Math 446: Homework 1
Due: Wednesday, January 28th

1 (–) Let $x \in G$ where $(G, \cdot)$ is a group with identity element e. If $x^2 \neq e$ and $x^6 = e$, prove that that $x^4 \neq e$ and $x^5 \neq e$.

2 (26ζ) Let G and G' be groups. Define a product operation on the set $G \times G'$ by the rule $(a, a')(b, b') = (ab, a'b')$. Show that $G \times G'$ is a group under this product. $G \times G'$ is called the **direct product** of G and G'.

3 (28α) Give a rigorous proof that for any elements $a_1, a_2, \ldots, a_n$ of a group,

$$(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1} a_1^{-1}.$$

Hint: Use induction.

4 (–) Define the **order** of an element of a multiplicative group to be the smallest positive integer n such that x^n is the identity element, and denote this integer by $o(n)$. If no positive power of x is the identity, then the order of x is defined to be infinity. The order of an element for additive groups is defined in a similar fashion but with exponentiation replaced with multiplication.

(a) Find the order of each element of the additive group $\mathbb{Z}_{12}$. Explain your reasoning.

(b) Find the order of each element of the multiplicative group $\mathbb{Z}_{12}^*$. Explain your reasoning.

(c) Based on the computations above, make a conjecture.

5 (28γ) Prove that isomorphisms preserve identity elements and inverses.

6 (–) Show that the mapping $\varphi(x) = \ln(x)$ is an isomorphism from $(\mathbb{R}^+, \cdot)$ (the group of positive real numbers under multiplication) to $(\mathbb{R}, +)$ (the group of real numbers under addition) is an isomorphism. What is the inverse of φ?

Explain how this isomorphism illustrates the statement that "logarithms give a method of turning multiplication questions into addition questions."

7 (–) Prove or disprove that $\mathbb{Z}_4 \simeq \mathbb{Z}_5^*$.

8 (–) An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out? Explain how you reached your conclusion.

9 (–) Let G be a group with the following property: Whenever a, b, and c belong to G and $ab = ca$, then $b = c$. Prove that G is abelian.

10 **(30$\boldsymbol{\beta}$)** Prove that S_n is not abelian for $n > 2$.

11 **(30$\boldsymbol{\gamma}$)** Construct an isomorphism of the symmetric group S_3 with the dihedral group D_3. Be sure to explain why your map is in fact an isomorphism.

12 (–) Show that the quaternion group Q is not isomorphic to the dihedral group D_4. See **26$\boldsymbol{\theta}$** and **28$\boldsymbol{\gamma}$**.