Math 581: Homework 1
Due: Friday, April 9th

1) Prove that $\mathbb{Z} \times \mathbb{Z}$ is a ring. Also prove that $\mathbb{Z} \times \mathbb{Z}$ is not a domain.

2) Prove that $\mathbb{Q}[\sqrt{2}]$ is a field.

3) Prove or disprove that $\mathbb{Z}_5[i]$ is a field.

4) Let F be a field. Find some quality inherent in F that is sufficient to make the set of ordered pairs in $F \times F$ a field when "addition" is defined by $\maltese$ and "multiplication" is defined by $\star$:

$$(a, b) \maltese (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b) \star (c, d) = (ac - bd, ad + bc).$$

Hint: What if $F = \mathbb{R}$? What if $F = \mathbb{C}$? Keep on going!

5) Find the units in $\mathbb{Z}_n$ for $n = 2, 3, 4, 5, 6, 7, 8$. Make a conjecture.

6) Prove that in a domain of characteristic p, the so-called *Freshman Binomial Theorem* holds:

$$(a + b)^p = a^p + b^p$$

7) Solve the following equation

$$x^3 + x^2 + x + 1$$

over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\mathbb{Z}_5$, $\mathbb{Z}_3$, and $\mathbb{Z}_2$.

8) Let R be an integral domain and let $p(x)$ and $q(x)$ be nonzero elements of $R[x]$. Prove that

$$\deg(p(x) \cdot q(x)) = \deg(p(x)) + \deg(q(x)).$$

Is this result necessarily true if R is not an integral domain? Give a proof or counterexample to justify your claim.