Math 331: Homework 1
Due: Friday, September 5th

1 (2.1.7) Prove that if $P$ and $Q$ are distinct points in $\mathbb{H}$, then they cannot lie simultaneously on both $aL$ and $cL_r$ for some choice of $a$, $c$, and $r$.

2 (2.1.19) Some finite geometries are defined pictorially:

![Diagram of geometries](image)

Answer the following questions:

(a) In each example list the set of lines.

(b) Which of these geometries are abstract geometries?

(c) Which of these geometries are incidence geometries?

3 (2.1.20) Let $\{S, \mathcal{L}\}$ be an abstract geometry and assume that $S_1 \subseteq S$. We define an $S_1$-line to be any subset of $S_1$ of the form $l \cap S_1$ where $l \in \mathcal{L}$ and where $l \cap S_1$ has at least two points. Let $\mathcal{L}_1$ be the collection of all $S_1$-lines. Prove that $\{S_1, \mathcal{L}_1\}$ is an abstract geometry. $\{S_1, \mathcal{L}_1\}$ is called the geometry induced from $\{S, \mathcal{L}\}$.

4 (2.1.21) If $\{S_1, \mathcal{L}_1\}$ is the geometry induced from an incidence geometry $\{S, \mathcal{L}\}$, prove that $\{S_1, \mathcal{L}_1\}$ is an incidence geometry if $S_1$ has a set of three non-collinear points.

5 (2.1.24) Let $\{S, \mathcal{L}\}$ be an abstract geometry. If $l_1$ and $l_2$ are lines in $\mathcal{L}$, we write $l_1 \sim l_2$ if $l_1$ is parallel to $l_2$. Prove or disprove that $\sim$ is an equivalence relation. If $\{S, \mathcal{L}\}$ is the Cartesian Plane then each equivalence class can be characterized by a real number or infinity. What is this number?
6 (2.2.2) Prove that the function \( d_H \) defined by Equations (2-2) and (2-3) on page 28 of your text satisfies axioms (i) and (iii) of the definition of a distance function.

7 (2.2.8) Prove that the function \( g : \mathbb{R} \to \mathbb{R} \) given by \( g(a, y) = \ln(y) \) is a bijection and that it satisfies the Ruler Equation. Also show that the inverse of \( g \) is given by \( g^{-1}(t) = (a, e^t) \).

8 (2.2.18) Define the \textbf{max distance} (or \textbf{supremum distance}), \( d_S \) on \( \mathbb{R}^2 \) by

\[
    d_S(P, Q) = \max\{|x_1 - x_2|, |y_1 - y_2|\}
\]

where \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \).

(a) Prove that \( d_S \) is a distance function.

(b) Prove that \( \{\mathbb{R}^2, \mathcal{L}, d_S\} \) is a metric geometry.

9 (2.2.19) In a metric geometry \( \{S, \mathcal{L}, d\} \) if \( P \in S \) and \( r > 0 \), then the \textbf{circle with center} \( P \) and \textbf{radius} \( r \) is

\[
    \mathcal{C} = \{Q \in S : d(P, Q) = r\}.
\]

Draw a picture of the circle of radius 1 and center \((0, 0)\) in \( \mathbb{R}^2 \) for each of the distances \( d_E, d_T, \) and \( d_S \).

10 (2.3.2) In the Poincaré Plane find a ruler \( f \) with \( f(P) = 0 \) and \( f(Q) > 0 \) for the given pair \( P \) and \( Q \):

(a) \( P = (2, 3), Q = (2, 1) \).

(b) \( P = (2, 3), Q = (-1, 6) \).

11 (2.3.5) Prove that a line in a metric geometry has infinitely many points.