IMMERSE 2008: Assignment 2

2.1) (a) Prove that M is an abelian group if and only if M is a $\mathbb{Z}$-module.

(b) Is it true that every abelian group is also a $\mathbb{Q}$-module? If so give a proof. If not give a counterexample.

2.2) Let A be a ring and let F be an A-module which has a basis $X = \{f_1, \dots, f_i, \dots\}$. Prove that the map

$$\varphi : F \to \bigoplus_{|X|} A$$
$$f_i \mapsto (0, \dots, 0, \underbrace{1}_{i\text{th spot}}, 0, \dots)$$

is an isomorphism of A-modules.

2.3) Let M and N be A-modules and let $\alpha : M \to N$ and $\beta : N \to M$ be A-module homomorphisms. Prove that if $\beta \circ \alpha = \mathbb{1}_M$ and $\alpha \circ \beta = \mathbb{1}_N$, then α and β are isomorphisms.

2.4) If M and N are submodules of an R-module K prove that

(a) $M \cap N$ is a submodule of K, and

(b) $M + N = \{m + n : m \in M, n \in N\}$ is a submodule of K.

(c) If $M+N$ and $M \cap N$ are finitely generated then M and N are finitely generated.

2.5) Let M be an R-module and N a submodule. Prove that if N and M/N are finitely generated then M is finitely generated.

2.6) Let M and N be submodules of K such that $K = M \oplus N$. Show that M and N are finitely generated if and only if K is finitely generated.

2.7) Let M be an R-module and let $\text{Ann}(M) = \{x \in R : xM = 0\}$. Prove that $\text{Ann}(M)$ is an ideal of R.

2.8) Let M be an R-module and let I and J be ideals of R. Prove that

$$I\left(\frac{M}{JM}\right) \simeq \frac{(I+J)M}{JM}.$$

2.9) Let R be a ring and let I and J be ideals of R. If R/I and R/J are isomorphic as rings, prove $I \simeq J$ as R-modules.

2.10) Let R be a ring and let I and J be ideals of R. If the R-modules R/I and R/J are isomorphic, prove $I = J$. Note, we really do mean *equals*!

2.11) Let V be the set of all infinite sequences of real numbers. In other words, any $\mathbf{v} \in V$ is of the form $(v_1, v_2, v_3, \dots)$ with $v_i \in \mathbb{R}$. Define addition by

$$(u_1, u_2, u_3, \dots) + (v_1, v_2, v_3, \dots) = (u_1 + v_1, u_2 + v_2, u_3 + v_3, \dots)$$

and scalar multiplication by

$$r(u_1, u_2, u_3, \dots) = (ru_1, ru_2, ru_3, \dots)$$

for $r \in \mathbb{R}$. Then V is a vector space over $\mathbb{R}$. Determine whether or not each of the following subsets of V is a subspace of V:

(a) All sequences containing only a finite number of nonzero terms.

(b) All sequences of the form $(u_1, u_2, u_3, \dots, u_n, 0, 0, \dots)$ where n is a fixed positive integer.

(c) All decreasing sequences, i.e., sequences where $u_{i+1} \leq u_i$ for all i.

(d) All convergent sequences, i.e., sequences for which $\lim_{i\to\infty} u_i$ exists.

2.12) Is $\mathbb{Q} \simeq \mathbb{Q} \oplus \mathbb{Q}$ as a $\mathbb{Z}$-module? Justify your answer with a proof.

2.13) Is $\mathbb{R} \simeq \mathbb{R} \oplus \mathbb{R}$ as a $\mathbb{Z}$-module? Justify your answer with a proof.

2.14) Let F be a field.

(a) Prove that every field is either of characteristic 0 or is of characteristic p, where p is a prime integer.

(b) Prove that a finite field has characteristic p, where p is a prime integer.

(c) Prove that every field is either a $\mathbb{Z}_p$-vector space or is a $\mathbb{Q}$-vector space.

(d) Suppose further that F is a finite field. Show that the order of F is p^n.

(e) Can there be a finite field of order 6?

2.15) Let R be a ring. Prove that if every R-module is free, then R must be a field. [Hint: it's enough to prove that R has no proper non-zero ideals.]

2.16) Compute the kernel and image of the following homomorphism:

$$\Phi = \begin{bmatrix} xy & xz & yz \\ x-1 & y-1 & z-1 \end{bmatrix}$$

$$A^3 \xrightarrow{\Phi} A^2$$

where $A = \mathbb{Q}[x, y, z]$.

2.17) Show that $\mathbb{Q}[x]$ is a $\mathbb{Q}$-vector space, that is, a free $\mathbb{Q}$-module. Additionally, show the following:

(a) Given $p_0(x), p_1(x), p_2(x), \dots \in \mathbb{Q}[x]$ such that $\deg(p_n) = n$, show that $p_0, p_1, p_2, ...$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[x]$.

(b) Show that $p_n(x) = \binom{x+n}{n}$ for $n \in \mathbb{Z}$, $n \geq 0$ is a $\mathbb{Q}$-basis for $\mathbb{Q}[x]$.

2.18) A polynomial $p(x) \in \mathbb{Q}[x]$ is called **integer-valued** if $p(x) \in \mathbb{Z}$ for all $x \in \mathbb{Z}$. We will denote the set of integer valued polynomials by $\mathbb{IP}$.

(a) Explain why $\mathbb{IP} \neq \mathbb{Z}[x]$.

(b) Prove that $\mathbb{IP}$ is a $\mathbb{Z}$-module.

(c) Show that any $p(x) \in \mathbb{IP}$ of degree d may be written as:

$$p(x) = \sum_{n=0}^{d} a_n \binom{x+n}{n} \qquad a_i \in \mathbb{Q}$$

(d) Define $\Delta p(x) := p(x+1) - p(x)$ and define $\Delta^i p(x)$ to be the ith iteration of Δ. Show that

$$\Delta^i p(x) = \sum_{n=0}^{d-i} a_{n+i} \binom{x+i+n}{n}$$

and conclude that $\Delta^n p(-i-1) = a_i$.

(e) Show that the set of integer-valued polynomials is a free $\mathbb{Z}$-module with basis:

$$\left\{ \binom{x+n}{n} : n \in \mathbb{N} \right\}$$

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2.19) Let R be a ring and A and B be R-algebras.

(a) If $A \simeq B$ as R-modules, is $A \simeq B$ as rings? Prove or disprove your conclusion.

(b) If $A \simeq B$ as rings, is $A \simeq B$ as R-modules? Prove or disprove your conclusion.

2.20) Let F be a field and let $E = F \times F$. Define addition and multiplication in E by the rules:

$$(a,b) + (c,d) = (a+c, b+d) \qquad \text{and} \qquad (a,b)(c,d) = (ac-bd, ad+bc)$$

(a) Determine conditions on F under which E is a field.

(b) Explain how one can view $\mathbb{Z}_3 \times \mathbb{Z}_3$ as a $\mathbb{Z}_3$-algebra.

(c) Will the same trick work for $\mathbb{Z}_2$?