IMMERSE 2008: Assignment 4

4.1) Let $A$ be a ring and set $R = A[x_1, \ldots, x_n]$. For each \( a = (a_1, \ldots, a_N) \in \mathbb{N}^N \) let \( R_a = A \cdot x_1^{a_1} \cdots x_N^{a_N} \). Prove that \[ R = \bigoplus_{a \in \mathbb{N}^N} R_a \]
is an \( \mathbb{N}^N \)-graded ring.

**Proof:** It is necessary to show that (a) each component of the direct sum, \( R_a \), is an additive subgroup, (b) the multiplication described in the definition of a graded ring holds, and that (c) \( R = \bigoplus_{a \in \mathbb{N}^N} R_a \).

(a) First we will show that \( R_a \) is nonempty, closed under addition, and contains inverses. Since \( A \) is a ring, \( 0 \in A \). Thus \( 0 \cdot x_1^{a_1} \cdots x_N^{a_N} = 0 \in R_a \). Hence \( R_a \) is nonempty.

Now assume that \( r_a \) and \( s_a \in R_a \), and let \( r_a = r \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \) and \( s_a = s \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \), where \( r, s \in A \). Thus, since \( A \) is a ring, \( r \cdot s \in A \). Hence \( r_a + s_a = (r + s)(x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}) \in R_a \).

It remains to show that for all \( r_a = r \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \in R_a \), there exists \( s_a \in R_a \) such that \( r_a + s_a = 0 \). Since \( r \in R \), there exists an additive inverse \( s \in R \). Let \( s_a = s \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \in R_a \). Then \( r_a + s_a = (r + a)(x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N}) = 0 \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} = 0 \).

Therefore \( R_a \) contains additive inverses. Therefore \( R_a \) is an additive subgroup for \( a \in \mathbb{N}^N \).

(b) Next we must show that \( R_a \cdot R_b \subseteq R_{a+b} \).

Let \( r_a = r \cdot x_1^{a_1} x_2^{a_2} \cdots x_N^{a_N} \in R_a \) and \( s_b = s \cdot x_1^{b_1} x_2^{b_2} \cdots x_N^{b_N} \in R_b \), where \( r, s \in A \), then \( r_a \cdot s_b \in R_a \cdot R_b \). Then we have \( r_a \cdot s_b = rs \cdot x_1^{a_1+b_1} x_2^{a_2+b_2} \cdots x_N^{a_N+b_N} \in R_{a+b} \).

Hence the multiplication property of graded rings holds, so \[ \bigoplus_{a \in \mathbb{N}^N} R_a \]
is an $\mathbb{N}^N$-graded ring.
(c) Finally, we will show that $A[x_1, ..., x_n] = \bigoplus_{a \in \mathbb{N}^N} R_a$
by demonstrating set inclusions and that $R_a \cap R_b = 0$ for $a \neq b$. Let
$$
\sum_{i,j=1}^{n} (\alpha_i x_j^i)
$$
be an element of $R$. Since $\alpha_i \in A$ for all $i \in \mathbb{N}$ and $x_j^i \in R_{a_j}$ for some $a_j \in \mathbb{N}^N$, we
have $\alpha_i x_j^i \in R_{a_j}$. Therefore
$$
A[x_1, ..., x_n] \subseteq \bigoplus_{a \in \mathbb{N}^N} R_a
$$
Next consider an element $r$ in $\bigoplus_{a \in \mathbb{N}^N} R_a$. Thus
$$
r = \sum_{i=1}^{n} A \cdot x_i^{a_i},
$$
which is an element of the ring $A[x_1, ..., x_n]$. Therefore $\bigoplus_{a \in \mathbb{N}^N} R_a \subseteq A[x_1, ..., x_n]$.
Now let $a, b \in \mathbb{N}^N$ such that $a \neq b$. Moreover, $a_i \neq b_i$ for $i \geq 1$. Thus $R_a \cap R_b = 0$.
Hence
$$
R = A[x_1, ..., x_n] = \bigoplus_{a \in \mathbb{N}^N} R_a
$$
is an $\mathbb{N}^N$-graded ring.

4.2) Let $R$ be a graded ring. Prove that if $I$ is a homogeneous ideal of $R$, then $R/I$ is
a homogeneous $R$-module. That is, show that $R/I$ is generated by homogeneous elements
and is hence graded with the inherited grading.

Proof: Let $R$ be a graded ring such that $R = \bigoplus_{i=0}^{\infty} R_i$. Let $I$ be a homogeneous
ideal of $R$. We want to show that $R = \bigoplus_{i=0}^{\infty} R_i/R_i \cap I = \bigoplus_{i=0}^{\infty} R_i$ (i.e. that $R/I$ is
graded with the inherited grading from $R$). To do this we must prove four things:
1) Each $\overline{R}_i$ is a subgroup of $\overline{R}$;
2) $\overline{R} = \bigoplus_{i=0}^{\infty} \overline{R}_i$;
3) $\overline{R}_i \cap \overline{R}_j = \{0\}$.
4) $\overline{R}_m \cdot \overline{R}_n = \overline{R}_{m+n}$

Proof of (1)
Note that $I$ is an ideal of $R$ a ring, so $I$ must be an abelian group. Also, $R_i$ is a
abelian subgroup of $R$. So $R_i \cap I$ must also be an abelian group. So $R_i/R_i \cap I$ is
also an abelian group. Therefore each $\overline{R}_i$ is an abelian subgroup of $\overline{R}$.

Proof of (2)
(⊆) Let \( \bar{x} \in \bar{R} = R/I \). Then \( x \in R \) so \( x = \sum_{i=0}^{\infty} r_i \) for some \( r_i \in R_i \). Then \( \bar{x} = \sum_{i=0}^{\infty} \bar{r}_i = \sum_{i=0}^{\infty} r_i \). So \( \bar{x} \in \bigoplus_{i=0}^{\infty} \bar{R}_i \). Hence, \( \bar{R} \subseteq \bigoplus_{i=0}^{\infty} \bar{R}_i \).

(⊇) Let \( \bar{x} \in \bigoplus_{i=0}^{\infty} \bar{R}_i \). Then \( \bar{x} = \sum_{i=0}^{\infty} \bar{r}_i \) for some \( \bar{r}_i \in \bar{R}_i \). So \( \bar{x} = \sum_{i=0}^{\infty} r_i \) which implies that \( x = \sum_{i=0}^{\infty} r_i \). So \( x \in R \). So \( \bar{x} \in R \). Hence \( \bar{R} \supseteq \bigoplus_{i=0}^{\infty} \bar{R}_i \).

Therefore \( \bar{R} = \bigoplus_{i=0}^{\infty} \bar{R}_i \).

Proof of (3)
Seeking a contradiction, assume that there exists some \( x \in R_i \cap R_j \). Then \( x \in R_i \) and \( x \in R_j \). So \( x \in R_i \cap R_j \). But this contradicts the assumption that \( R \) is a graded ring. Hence there is no \( x \in R_i \cap R_j \). Therefore \( R_i \cap R_j = \{0\} \).

Proof of (4)
Note that \( R_m \cdot R_n \subseteq R_{m+n} \) since \( R \) is a graded ring. So let’s look at an element from \( R_m/R_m \cap I \cdot R_n/R_n \cap I \), \( x = \bar{r}_n \cdot \bar{r}_m \in R_m/R_m \cap I \cdot R_n/R_n \cap I \). Then
\[
\bar{r}_n \cdot \bar{r}_m = \{ (r_n + i_n)(r_m + i_m) | i_n \in I_n, i_m \in I_m \} = \{ (r_n r_m + i_n r_m + r_n i_m + i_n i_m) | i_n \in I_n, i_m \in I_m \}.
\]

So \( \bar{r}_n \cdot \bar{r}_m \in R_{m+n}/I_{m+n} \). So \( \bar{R}_m \cdot \bar{R}_n \subseteq \bar{R}_{m+n} \).

Therefore, \( R/I \) is graded.

4.3) Let \( K \) be a field and \( R = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \) where we set \( \deg(x_i) = (1, 0) \) and \( \deg(y_j) = (0, 1) \). Let \( I \) be an ideal generated by finitely many monomials. By the previous exercise, \( A = R/I \) is a graded \( R \)-module. Prove that the monomials of degree \((\lambda, \nu)\) form a basis for \( A_{(\lambda, \nu)} \) over \( K \).

4.4) Let \( S = k[x_1, \ldots, x_n] \) be a standard graded ring and \( f_1, \ldots, f_d \) be homogeneous elements of \( S \) of degrees \( \alpha_1, \ldots, \alpha_d \) respectively. Prove that \( R = S_0[f_1, \ldots, f_d] \) is an \( \mathbb{N} \)-graded ring where
\[
R_n = \left\{ \sum_{m \in \mathbb{N}^d} r_m f_1^{m_1} \cdots f_d^{m_d} : r_m \in S_0 \text{ and } \alpha_1 m_1 + \ldots + \alpha_d m_d = n \right\}.
\]

**Proof:** For ease of notation, let
\[
f_{\bar{\alpha}, \bar{\beta}} = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n}.
\]
To begin, notice that the set of monomials in \( R \) of degree \((\lambda, \nu)\) are the set
\[
F = \left\{ f_{\bar{\alpha}, \bar{\beta}} \text{ such that } \sum_{i=1}^{n} \alpha_i = \lambda, \sum_{i=1}^{n} \beta_i = v \right\}.
\]

Finally, let
\[
\overline{F} = \left\{ \frac{f_{\bar{\alpha}, \bar{\beta}}}{I} \in R/I \text{ such that } \sum_{i=1}^{n} \alpha_i = \lambda, \sum_{i=1}^{n} \beta_i = v \text{ and } f_{\bar{\alpha}, \bar{\beta}} \notin I \right\}.
\]

We will show that \( \overline{F} \) is a basis for \( A_{(\lambda, \nu)} \). To do this, we must show that \( \overline{F} \) generates \( A_{(\lambda, \nu)} \) and is linearly independent.
Any element in $A_{(\lambda, \nu)}$ is some $\overline{h}$ where $h$ is in $R_{(\lambda, \nu)}$. Notice that $F$ is a finite set, and that it is all the monomials from $R_{(\lambda, \nu)}$ that are not sent to zero in the quotient module $R/I$. Thus, since $F$ generates $R_{(\lambda, \nu)}$, $F$ spans $A_{(\lambda, \nu)}$.

To show that the set is linearly independent, let $\sum c_if_i = 0$ for a set of coefficients $C_i \in R$ and we will show that $c_i = 0$ for every $i$. We then know that $\sum c_if_i = 0$, so $\sum c_if_i \in R_{(\lambda, \nu)} \cap I$. Obviously, $\sum c_if_i \in R_{(\lambda, \nu)}$. However, since it is also an element of $I$, a finitely generated ideal, given $G = \{g_1 \cdots g_m\}$ a generating set for $I$, there exist $b_i \in R$ such that

$$\sum c_if_i = \sum b_ig_i$$

and so

$$\sum c_if_i - \sum b_ig_i = 0.$$

However, by the way we defined the set $F$, the $f_i$'s we have above cannot be in $I$, and so $f_i \neq g_j$ for any $i$ and $j$. Thus, all coefficients are 0, and particularly, $c_i = 0$ for every $i$.

Thus $F$ is linearly independent and generates $A_{(\lambda, \nu)}$, so it is a basis. \[\square\]

4.5) Let $k$ be a field and $R = k[x]$. Set

$$R_n = \{c(x-1)^n : c \in k\}$$

for all $n \in \mathbb{N}$.

(a) Prove that $R$ is an $\mathbb{N}$-graded ring.

Note: This looks like a monomial ideal; however, it is not with this grading.

**Proof:** For $R$ to be an $\mathbb{N}$-graded ring, we must show that it has additive subgroups $R_n$ such that $R = \bigoplus_{n \in \mathbb{N}} R_n$, and $R_n \cdot R_m \subseteq R_{m+n}$. First, we will show that $R_n$ as defined is an additive subgroup of $R$.

To show $R_n$ is an additive subgroup of $R$, we must show that is nonempty, closed under addition, and has inverses. Since $k$ is a field, $0 \in k$, so $0(x-1)^n = 0 \in R_n$.

Where $a \in k$ and $b \in k$,

$$a(x-1)^n + b(x-1)^n = (a+b)(x-1)^n,$$

and since $(a+b)$ is in $k$, $(a+b)(x-1)^n \in R_n$, thus $R_n$ is closed under addition. Again, because $k$ is a field, each $c \in k$ has an inverse $-c$, and

$$c(x-1)^n + (-c(x-1)^n) = c(1-x)^n - c(1-x)^n = 0.$$

So everything in $R_n$ has an inverse. Thus $R_n$ is an additive subgroup of $R$.

We next show that $R = \bigoplus_{n \in \mathbb{N}} R_n$, by subset inclusion. We know that $\bigoplus_{n \in \mathbb{N}} R_n$ is a polynomial in $x$ over elements of $k$, so it is clearly contained in $R$. We also know that $c(1-x)^0 = c \in R_0$, and $1(1-x)^1 = (x-1) \in R_1$, and since $1 \in R_0$ and $(x-1) \in R_1$, $(x-1) + 1 = x \in R_0 \otimes R_1$. Given any polynomial in $R$, we can find a linear combination of $c$ and $x$ to mirror it in $\bigoplus_{n \in \mathbb{N}} R_n$. Thus $R$ is contained in $\bigoplus_{n \in \mathbb{N}} R_n$, and by subset inclusion, $R = \bigoplus_{n \in \mathbb{N}} R_n$.\[4\]
Finally, we will show that $R_n \cdot R_m \subseteq R_{m+n}$. Let $c(1-x)^n \in R_n$ and $d(1-x)^m \in R_m$. Then
\[ c(1-x)^n \cdot d(1-x)^m = c \cdot d(1-x)^{m+n} \in R_n \cdot R_m. \]

But since $c \cdot d \in k$, $c \cdot d(1-x)^{m+n} \in R_{m+n}$. Thus $R_n \cdot R_m \subseteq R_{m+n}$.

Since there exist additive subgroups $R_n$ of $R$ such that $R = \bigoplus_{n \in \mathbb{N}} R_n$ and $R_n \cdot R_m \subseteq R_{m+n}$, $R$ is an $\mathbb{N}$-graded ring. 

(b) Prove that $I = (x)$ is not an homogeneous ideal of $R$.

Note: This looks like a monomial ideal; however, it is not with this grading.

**Proof:** An ideal is homogeneous if and only if for every $x \in I$ the homogeneous components of $x$ are contained in $I$. Since $1(x-1)^1 = (1-x) \in R_1$ and $1(x-1)^0 = 1 \in R_0$, and $(x-1) + 1 = x \in (x)$, it follows that $x-1 \in I$, but this is not the case. Therefore $(x)$ is not a homogeneous ideal of $R$. 

4.6) Assuming that all units in a $\mathbb{Z}$-graded domain are homogeneous, prove that if $R$ is a $\mathbb{Z}$-graded field, then $R$ is concentrated in degree 0, meaning $R_0 = R$ and $R_n = 0$ for all $|n| \geq 1$.

4.7) Let $R$ be a $\mathbb{Z}$-graded ring and $I$ be an ideal of $R_0$. Prove that $IR \cap R_0 = I$.

**Solution.** Since $R$ is a $\mathbb{Z}$-graded ring, we have
\[ R = \bigoplus_{n \in \mathbb{Z}} R_n \]
and $R_n \cdot R_m \subseteq R_{n+m}$. Let $x \in IR \cap R_0$. Then $x \in IR$ and $x \in R_0$. Since $x \in IR$, $x$ is a finite sum of elements of the form $ir$ where $i \in I$, $r \in R$. So we start by considering elements of the form $ir$. Since $R$ is $\mathbb{Z}$-graded, we can write $x = i(r_{-2}, r_{-1}, r_0, r_1, r_2, \ldots)$ where $r_i \in R_i$. So $x = \sum ir_{-2} + ir_{-1} + irr_0 + irr_1 + irr_2 + \ldots$. Note that since any element of $I$ has degree 0, $ir_i$ only has degree 0 when $r_i \in R_0$. Since $x \in R_0$, $x = ir_0$ because $ir_i = 0$ for $i \neq 0$. Thus $x \in I$ since $I$ is an ideal of $R_0$. Now we consider elements of the form $x = \sum i_n r_n$ where the sum is finite. Since each $i_n r_n \in I$, $x = \sum i_n r_n \in I$.

Now let $x \in I$. Since $R$ is a commutative ring with unity, we can write $x = x \cdot 1 \in IR$. Since $I$ is an ideal of $R_0$, $I \subseteq R_0$, so $x \in R_0$. Therefore, $x \in IR \cap R_0$. This proves that $IR \cap R_0 = I$.

4.8) Let $R$ be a nonnegatively graded ring and $I_0$ an ideal of $R_0$. Prove that
\[ I = I_0 \oplus R_1 \oplus R_2 \oplus \cdots \]
is an ideal of $R$. Also, show that $\mathfrak{M}$ is a homogeneous maximal ideal of $R$ if and only if
\[ \mathfrak{M} = m \oplus R_1 \oplus R_2 \oplus \cdots \]
for some maximal ideal $m$ of $R_0$. 

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Proof: Let $x = \sum_i x_i, y = \sum_i y_i$ be elements of $I$ where $x_0, y_0$ are in $I_0$ and $x_i, y_i$ are in $R_i$ for $i > 0$ and let $r$ be an element of $R$. Since $I_0$ is an ideal of $R_0$ and $R_i$ is an abelian group we know that $x_0 + y_0$ is in $I_0$ and $x_i + y_i$ is in $R_i$ for $i > 0$ and hence $x + y = \sum_i x_i + y_i$ is an element of $I$. Now consider the product $rx$. Given that we have shown $I$ to be closed under addition we may assume that $r$ is homogeneous and contained in $R_n$. If $n = 0$ then since $I_0$ is an ideal of $R_0$ we know $rx_0$ is an element of $I_0$ and $rx_i$ is an element of $R_i$ so $rx = \sum_i rx_i$ is in $I$. If $n > 1$ then $rx_i \in R_{n+i} \neq R_0$ so $rx = \sum_i rx_i$ is in $I$ as well. Thus $I$ is an ideal.

Now suppose $\mathfrak{M}$ is of the form above, then consider the quotient $R/\mathfrak{M} = \bigoplus_{i} R_i/\mathfrak{M}_i$ where $M_i$ is the graded component of $\mathfrak{M}$. If we notice that

$$\bigoplus_{i} R_i/\mathfrak{M}_i \cong R_0/\mathfrak{m} \oplus R_1/R_1 \oplus R_2/R_2 \oplus \cdots$$

then $R/\mathfrak{M}$ is a field since $R_i/R_i \cong 0$ and $R_0/\mathfrak{m}$ is a field because $\mathfrak{m}$ is maximal in $R_0$. We conclude that $\mathfrak{M}$ is maximal in $R$.

Conversely suppose that $\mathfrak{M}$ is a homogeneous maximal ideal in $R$. Since $I$ is homogeneous we can write

$$\mathfrak{M} = R_0 \cap \mathfrak{M} \oplus R_1 \cap \mathfrak{M} \oplus \cdots.$$ 

But then we know $R/\mathfrak{M}$ is a field since $\mathfrak{M}$ is maximal and from a previous problems (4.2, 4.6) we know that $R/\mathfrak{M}$ has an induced grading from $R$ and hence the quotient ring is concentrated in degree zero, that is

$$R/\mathfrak{M} = R_0/(R_0 \cap \mathfrak{M}).$$

We can then conclude that the zero degree component of $\mathfrak{M}$, $R_0 \cap \mathfrak{M}$, is maximal in $R_0$ and that positive degree components are just

$$R_i \cap \mathfrak{M} = R_i \quad i > 0.$$ 

\[\square\]

4.9) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the integer function defined by $f(n) = n!$ for $n > 1$ and $f(n) = 0$ for $n \leq 0$. Show that $f$ is not of polynomial type.

4.10) Let $k$ be a field. Suppose the following rings have the standard grading.

(a) If $R = k[x, y, z]$, compute $HF_R(n)$ for all $n \geq 0$.

(b) If $R = k[x, y, z, w]$, compute $HF_R(n)$ for all $n \geq 0$.

(c) If $R = k[x_1, \ldots, x_i]$, compute $HF_R(n)$ for all $n \geq 0$.

For each of the cases above, what is the respective Hilbert polynomial and Hilbert series?
Solution.: In order to see examples, we will pick our favorite field to work with and the following are the commands that should be entered into MacCauly 2:

a) \( k = \mathbb{Z}/5 \)
\( R = \mathbb{Z}/5[x, y, z] \)

\( \text{hilbertFunction}(0, R) \)
\( \text{hilbertPolynomial}(R) \)
\( \text{hilbertSeries}(R) \)

note: The function command returns how many generators are there for that degree and you can find another degree, say 2, by replacing the 0. The \( \text{hilbertSeries} \) is \( \frac{1}{1-t^3} \) and the \( \text{hilbertPolynomial} \) is \( \frac{1}{6}t^3 + \frac{3}{2}t + 1 \).

b) \( k = \mathbb{Z}/5 \)
\( R = \mathbb{Z}/5[x, y, z, w] \)

Notice the only thing changing in the commands is that there are four variables now, but the commands will be the exact same! The \( \text{hilbertSeries} \) is \( \frac{1}{1-t^4} \) and the \( \text{hilbertPolynomial} \) is \( \frac{1}{6}t^3 + t^2 + \frac{11}{6}t + 1 \).

c) This one can’t be done in MacCauly 2 since it doesn’t read an infinite number of variables, but you play around and input all different sizes of variables to try to catch the pattern. The \( \text{hilbertSeries} \) will be \( \frac{1}{(1-t)^i} \) where \( i \) is the number of variables you input.

4.11) Let \( k \) be a field. Suppose the following rings have the standard grading.

(a) If \( R = k[x^3] \), compute \( HF_R(n) \) for all \( n \geq 0 \).
(b) If \( R = k[x^3, x^5] \), compute \( HF_R(n) \) for all \( n \geq 0 \).
(c) If \( R = k[x, y^2] \), compute \( HF_R(n) \) for all \( n \geq 0 \).

For each of the cases above, what is the respective Hilbert series?

Solution.: In this problem, we use Macaulay 2 to help find solutions. We have to designate \( k \) to be a particular field; we used \( \mathbb{Z}/2 \). The command for computing the Hilbert function is \( \text{hilbertFunction}(n) \), where the polynomial ring is assumed, and \( n \) is the degree.

(a) In this ring, since we have the standard grading, all elements will have degree which is a multiple of 3. Using Macaulay 2 to test our intuition, we find that we are correct, and

\[
HF_R(n) = \begin{cases} 
1 & n \equiv 0 \pmod{3} \\
0 & \text{else}.
\end{cases}
\]

In computing the Hilbert series, we input \( \text{hilbertSeries}(R) \) into Macaulay 2, and what we get is:

\[
\text{HS}(R) = \frac{1}{(1-t)^3}.
\]

Note that in problem 10, we computed the Hilbert polynomial to find a general formula for the Hilbert function. That only works in Macaulay 2 when the variables have degree one.
(b) To compute the Hilbert series, first input the ring: \( R = \mathbb{Z}/2[x,x,\text{Degrees}\Rightarrow 3,5] \).
Then, compute the Hilbert series: \( \text{HS} = \text{hilbertSeries}(R) \) to obtain
\[
\text{HS}(R) = \frac{1}{(1-t^3)(1-t^5)}.
\]
(c) We input this ring into Macaulay 2 using the command \( R = \mathbb{Z}/2[x,y,\text{Degrees}\Rightarrow 1,2] \) to designate the degrees of the variables. We then compute the Hilbert function for several values of \( n \), and see a pattern begin to emerge. We get the equation
\[
\text{HF}_R(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1,
\]
where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).

4.12) Let \( R \) be a graded ring and \( M = \bigoplus_{i=1}^{\infty} M_i \) a finitely generated graded \( R \)-module. Prove \( \text{Ann}(M) \) is a homogeneous ideal.

**Proof:** Let \( R \) be a graded ring and \( M = \bigoplus_{i=1}^{\infty} M_i \) a finitely generated graded \( R \)-module. Then there exists a homogeneous generating set \((m_1, m_2, ..., m_l)\), where \( m_i \in R_i \). From a theorem in class it is enough to show that if \( x \in \text{Ann}(M) \) and we write \( x = \sum_{i=0}^{t} x_i \), where all \( x_i \in R_i \), \( \text{Ann}(M) \) is a homogeneous ideal if \( x_i \in \text{Ann}(M) \) for all \( 0 \leq i \leq t \). Note that \( x \) must be composed of only finitely many \( x_i \in R_i \), since \( M \) is an infinite sum and not product. Next let \( x \in \text{Ann}(M) \). Also note that \( xm_i = 0 \) for all the \( m_i \) in the homogeneous generating set of \( M \). So let’s take the first generator \( m_0 \),
\[
xm_0 = (x_0m_0 + x_1m_0 + ... + x_tm_0) = 0, \quad \text{with } x_i \in R_i, \text{ and } m_0 \in M_0.
\]
If we look at the degree of each part of the sum we see \( \deg(x_{i-1}m_0) < \deg(x_im_0) \) for all \( 1 \leq i \leq t \). Therefore \( x_im_0 = 0 \) for all \( 0 \leq i \leq t \), and hence \( x_i \in \text{Ann}(M) \) for all \( 0 \leq i \leq t \).

4.13) Let \( H(t) = \sum_{n=0}^{\infty} a_n t^n \) be an infinite series with nonnegative integer coefficients, and assume that \( H(t) = \frac{L(t)}{(1-t)^s} \), where \( L(1) \neq 0 \) and \( L(t) = b_s t^s + b_{s+1} t^{s+1} + \cdots + b_r t^r \), with each \( b_i \in \mathbb{Z} \), \( b_s \neq 0 \), \( b_r \neq 0 \). Prove that \( a_n = 0 \) for all \( n < s \) and there exists a polynomial \( P(t) \) such that \( P(n) = a_n \) for all \( n \geq r \).

4.14) Let \( R \) be an \( \mathbb{N} \)-graded ring that is generated in degree one. For an ideal \( I \) of \( R \), let \( I^s \) denote the ideal of \( R \) generated by the homogeneous elements of \( I \). Prove that if \( P \) is a prime ideal then \( P^s \) is a prime ideal.

4.15) Let \( R \) be a graded ring and
\[
0 \rightarrow M_k \rightarrow M_{k-1} \rightarrow \cdots \rightarrow M_0 \rightarrow 0
\]
an exact sequence of graded $R$-modules with degree 0 maps. Prove that $\sum_i (-1)^i HS_{M_i}(t) = 0$.

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4.16) Prove that all units in a $\mathbb{Z}$-graded domain are homogeneous.

4.17) Suppose $I$ is a homogeneous ideal of a $\mathbb{Z}$-graded ring $R$. Prove that $\sqrt{I}$ is homogeneous.