Geometry and Arithmetic
Due: Friday, April 30th

> As long as algebra and geometry have been separated, their progress have been slow and their uses limited; but when these two sciences have been united, they have lent each mutual forces, and have marched together towards perfection.
>
> —Joseph Louis Lagrange

About a century before the time of Euclid, Plato—a student of Socrates—declared that the compass and straightedge should be the only tools of the geometer. Why would he do such a thing? For one thing, both the the compass and straightedge are fairly simple instruments. One draws circles, the other draws lines—what else could possibly be needed to study geometry? Moreover, rulers and protractors are far more complex in comparison and people back then couldn't just walk to the campus bookstore and buy whatever they wanted. However, there are other reasons:

(i) Compass and straightedge constructions are **independent of units**.

(ii) Compass and straightedge constructions are **theoretically correct**.

(iii) Combined, the compass and straightedge seem like **powerful tools**.

Compass and straightedge constructions are independent of units. Whether you are working in centimeters or miles, compass and straightedge constructions work just as well. By not being locked to set of units, the constructions given by a compass and straightedge have certain generality that is appreciated even today.

Compass and straightedge constructions are theoretically correct. In mathematics, a correct method to solve a problem is more valuable than a correct solution. In this sense, the compass and straightedge are ideal tools for the mathematician. Easy enough to use that the rough drawings that they produce can be somewhat relied upon, yet simple enough that the tools themselves can be described theoretically. Hence it is usually not too difficult to connect a given construction to a formal proof showing that the construction is correct.

Combined, the compass and straightedge seem like powerful tools. No tool is useful unless it can solve a lot of problems. Without a doubt, the compass and straightedge combined form a powerful tool. Using a compass and straightedge, we are able to solve many problems exactly. Of the problems that we cannot solve exactly, we can always produce an approximate solution.

For the ancient Greeks, arithmetic was done geometrically. When one reads *The Elements* by Euclid, one gets the feeling that not only is the author trying to lay a rigorous foundation to geometry—he is also trying to make numbers real objects, realized as lengths. Isn't it wonderful that a book which seems rather abstract to most, actually has such a concrete goal? In this project, our goal is to start to understand what numbers can be obtained through compass and straightedge alone.

The discussion that follows is modeled off of the presentation and exercises found in the following texts, I encourage you to investigate them:

- *Elements of Abstract Algebra* by Allan Clark.
- *What is Mathematics?* by Richard Courant and Herbert Robbins.
- *Abstract Algebra* by Ronald Solomon.

We'll start by giving the rules of compass and straightedge constructions:

Rules for Compass and Straightedge Constructions

(i) You may only use a compass and straightedge.

(ii) You must have two points to draw a line.

(iii) You must have a point and a line segment to draw a circle. The point is the center and the line segment gives the radius.

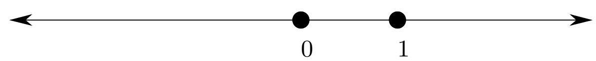
(iv) Points can only be placed in two ways:

  (a) As the intersection of lines and/or circles.

  (b) As a **free point**, meaning the location of the point is not important for the final outcome of the construction.

Constructible Numbers

What figures can we construct with compass and straightedge alone? This this a difficult problem to think about. Instead, we will think about *constructible numbers*. Imagine a line with two points on it:

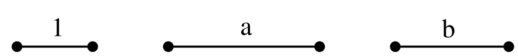


Label the left point 0 and the right point 1. If we think of this as a starting point for a number line, then a *constructible number* is nothing more than a point we can obtain on the above number line using one of the construction techniques above starting with the points 0 and 1.

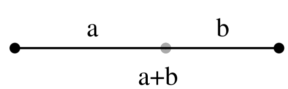
Definition. Denote the set of numbers constructible by compass and straightedge with $\mathcal{C}$. We'll call $\mathcal{C}$ the set of **constructible numbers**.

We seek the answer to the following question: Exactly what numbers are in $\mathcal{C}$? We won't completely answer this question, but we'll make a good start!

If we could use constructions to make the operations $+$, $-$, $\cdot$, and $\div$, then we would be able to say a lot more. In fact we will do just this. In the following constructions, the segments of length 1, a, and b are as given below:



Construction (Addition) *Adding is simple, use the compass to extend the given line segment as necessary.*



Construction (Subtraction) *Subtracting is easy too:*

Exercise 1 *What does our number line look like at this point?*

We still have some more operations:

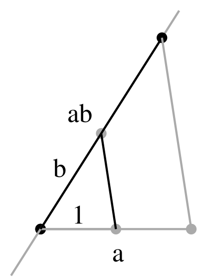
Construction (Multiplication) *This construction is based on the idea of similar triangles. Start with given segments of length a, b, and 1:*

(i) *Make a small triangle with the segment of length* 1 *and segment of length b.*

(ii) *Now place the segment of length a on top of the unit segment with one end at the vertex.*

(iii) *Draw a line parallel to the segment connecting the unit to the segment of length b starting at the other end of segment of length a.*

(iv) *The length from the vertex to the point that the line containing b intersects the line drawn in Step* (iii) *is of length $a \cdot b$.*



Exercise 2 *Prove that this construction actually multiplies two numbers. Give a relevant and revealing example.*

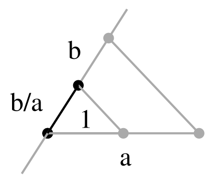
Construction (Division) *This construction is also based on the idea of similar triangles. Again, you start with given segments of length a, b, and* 1*:*

(i) *Make a triangle with the segment of length a and the segment of length b.*

(ii) *Put the unit along the segment of length a starting at the vertex where the segment of length a and the segment of length b meet.*

(iii) *Make a line parallel to the third side of the triangle containing the segment of length a and the segment of length b starting at the end of the unit.*

(iv) *The distance from where the line drawn in Step* (iii) *meets the segment of length b to the vertex is of length b/a.*



Exercise 3 *Prove that this construction actually divides two numbers. Give a relevant and revealing example.*

Exercise 4 *What does our number line look like at this point?*

Exercise 5 *Explain why the constructible numbers form a field. What can you tell me about this field at this point?*

Construction (Square-Roots) *Start with given segments of length a and 1:*

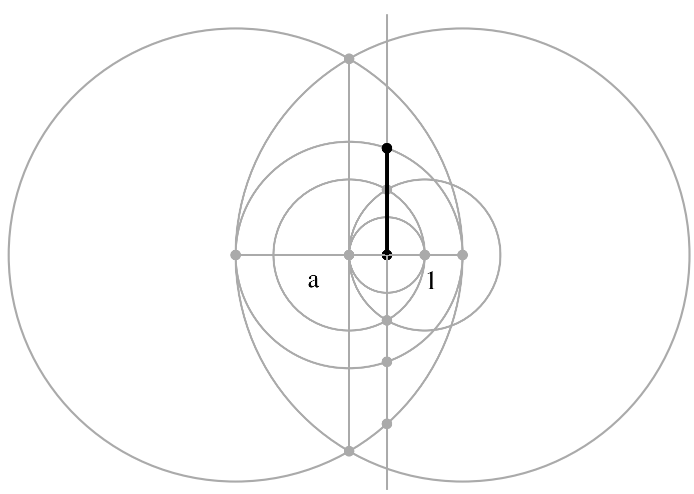
(i) *Put the segment of length a immediately to the left of the unit segment on a line.*

(ii) *Bisect the segment of length $a + 1$.*

(iii) *Draw an arc centered at the bisector that starts at one end of the line segment of length $a + 1$ and ends at the other end.*

(iv) *Construct the perpendicular at the point where the segment of length a meets the unit.*

(v) *The line segment connecting the meeting point of the segment of length a and the unit to the arc drawn in Step* (iii) *is of length $\sqrt{a}$.*



Exercise 6 *Prove that this construction actually produces a square-root. Give a relevant and revealing example.*

Exercise 7 *Prove that* $\sqrt{2}$ *is not rational.*

Exercise 8 *Prove that* $\sqrt{p}$ *is not rational for every integer prime* p*.*

OK, so how do we talk about a field that contains both $\mathbb{Q}$ and $\sqrt{2}$? Simple, use this notation:

$$\mathbb{Q}(\sqrt{2}) = \{\text{the smallest field containing both } \mathbb{Q} \text{ and } \sqrt{2}\}$$

Exercise 9 *Let* F *be a field. Prove that if* $x \in F$ *and* $\sqrt{x} \notin F$*, then*

$$F[\sqrt{x}] = \{a + b\sqrt{x} : a, b \in F\}$$

is a field. Prove further that $F[\sqrt{x}] = F(\sqrt{x})$*.*

Theorem 1 *The use of compass and straightedge alone on a field* F *can at most produce numbers in a field* $F(\sqrt{\alpha})$ *where* $\alpha \in F$*.*

Exercise 10 *Can you explain why the above theorem is true? Hint, there are three cases to consider:*

(i) *The intersection of two lines.*

(ii) *The intersection of a line and a circle.*

(iii) *The intersection of two circles.*

Exercise 11 *Which of the following numbers are constructible?*

$$3.1415926, \qquad \sqrt[16]{5}, \qquad \sqrt[3]{27}, \qquad \sqrt[6]{27}.$$

$\sqrt[3]{2}$ is Not Constructible

We are going to give an elementary proof explaining why the cube root of two is not constructible with compass and straightedge alone. In a future mathematics course you might see another proof.

Exercise 12 *Sketch a plot of* $x^3 - 2$*.*

Exercise 13 *How many real roots does* $x^3 - 2$ *have? How many complex roots does* $x^3 - 2$ *have?*

Exercise 14 *Prove that* $\sqrt[3]{2}$ *is not rational.*

We know that $\sqrt[3]{2} \notin \mathbb{Q}$. We also know that we can construct larger and larger fields:

$$\mathbb{Q} = F_0 \subset F_1 \subset F_2 \cdots \subset F_\ell \subset \cdots \subset \mathcal{C}$$

where each $F_{i+1} = F_i(\sqrt{\alpha_i})$ with $\alpha_i \in F_i$ and $\sqrt{\alpha_i} \notin F_i$.

Seeking a contradiction, we will suppose that there is some $\delta \in \mathcal{C}$ with $\delta^3 - 2 = 0$. Let ℓ be the least positive integer such that there is some F_ℓ with $\delta \in F_\ell$.

Exercise 15 *Explain why every number in F_ℓ can be expressed as*

$$p + q\sqrt{\alpha}$$

where $p, q \in F_{\ell-1}$, $\alpha \in F_{\ell-1}$, with $\sqrt{\alpha} \notin F_{\ell-1}$.

Hence we may write δ as

$$\delta = p + q\sqrt{\alpha}$$

Exercise 16 *Use the Binomial Theorem to expand $\delta^3 - 2$:*

$$(p + q\sqrt{\alpha})^3 - 2$$

Exercise 17 *Collect like terms to write*

$$(p + q\sqrt{\alpha})^3 - 2$$

as

$$P + Q\sqrt{\alpha}$$

Exercise 18 *A little bird told me that at this point we must conclude that:*

$$P + Q\sqrt{\alpha} = 0$$

Remind me, why is the above statement true again?

Exercise 19 *If $P + Q\sqrt{\alpha} = 0$ and $Q \neq 0$, then we can write*

$$\sqrt{\alpha} = \frac{-P}{Q}$$

is this a problem? What must we conclude about Q? Now what must we conclude about P?

Exercise 20 *Repeat Exercises 16–19 above, this time replacing δ with*

$$\gamma = p - q\sqrt{\alpha}.$$

Exercise 21 *We must conclude that δ and γ are both real roots of $x^3 - 2$. Is this a problem? Where do we go from here?*

Exercise 22 *Give a brief summary of how we proved that the cube-root of two is not constructible.*

Final thoughts

We have hopefully given you a taste of compass and straightedge constructions and also proved that the cube-root of two is not constructible. The proof we gave was elementary in nature, all you needed to complete the proof was guts and basic algebra skill. Perhaps in a future mathematics course you will see a more elegant and powerful approach to this problem.