Math 446: Homework 4
Due: Friday, March 13th

1 (87δ) If a and b are elements of a field F and $b \neq 0$, let a/b denote ab^{-1}. Prove that when $a \neq 0$:

$$\frac{1}{a/b} = \frac{b}{a}$$

2 (87ζ) Let F be a field and let $E = F \times F$. Define addition and multiplication in E by the rules:

$$(a,b) + (c,d) = (a+c, b+d) \qquad \text{and} \qquad (a,b)(c,d) = (ac - bd, ad + bc)$$

Determine conditions on F under which E is a field.

3 (87η) Show that a field homomorphism is always one-to-one or trivial (every element is mapped to zero). Explain why an onto field homomorphism is a field isomorphism.

4 (88α) Show that a subset F of a field E is a subfield if and only if F^* is nonempty and $a, b \in F$ implies $a - b \in F$ and when $b \neq 0$ that $a/b \in F$.

5 (88δ) Prove that every number field contains $\mathbb{Q}$.

6 (–) Consider the following $\mathbb{Q}$-vector space:

$$\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{z \in \mathbb{C} : z = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \text{ where } a, b, c, d \in \mathbb{Q}\}$$

Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a field.

7 (92γ) Let $T : E \to E'$ be a linear transformation of F-vector spaces. Show that the sets

$$\text{Ker}(T) = \{\alpha \in E : T(\alpha) = 0\}$$

and

$$\text{Im}(T) = \{\alpha' \in E' : \alpha' = T(\alpha) \text{ where } \alpha \in E\}$$

are subspaces of E and E' respectively.

8 (95β) Prove that a subspace E' of a finite dimensional vector space E over F is again finite dimensional and that $[E' : F] \leqslant [E : F]$.

9 (95γ) Let E' be a subspace of a finite dimensional vector space E over F. Prove that when E is finite dimensional the dimension of E is the sum of the dimensions of E' and E/E'.

10 (96δ) Show that a finite field of characteristic p, see **89α**, has p^n elements for some natural number n. Explain why there is no field of order 6.