IMMERSE 2008: Assignment 1

1.1) Let $R$ be a ring. Prove
(a) $0a = a0 = 0$ for all $a \in R$.
(b) $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.
(c) $(-a)(-b) = ab$ for all $a, b \in R$.
(d) If $R$ has an identity 1, then the identity is unique and $-a = (-1)a$.

1.2) Problems involving zerodivisors:
(a) Prove that a unit element of a ring cannot be a zerodivisor.
(b) Let $a$ and $b$ be elements of a ring whose product $ab$ is a zerodivisor. Show that either $a$ or $b$ is a zerodivisor.
(c) Is the sum of two zerodivisors necessarily a zerodivisor? If so, give a prove. If not, give a counterexample.

1.3) Let $R$ be an integral domain. Determine the units of $R[x]$.

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1.5) Let $A$ be the ring of all functions from $[0, 1]$ to $\mathbb{R}$.
(a) What are the units of $A$?
(b) Prove that if $f$ is not a unit and not zero, then $f$ is a zero divisor.

1.6) Let $A$ be the ring of all continuous functions from $[0, 1]$ to $\mathbb{R}$.
(a) What are the units of $A$?
(b) Give an example of an element which is neither a unit nor a zero divisor.
(c) Give an example of a zero divisor in $A$.

1.7) Determine whether the following polynomials are irreducible in the rings indicated. For those that are reducible, determine their factorization into irreducibles. The notation $\mathbb{F}_p$ denotes the finite field $\mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime.
(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$.
(b) $x^3 + x + 1$ in $\mathbb{F}_3[x]$.
(c) $x^4 + 1$ in $\mathbb{F}_5[x]$.
(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$.

1.8) Let $R$ be a non-zero ring. Prove that the following are equivalent:
(a) $R$ is a field.
(b) The only ideals in $R$ are $(0)$ and $(1)$.
(c) Every homomorphism of $R$ into a non-zero ring $B$ is injective.
1.9) Let $f : R \rightarrow S$ be a ring homomorphism.

(a) Prove that $\ker f$ is an ideal of $R$.
(b) Prove that if $J$ is an ideal of $S$ then $f^{-1}(J)$ is an ideal of $R$ that contains $\ker f$.
(c) Prove that if $P$ is a prime ideal of $S$ then $f^{-1}(P)$ is a prime ideal of $R$.

1.10) Let $p$ be a prime and consider the ring of polynomials in $x$ with coefficients in $\mathbb{F}_p$. This ring is denoted by $\mathbb{F}_p[x]$. Let $\varphi : \mathbb{F}_p[x] \rightarrow \mathbb{F}_p[x]$ be the map given by $\varphi(f) = f^p$. Prove that $\varphi$ is an endomorphism. This map is called the Frobenius endomorphism.

1.11) Let $S \subseteq R$ and let $I$ be an ideal of $R$. Prove that the following statements are equivalent:

(a) $S \subseteq I$.
(b) $(S) \subseteq I$.

This fact is useful when you want to show that one ideal is contained in another.

1.12) Prove the following equalities in the polynomial ring $R = \mathbb{Q}[x, y]$:

(a) $(x + y, x - y) = (x, y)$.
(b) $(x + xy, y + xy, x^2, y^2) = (x, y)$.
(c) $(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = (x^2 - 4, y^2 - 1)$.

This illustrates that the same ideal can have many different generating sets and that different generating sets may have different numbers of elements.

1.13) Let $R$ be a ring and let $I, J$ and $K$ be ideals of $R$.

(a) Prove that $I \cap J$ is an ideal of $R$.
(b) Prove that $I(J + K) = IJ + IK$.
(c) Prove that if either $J \subseteq I$ or $K \subseteq I$ then $I \cap (J + K) = I \cap J + I \cap K$. (modular law)

1.14) In the ring of integers $\mathbb{Z}$ compute the ideals:

(a) $(2) + (3)$,
(b) $(2) + (4)$,
(c) $(2)((3) + (4))$,
(d) $(2)(3) \cap (2)(4)$,
(e) $(6) \cap (8)$,
(f) $(6)(8)$

1.15) Let $\mathbb{Q}[x, y]$. Compute the ideals:

(a) $(x) \cap (y)$,
(b) $(x + y)^2$,
(c) $(x, y)^2$,
(d) $(x^2) \cap (x, y)$,
1.16) Let $R$ be a ring. The nilradical $\sqrt{0}$ of $R$ is the set of nilpotent elements of $R$. Prove that $\sqrt{0}$ is an ideal, and if $\overline{x}^n = 0$ in $R/\sqrt{0}$ for some $n$ then $\overline{x} = 0$.

1.17) Let $R$ be a ring, and $I \subseteq \sqrt{0}$ an ideal, where $\sqrt{0}$ is the nilradical of $R$. Prove that if $\overline{x}$ is a unit of $R/I$ then $x$ is a unit of $R$.

1.18) Let $R$ be a ring and $P$ a prime ideal of $R$. Let $I$ be the ideal generated by all the idempotent elements of $P$. Prove that $R/I$ has no non-trivial idempotents.

**IMMERSE 2008: Extras 1**

1.19) A (not necessarily commutative) ring $R$ is *boolean* if $x^2 = x$ for all $x \in R$. If $R$ is a boolean ring show that

(a) $2x = 0$ for all $x \in R$,
(b) $R$ is commutative,
(c) every prime ideal $p$ is maximal, and $R/p$ is a field with two elements, and
(d) every finitely generated ideal is principal.

1.20) Let $R$ be a ring in which every ideal of $R$ except (1) is prime. Prove that $R$ is a field.

1.21) For ideals $I$ and $J$ in a ring $R$ their ideal quotient is

$$(I :_R J) = \{x \in R \mid xJ \subseteq I\}.$$  

Let $R$ be a ring and let $P$ be a finitely generated prime ideal of $R$ with $(0 :_R P) = 0$. Prove that $(P :_R P^2) = P$.

1.22) Let $K$ be a field and let $R$ be the ring of polynomials in $x$ over $K$ subject to the condition that they contain no terms in $x$ or $x^2$. Let $I$ be the ideal in $R$ generated by $x^3$ and $x^4$. Prove that $x^5 \not\in I$ and $x^5I \subseteq I^2$. (This shows that the assumption that $P$ is prime in 1.21 is necessary.)

1.23) Let $F$ be a field and let $E = F \times F$. Define addition and multiplication in $E$ by the rules:

$$(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b)(c, d) = (ac - bd, ad + bc)$$

Determine conditions on $F$ under which $E$ is a field.

1.24) Find all the monic irreducible polynomials of degree less than or equal to 3 in $\mathbb{F}_2[x]$, and the same in $\mathbb{F}_3[x]$.  

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1.25) Construct fields of each of the following orders:

(a) 9
(b) 49
(c) 8
(d) 81

1.26) Exhibit all the ideals in the ring $F[x]/(p(x))$, where $F$ is a field and $p(x)$ is a polynomial in $F[x]$ (describe them in terms of the factorizations of $p(x)$).

1.27) An element $x$ of a ring is idempotent if $x^2 = x$. Prove that a local ring contains no non-trivial idempotents. (The trivial idempotents are 0 and 1.)