# IMMERSE 2008: Assignment 1

**1.1)** Let $R$ be a ring. Prove

(a) $0a = a0 = 0$ for all $a \in R$.

(b) $(-a)b = a(-b) = -(ab)$ for all $a, b \in R$.

(c) $(-a)(-b) = ab$ for all $a, b \in R$.

(d) If $R$ has an identity 1, then the identity is unique and $-a = (-1)a$.

**1.2)** Problems involving zerodivisors:

(a) Prove that a unit element of a ring cannot be a zerodivisor.

(b) Let $a$ and $b$ be elements of a ring whose product $ab$ is a zerodivisor. Show that either $a$ or $b$ is a zerodivisor.

(c) Is the sum of two zerodivisors necessarily a zerodivisor? If so, give a prove. If not, give a counterexample.

**1.3)** Let $R$ be an integral domain. Determine the units of $R[x]$.

**1.4)** Let $R$ be an integral domain. Determine the units of $R[\![x]\!]$.

**1.5)** Let $A$ be the ring of all functions from $[0, 1]$ to $\mathbb{R}$.

(a) What are the units of $A$?

(b) Prove that if $f$ is not a unit and not zero, then $f$ is a zero divisor.

**1.6)** Let $A$ be the ring of all continuous functions from $[0, 1]$ to $\mathbb{R}$.

(a) What are the units of $A$?

(b) Give an example of an element which is neither a unit nor a zero divisor.

(c) Give an example of a zero divisor in $A$.

**1.7)** Determine whether the following polynomials are irreducible in the rings indicated. For those that are reducible, determine their factorization into irreducibles. The notation $\mathbb{F}_p$ denotes the finite field $\mathbb{Z}/p\mathbb{Z}$, where $p$ is a prime.

(a) $x^2 + x + 1$ in $\mathbb{F}_2[x]$.

(b) $x^3 + x + 1$ in $\mathbb{F}_3[x]$.

(c) $x^4 + 1$ in $\mathbb{F}_5[x]$.

(d) $x^4 + 10x^2 + 1$ in $\mathbb{Z}[x]$.

**1.8)** Let $R$ be a non-zero ring. Prove that the following are equivalent:

(a) $R$ is a field.

(b) The only ideals in $R$ are $(0)$ and $(1)$.

(c) Every homomorphism of $R$ into a non-zero ring $B$ is injective.

**1.9)** Let $f : R \to S$ be a ring homomorphism.

(a) Prove that $\operatorname{Ker} f$ is an ideal of $R$.

(b) Prove that if $J$ is an ideal of $S$ then $f^{-1}(J)$ is an ideal of $R$ that contains $\operatorname{Ker} f$.

(c) Prove that if $P$ is a prime ideal of $S$ then $f^{-1}(P)$ is a prime ideal of $R$.

**1.10)** Let $p$ be a prime and consider the ring of polynomials in $x$ with coefficients in $\mathbb{F}_p$. This ring is denoted by $\mathbb{F}_p[x]$. Let $\varphi : \mathbb{F}_p[x] \to \mathbb{F}_p[x]$ be the map given by $\varphi(f) = f^p$. Prove that $\varphi$ is an endomorphism. This map is called the **Frobenius endomorphism**.

**1.11)** Let $S \subseteq R$ and let $I$ be an ideal of $R$. Prove that the following statements are equivalent:

(a) $S \subseteq I$.

(b) $(S) \subseteq I$.

This fact is useful when you want to show that one ideal is contained in another.

**1.12)** Prove the following equalities in the polynomial ring $R = \mathbb{Q}[x, y]$:

(a) $(x + y, x - y) = (x, y)$.

(b) $(x + xy, y + xy, x^2, y^2) = (x, y)$.

(c) $(2x^2 + 3y^2 - 11, x^2 - y^2 - 3) = (x^2 - 4, y^2 - 1)$.

This illustrates that the same ideal can have many different generating sets and that different generating sets may have different numbers of elements.

**1.13)** Let $R$ be a ring and let $I, J$ and $K$ be ideals of $R$.

(a) Prove that $I \cap J$ is an ideal of $R$.

(b) Prove that $I(J + K) = IJ + IK$.

(c) Prove that if either $J \subseteq I$ or $K \subseteq I$ then $I \cap (J + K) = I \cap J + I \cap K$. (modular law)

**1.14)** In the ring of integers $\mathbb{Z}$ compute the ideals:

(a) $(2) + (3)$,

(b) $(2) + (4)$,

(c) $(2)((3) + (4))$,

(d) $(2)(3) \cap (2)(4)$,

(e) $(6) \cap (8)$,

(f) $(6)(8)$

**1.15)** Let $\mathbb{Q}[x, y]$. Compute the ideals:

(a) $(x) \cap (y)$,

(b) $(x + y)^2$,

(c) $(x, y)^2$,

(d) $(x^2) \cap (x, y)$,

(e) $(x^2 + xy) \cap (xy + y^2)$,
(f) $(x) + (y)$,
(g) $(x + 1) + (x)$,
(h) $(x^2 + xy)(x - y)$,
(i) $(x^2) \cap ((xy) + (y^2))$,
(j) $(x - y)((x) + (y^2))$

**1.16)** Let $R$ be a ring. The nilradical $\sqrt{0}$ of $R$ is the set of nilpotent elements of $R$. Prove that $\sqrt{0}$ is an ideal, and if $\overline{x}^n = 0$ in $R/\sqrt{0}$ for some $n$ then $\overline{x} = 0$.

**1.17)** Let $R$ be a ring, and $I \subseteq \sqrt{0}$ an ideal, where $\sqrt{0}$ is the nilradical of $R$. Prove that if $\overline{x}$ is a unit of $R/I$ then $x$ is a unit of $R$.

**1.18)** Let $R$ be a ring and $P$ a prime ideal of $R$. Let $I$ be the ideal generated by all the idempotent elements of $P$. Prove that $R/I$ has no non-trivial idempotents.

## IMMERSE 2008: Extras 1

**1.19)** A (not neccessarily commutative) ring $R$ is *boolean* if $x^2 = x$ for all $x \in R$. If $R$ is a boolean ring show that

(a) $2x = 0$ for all $x \in R$,
(b) $R$ is commutative,
(c) every prime ideal $p$ is maximal, and $R/p$ is a field with two elements, and
(d) every finitely generated ideal is principal.

**1.20)** Let $R$ be a ring in which every ideal of $R$ except (1) is prime. Prove that $R$ is a field.

**1.21)** For ideals $I$ and $J$ in a ring $R$ their ideal quotient is

$$(I :_R J) = \{x \in R \mid xJ \subseteq I\}.$$

Let $R$ be a ring and let $P$ be a finitely generated prime ideal of $R$ with $(0 :_R P) = 0$. Prove that $(P :_R P^2) = P$.

**1.22)** Let $K$ be a field and let $R$ be the ring of polynomials in $x$ over $K$ subject to the condition that they contain no terms in $x$ or $x^2$. Let $I$ be the ideal in $R$ generated by $x^3$ and $x^4$. Prove that $x^5 \notin I$ and $x^5 I \subseteq I^2$. (This shows that the assumption that $P$ is prime in 1.21 is necessary.)

**1.23)** Let $F$ be a field and let $E = F \times F$. Define addition and multiplication in $E$ by the rules:

$$(a, b) + (c, d) = (a + c, b + d) \qquad \text{and} \qquad (a, b)(c, d) = (ac - bd, ad + bc)$$

Determine conditions on $F$ under which $E$ is a field.

**1.24)** Find all the monic irreducible polynomials of degree less than or equal to 3 in $\mathbb{F}_2[x]$, and the same in $\mathbb{F}_3[x]$.

**1.25)** Construct fields of each of the following orders:

(a) 9

(b) 49

(c) 8

(d) 81

**1.26)** Exhibit *all* the ideals in the ring $F[x]/(p(x))$, where $F$ is a field and $p(x)$ is a polynomial in $F[x]$ (describe them in terms of the factorizations of $p(x)$).

**1.27)** An element $x$ of a ring is *idempotent* if $x^2 = x$. Prove that a local ring contains no non-trivial idempotents. (The trivial idempotents are 0 and 1.)