Math 446: Homework 5
Due: Monday, April 6th

1 (100β) Show that a polynomial f over a field F and its formal derivative f' have a common root α in F if and only if α is a *multiple root* of f, that is, $(x-\alpha)^2$ divides f.

2 (100δ) Show that every element of a finite field with q elements is a root of the polynomial $x^q - x$.

3 (100ι) Let $f(x)$ be a polynomial over a field F whose derivative is 0. Show that if $\text{char}(F) = 0$, then $f(x)$ is a constant polynomial. What can one say in the case when $\text{char}(F) = p$?

4 (101α) Show that every polynomial of positive degree over $\mathbb{R}$ can be factored as a product of polynomials over $\mathbb{R}$ with degrees 1 or 2.

5 (–) A field F is called **algebraically closed** if every polynomial in $F[x]$ whose degree is one or greater has a root in F. For example, $\mathbb{C}$ is algebraically closed but $\mathbb{R}$ is not. Prove that every algebraically closed field has an infinite number of elements.

6 (102γ) Prove that there are an infinite number of irreducible polynomials over any field.

7 (102δ) Compute the number of irreducible polynomials of degrees 1, 2, and 3 over $\mathbb{Z}_p$ where p is a prime.

8 (102ζ) Prove that a polynomial $f(x)$ over a field is irreducible if and only if the polynomial $g(x)$ defined by $g(x) = f(x+a)$ is irreducible over the same field.

9 (103ε) Show that the field $\mathbb{Q}[x]/(x^2-2)$ is isomorphic to the field $\mathbb{Q}(\sqrt{2})$.

10 (103η) Construct a field with 9 elements. Explain your construction.

11 (–) Determine whether the following polynomials are irreducible. For those are are reducible, give their factorizations into irreducible polynomials. For those that are irreducible, give a proof explaining your conclusion.

(a) $x^2 + x + 1$ over $\mathbb{Z}_2$.

(b) $x^3 + x + 1$ over $\mathbb{Z}_3$.

(c) $x^4 + 1$ over $\mathbb{Z}_5$.

(d) $x^2 + x + 4$ over $\mathbb{Z}_{11}$.

(e) $x^5 + 9x^4 + 12x^2 + 6$ over $\mathbb{Q}$.

(f) $17x^3 - 7x^2 + 34x + 1$ over $\mathbb{Q}$.

(g) $x^4 + 10x + 1$ over $\mathbb{Q}$.

(h) $x^4 + 10x^2 + 1$ over $\mathbb{Q}$.

(i) $4x^3 - 3x - 1/2$ over $\mathbb{Q}$.