Math 446: Homework 4
Due: Friday, March 13th

1 (87δ) If $a$ and $b$ are elements of a field $F$ and $b \neq 0$, let $a/b$ denote $ab^{-1}$.
Prove that when $a \neq 0$:
\[
\frac{1}{a/b} = \frac{b}{a}
\]

2 (87ζ) Let $F$ be a field and let $E = F \times F$. Define addition and multiplication in $E$ by the rules:
\[
(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b)(c, d) = (ac - bd, ad + bc)
\]
Determine conditions on $F$ under which $E$ is a field.

3 (87η) Show that a field homomorphism is always one-to-one or trivial (every element is mapped to zero). Explain why an onto field homomorphism is a field isomorphism.

4 (88α) Show that a subset $F$ of a field $E$ is a subfield if and only if $F^*$ is nonempty and $a, b \in F$ implies $a - b \in F$ and when $b \neq 0$ that $a/b \in F$.

5 (88δ) Prove that every number field contains $\mathbb{Q}$.

6 (−) Consider the following $\mathbb{Q}$-vector space:
\[
\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{ z \in \mathbb{C} : z = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{2}\sqrt{3} \text{ where } a, b, c, d \in \mathbb{Q} \}
\]
Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is a field.

7 (92γ) Let $T : E \to E'$ be a linear transformation of $F$-vector spaces. Show that the sets
\[
\ker(T) = \{ \alpha \in E : T(\alpha) = 0 \}
\]
and
\[
\text{im}(T) = \{ \alpha' \in E' : \alpha' = T(\alpha) \text{ where } \alpha \in E \}
\]
are subspaces of $E$ and $E'$ respectively.

8 (95β) Prove that a subspace $E'$ of a finite dimensional vector space $E$ over $F$ is again finite dimensional and that $[E' : F] \leq [E : F]$.

9 (95γ) Let $E'$ be a subspace of a finite dimensional vector space $E$ over $F$. Prove that when $E$ is finite dimensional the dimension of $E$ is the sum of the dimensions of $E'$ and $E/E'$.

10 (96δ) Show that a finite field of characteristic $p$, see 89α, has $p^n$ elements for some natural number $n$. Explain why there is no field of order 6.