Math 581: Homework 5
Due: Friday, June 4th

1) Prove the Chinese Remainder Theorem: Let $m, n \in \mathbb{N}$ with $(m, n) = 1$. Then given $y, z \in \mathbb{Z}$ there is a unique integer x between 1 and mn satisfying

$$x \equiv y \pmod{m} \qquad \text{and} \qquad x \equiv z \pmod{n}.$$

Hint: Use the idea from the proof that when $(m, n) = 1$, $\phi(mn) = \phi(m)\phi(n)$.

2) Find an integer x such that

$$x \equiv 16 \pmod{523} \qquad \text{and} \qquad x \equiv 17 \pmod{541}.$$

3) Given two integers m and n such that $(m, n) = 1$, prove that

$$\mathbb{Z}/(mn) \simeq \mathbb{Z}/(m) \times \mathbb{Z}/(n).$$

Hints:

(a) Define a map $\varphi : \mathbb{Z} \to \mathbb{Z}/(m) \times \mathbb{Z}/(n)$ via

$$x \mapsto (x + (m), x + (n))$$

(b) Prove φ is a homomorphism of rings.

(c) Prove that φ is surjective.

(d) Compute $\mathrm{Ker}(\varphi)$.

(e) There is one more hint in the notes.

Use the First Isomorphism Theorem to complete the proof.

4) Find all solutions of the equation $x^3 - 6x + 4 = 0$. How many real solutions did you find? How many complex solutions did you find?

5) Find the minimal polynomials for:

(a) $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}$.

(b) $\sqrt{2} + \sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$.

6) Prove there are an infinite number of irreducible polynomials in $F[x]$ where F is any field.