Math 581: Homework 2
Due: Friday, April 23rd

1) Given a ring R, if $X \subseteq R$, we write

$$(X) \qquad \text{or} \qquad (X)R$$

as the smallest ideal of R containing the subset X. Prove that

$$(X) = \left\{ \sum_{i=1}^{n} a_i r_i : r_i \in R, a_i \in X, \text{ and } n \text{ is some nonnegative integer} \right\}.$$

We call X a **generating set** for the ideal. Note, X could be infinite!

2) Let $S^2 = \{(a,b,c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that

$$I = \{f(x,y,z) \in \mathbb{R}[x,y,z] : f(a,b,c) = 0 \text{ for all } (a,b,c) \in S^2\}$$

is an ideal of $\mathbb{R}[x,y,z]$.

3) Let I and J be ideas of a ring R. Prove that

$$IJ = \left\{ \sum a_i b_j : a_i \in I \text{ and } b_j \in J \right\}$$

is an ideal of R.

4) Let I and J be ideas of a ring R. Prove that $I \cap J$ is an ideal of R. Find a ring R along with two ideals I and J such that $I \cup J$ is **not** an ideal of R.

5) Consider $f, g \in R[x]$ where R is a domain.

(a) Prove that given two polynomials f and g

$$\deg(f+g) \leqslant \max\{\deg(f), \deg(g)\} \qquad \text{and} \qquad \deg(f \cdot g) = \deg(f) + \deg(g).$$

(b) Is the above result necessarily true if R is not a domain? Justify your claim.

(c) Prove or disprove: The ideal $(2, x)$ is not principal in $\mathbb{Z}[x]$.

(d) Prove or disprove: The ideal $(2, x)$ is not principal in $F[x]$ where F is a field.

6) Formulate and prove a version of the Division Theorem for integers allowing negative remainders. Will you be able to preserve uniqueness of the quotient and remainder? Give a proof or counterexample.

7) Prove that for $a, b \in \mathbb{Z}$, $((a,b),b) = (a,b)$.

8) Prove that $(a,b) = 1$ and $c|a$ imply that $(c,b) = 1$.