Math 581: Homework 4
Due: Friday, May 21st

1) Use the Euclidean algorithm to find the generator of the principal ideal $(x^3 + x, x^2 + x + 1) \subseteq \mathbb{Z}_2[x]$.

2) Let R be a nonzero ring. Prove that the following are equivalent:

(a) R is a field.

(b) The only ideals in R are (0) and (1).

(c) Every homomorphism of R into a nonzero ring S is injective.

3) Prove that $\mathbb{Z}_3 \times \mathbb{Z}_5 \simeq \mathbb{Z}_{15}$.

4) Prove or disprove that $\mathbb{Z}_3 \times \mathbb{Z}_6 \simeq \mathbb{Z}_{18}$.

5) Given a domain R, prove that $F(F(R)) \simeq F(R)$. Use this isomorphism to explain why

$$\frac{a/b}{c/b} = \frac{a}{c}.$$

6) Prove that $\mathbb{Z}_n \simeq \mathbb{Z}/(n)$. Carefully explain the distinction between $\mathbb{Z}_n$ and $\mathbb{Z}/(n)$.

7) Prove that $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

8) Give two different methods for finding $2^{999} \pmod 5$, one using Fermat's Little Theorem and the other using basic problem solving methods.