 - x_4, \\ y_3 &= 4x_1 + 2x_2 + 4x_3, \end{aligned}$$

each in F, and let S be the subgroup generated by y_1, y_2, y_3. Determine the structure of the finitely generated abelian group F/S.

73) Compute the following tensor product: $(\mathbb{Z} \oplus \mathbb{Q}) \otimes_{\mathbb{Z}} (\mathbb{Z}_2 \oplus \mathbb{Z}_4) \otimes_{\mathbb{Z}} (\mathbb{Z}_3 \oplus \mathbb{Z}_6)$.

74) Let G be a group of prime order p. Prove that $\mathbb{Q}G = I_1 \oplus I_2$ where I_1 and I_2 are ideals such that $I_1 \overset{\mathbb{Q}G}{\simeq} \mathbb{Q}$ and $I_2 \overset{\mathbb{Q}G}{\simeq} \mathbb{Q}(e^{\frac{2\pi i}{p}})$.

75) Let M be a finitely generated module over a PID R. Prove that there exist irreducible elements $p_1, \ldots, p_k$ of R and a positive integer ℓ such that

$$\bigcap_{i=1}^{k} p_i^{\ell} M$$

is a torsion-free module.

76) Use rational canonical form to find all similarity types of 4×4 rational matrices A such that $A^2 + A^4 = 0$.

77) Let $A = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R} \oplus \mathbb{Z}_6$ and $B = \mathbb{Q} \oplus \mathbb{Z}_4$. Compute the tensor product $A \otimes_{\mathbb{Z}} B$ in simplest terms.

78) Let M be a finitely generated module over a commutative noetherian ring R.

(a) Prove that M is a noetherian module.

(b) If N is a submodule such that $M \stackrel{R}{\simeq} M/N$, show that $N = 0$.

79) Let A be the $n \times n$ $(n > 1)$ matrix over $\mathbb{Q}$ all of whose entries equal 1

(a) Prove that the minimum polynomial of A is $x(x - n)$.

(b) Find the characteristic polynomial of A.

(c) Show that A is diagonalizable and find its diagonal form.

80) Let G be a cyclic group of order 6. Explain why the group algebra $\mathbb{Q}G$ is a direct sum of fields and identify the fields.

81) Let M be a finitely generated abelian group. State the Structure Theorem for M in terms of invariant factors.

82) Prove that every nonzero vector space has a basis.

83) Let V be an n-dimensional vector space over the field of p elements. If $v_1, \ldots, v_r$ in V are linearly independent, prove that there are $p^n - p^r$ vectors $v \in V$ such that $v_1, \ldots, v_r, v$ are linearly independent.

Let $GL_n(p)$ be the group of all nonsingular $n \times n$ matrices over the field of p elements. Deduce from above that $GL_n(p)$ has order $(p^n - 1)(p^n - p) \ldots (p^n - p^{n-1})$.

84) Let R and S be rings and let A, B, C be a right R-module, an (R, S)-bimodule, and a right S-module respectively. Prove that there is an isomorphism of abelian groups

$$\mathrm{Hom}_S(A \otimes_R B, C) \simeq \mathrm{Hom}_R(A, \mathrm{Hom}_S(B, C)).$$

85) Let R be a ring with identity.

(a) Let M_R and ${}_RN$ be a right and a left R-module and let M_0, N_0 be submodules of M, N respectively. Prove that

$$(M/M_0) \otimes_R (N/N_0) \simeq (M \otimes_R N)/S$$

where S is the subgroup generated by all $a \otimes b$ where $a \in M_0$ or $b \in N_0$.

(b) If I is a left ideal of R, prove that $M \otimes_R (R/I) \simeq M/MI$.

86) Define what it means to be an injective module and show the following

(a) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module.

(b) The module $\mathbb{Z}/n\mathbb{Z}$ is injective over itself as a ring, where $n > 1$.

87) Suppose V is a finite dimensional vector space over $\mathbb{C}$. Show that the ring $\mathrm{Hom}_{\mathbb{C}}(V, V)$ is simple.

88) Prove that the following are equivalent for a commutative ring R:

(a) R satisfies the ascending chain condition on ideals.

(b) Each ideal of R is finitely generated.

(c) Every set of ideals in R contains a maximal element with respect to set theoretic inclusion.

89) Suppose that R is a PID and that F is a finitely generated free R-module. Show that any R-submodule M of F is a finitely generated free R-module.

90) Let R be a PID and Q its field of quotients. Assume that R is not equal to Q, so $R \subsetneq Q$ and Q may be viewed as an R-module. Prove that Q is not a finitely generated R-module.

91) Let R be a ring with identity and $e \in R$ be an idempotent element—that is $e = e^2$. Prove that the left ideal Re is a projective module.

92) Let V be the $\mathbb{Q}$-vector space of 3-tuples of column vectors. Let T be the linear transformation defined on the natural basis by

$$T : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}.$$

Then V becomes a module over the polynomial ring $\mathbb{Q}[x]$ when (left) multiplication by x has the same effect as T: $xv = T(v)$.

(a) Describe the decomposition of V into a direct sum of indecomposable $\mathbb{Q}[x]$-submodules.

(b) Find all the $\mathbb{Q}[x]$ submodules of V. [Hint: There are 6 of them.]

93) Let R be a PID and $F = R^n$, the free module consisting of all n-tuples over R. Let $y = (a_1, \ldots, a_n)$ be an element of F. Show that there is some basis $x_1, \ldots, x_n$ of F over R with $x_1 = y$ if and only if the ideal $Ra_1 + \cdots + Ra_n$ equals R.

94) Let $y = (6, 5, 2)$ be an element of $\mathbb{Z}^3$. Find a basis of $\mathbb{Z}^3$ over $\mathbb{Z}$ that includes y.

95) Let A be a finite abelian group. State the Structure Theorem for M in terms of elementary divisors.

(a) State a condition on the elementary divisors of A that hold if and only if A is a cyclic group.

(b) Prove that the multiplicative group of nonzero elements of a finite field is a cyclic group.

96) Let R be a ring with identity. Prove that every left R-module is projective if and only if every left R-module is injective.

97) Let R be a ring with identity and M a left R-module. For any left R-modules A and B and R-homomorphism $f : A \to B$, there is a homomorphism of abelian groups

$$f^* : \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M)$$

defined by $f^*(h) = h \circ f$. If

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is an exact sequence of R-modules and R-module homomorphisms, prove

$$0 \to \mathrm{Hom}_R(C, M) \xrightarrow{g^*} \mathrm{Hom}_R(B, M) \xrightarrow{f^*} \mathrm{Hom}_R(A, M)$$

is an exact sequence of abelian groups.

98) Let $F = \mathbb{Z} \oplus \mathbb{Z}$ be the free abelian (additive) group on two generators and let R be the subgroup of F generated by

$$r_1 = (4, 6), \quad r_2 = (8, -9), \quad r_3 = (4, -36).$$

Express the quotient F/R as a direct sum of cyclic groups each of which is either infinite or finite of prime power order.

99) Suppose A is a principal ideal domain and that F is a finitely generated free A-module. Show that any A-submodule M of F is a finitely generated free A-module.

100) Let G be a cyclic group of order 5. Explain why the group algebra $\mathbb{Q}G$ is a direct sum of fields and identify the fields.

101) Prove that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$ and deduce that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \neq 0$.

102) Prove that $\mathrm{Hom}_R(M, \prod_{\lambda\in\Lambda} A_\lambda) \simeq \prod_{\lambda\in\Lambda} \mathrm{Hom}_R(M, A_\lambda)$ where A_λ, M are left R-modules.

103) Given rings R, S, T and bimodules ${}_SA_R, {}_TB_R$, say what the bimodule structure of $\mathrm{Hom}(A, B)$ is and prove your assertion.

104) Let R be a commutative ring with identity. Prove that if every R-module is free, then R must be a field. [Hint: it's enough to prove that R has no proper non-zero ideals.]

105) Let V be an n-dimensional vector space over a field K, and let R be the ring of all linear transformations on V. Prove that R has the invariant basis property. [Hint: $\dim_K(R) = n^2$.]

106) Let R be a ring with identity and let M be a direct summand of an injective R-module J. Prove that M is an injective R-module.

107) Let F be a free R-module where R is a ring with identity. Suppose that S is a finitely generated submodule of F. Prove that there exists a surjective R-module homomorphism $f : F \to F_1$ where F_1 is a finitely generated R-module and $\mathrm{Ker}(f) \cap S = 0$.

108) Suppose $P_1 \to Q_1 \to M_1 \to 0$ and $P_2 \to Q_2 \to M_2 \to 0$ are exact sequences of R-modules, where P_1 and Q_1 are projective. Suppose also that there is a R-module homomorphism $\mu : M_1 \to M_2$. Prove that there are homomorphisms $\alpha : P_1 \to P_2$ and $\beta : Q_1 \to Q_2$ making a commutative diagram. [Hint: Use the definition of projective modules.]

109) If F is a field of prime characteristic p and G is a non-trivial finite p-group, prove that the group algebra FG is not semisimple. [Hint: the center of G is non-trivial.]

110) Let F be the free abelian group with basis $\{x_1, x_2, \ldots, x_n\}$, and let $A = [a_{ij}]$ be any $m \times n$ integer matrix. Define m elements y_i of F by $y_i = \sum_{j=1}^{n} a_{ij} x_j$, for $i = 1, 2, \ldots, m$. Let $K = (y_1, y_2, \ldots, y_m)$ and put $G = F/K$, a finitely generated abelian group. The following procedure is an algorithm to determine the structure of G.

(a) Show that applying a ROW operation to A does not change K.

(b) Show that applying a COLUMN operation to A changes the basis but not K. [Note: Row and column operations for integer matrices are defined as in linear algebra, but rows and columns can only be multiplied by ± 1.]

(c) Show that by a suitable sequence of row and column operations A can be transformed into a diagonal matrix $N = \mathrm{diag}(d_1, d_2, \ldots, d_k, 0, \ldots, 0)$ where the d_i are positive integers.

(d) Prove that $G \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ where the number of $\mathbb{Z}$'s is $n - k$.

(e) Show that G is finite if and only if $k = n$.

(f) Let $n = 3$ and define $K = (2x_1 - 4x_2 + 6x_3, 3x_1 + 9x_2 - 9x_3, 7x_1 + x_2 + 3x_3)$. Use the above method to determine the structure of the group $G = F/K$.

[Comment: The procedure just described is an algorithm to determine the structure of an abelian group with a set of n generators subject to a given set of m relations. For a background see Hungerford, p. 335-345.]

111) Define "torsion-submodule".

112) Define "torsion-free module".

113) Let $A = \prod_p \mathbb{Z}_p$. Identify the torsion submodule.

114) Prove that if M is finitely generated torsion module over a PID, then $\mathrm{Ann}_R(M) \neq 0$.

115) Give an example of a torsion module M over a domain R such that $\mathrm{Ann}_R(M) = 0$.

116) Show that $\mathbb{Q}$ is NOT a free abelian group.

117) TRUE or FALSE: If M is a torsion-free quotient of a finitely generated module over a PID, then M is free.

118) List all isomorphism types of abelian groups of order 600.

119) What are the invariant factors of $\mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27} \oplus \mathbb{Z}_5$.

120) Let $R = F[x, y]$ where F is a field. Prove that $Rx + Ry$ is NOT a cyclic submodule of R (viewed as a left R-module).

121) List all similarity types of nilpotent 4×4 matrices.

122) Let R be a domain and F its field of fractions, regarded as an R-module in the natural way. If S is a finitely generated R-submodule of F, prove that $S \simeq I$ for some ideal I of R.

123) Let R and F be as in the previous problem. If F is R-isomorphic with an ideal of R, prove that $R = F$. [Hint: $F = rF$ if $0 \neq r \in R$.]

124) Let F be a field and let $f_1, f_2, \ldots, f_k$ be distinct monic irreducible polynomials in $F[t]$. Use the Structure Theorem for finitely generated modules over PID's to prove the following statement: there is an n-dimensional F-vector space V and an $\alpha \in \mathrm{End}_F(V)$ such that $f_1 f_2 \cdots f_k$ is the minimum polynomial of α if and only if $n = \ell_1 n_1 + \cdots + \ell_k n_k$ where $n_i = \deg(f_i)$ and the ℓ_i are positive integers.

125) Let A and B be torsion-free abelian groups with subgroups S and T respectively. Prove that the natural map $S \otimes_{\mathbb{Z}} T \to A \otimes_{\mathbb{Z}} B$ is injective. Deduce that $A \otimes B \neq 0$ if $A \neq 0$ and $B \neq 0$. [Hint: Flat modules.]

126) Suppose R is a commutative ring with identity where each prime ideal is finitely generated. Prove that R is noetherian.