Math 446: Homework 2
Due: Wednesday, February 11th

1 (36α) Describe the subgroup $n\mathbb{Z} \cap m\mathbb{Z}$ of $\mathbb{Z}$. Justify your answer with a proof.

2 (36β) Describe the subgroup of $\mathbb{Z}$ generated by n and m, that is by the subset $\{n, m\}$. Justify your answer with a proof.

3 (36γ) Describe all the subgroups of $\mathbb{Z}_n$. Justify your answer with a proof.

4 (36δ) Show that $\mathbb{Z} \times \mathbb{Z}$ has subgroups not of the form $n\mathbb{Z} \times m\mathbb{Z}$.

5 (–) Give the lattice of subgroups for the following groups:

(a) $\mathbb{Z}_{10}$

(b) $\mathbb{Z}_3 \times \mathbb{Z}_4$

(c) $\mathbb{Z}_{13}^*$

(d) V

(e) Q

6 (38α.1) Let H and K be subgroups of a group G. Show that HK is a subgroup of G if and only if $HK = KH$.

7 (38α.2) Let H and K be subgroups of a group G. Show that when HK is a finite subgroup of G

$$o(HK) = o(H)o(K)/o(H \cap K).$$

8 (38γ) Let G be a nontrivial group with no proper subgroups except the trivial one. Show that G is finite and that the order of G is prime.

9 (39α) Let H denote the subgroup of S_n consisting of all elements $\pi \in S_n$ such that $\pi(n) = n$. What is $[S_n : H]$? Justify your answer with a proof.

10 (41) Let G be a finite cyclic group with $a, b \in G$. Prove that

$$o(ab) \mid [o(a), o(b)]$$

where $[m, n]$ is the **least common multiple** of natural numbers m and n, see **23γ**. Additionally, explain why if

$$(o(a), o(b)) = 1, \qquad \text{then} \qquad o(ab) = o(a) \cdot o(b).$$

11 (41) In this problem we will prove that given a finite abelian group G with an element a of maximal order, then $o(b) \mid o(a)$ for all $b \in G$.

(a) Given any two natural numbers m and n, prove that you can find two more natural numbers p and q such that

$$\left(\frac{m}{p}, \frac{n}{q}\right) = 1 \qquad \text{and} \qquad \frac{m}{p} \cdot \frac{n}{q} = [m, n].$$

(b) Let G be a finite abelian group with $a, b \in G$ such that $o(a) = m$ and $o(b) = n$. Prove that G contains an element of the form $a^p b^q$ of order $[m, n]$.

(c) Prove that given a finite abelian group G with an element a of maximal order, then $o(b) \mid o(a)$ for all $b \in G$.

12 (41ζ) Prove there can only be two distinct groups with order 4, up to isomorphism.