IMMERSE 2008: Assignment 3

3.1) Let R be a commutative ring with identity. Consider an exact sequence of finitely generated free R-modules

$$0 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 0.$$

(a) Show that there exists an R-linear map $\gamma : H \to G$ such that $\beta\gamma = \mathrm{id}_H$.

(b) Show that $F \oplus H \simeq \alpha(F) \oplus \gamma(H) = G$.

3.2) Let

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ M' & \xrightarrow{g} & N' \end{array}$$

be a commutative diagram of R-modules and R-module homomorphisms.

(a) Prove that $f(\operatorname{Ker} u) \subseteq \operatorname{Ker} v$. Thus $f|_{\operatorname{Ker} u} : \operatorname{Ker} u \to \operatorname{Ker} v$ is a well-defined R-module homomorphism.

(b) Prove that if f is injective then $f|_{\operatorname{Ker} u}$ is injective.

(c) Let $\overline{g} : \operatorname{Coker} u \to \operatorname{Coker} v$ be defined by $\overline{g}(\overline{m}) = \overline{g(m)}$ for $\overline{m} \in \operatorname{Coker} u$. Prove that $\overline{g}$ is an R-module homomorphism.

(d) Prove that if g is surjective then $\overline{g}$ is surjective.

3.3) Show that the following rings are not Noetherian by exhibiting an explicit infinite increasing chain of ideals:

(a) The ring of continuous real valued functions on $[0, 1]$.

(b) The ring of all functions from any infinite set X to $\mathbb{Z}/2\mathbb{Z}$.

3.4) Prove that the subring $k[x, x^2y, x^3y^2, \ldots, x^iy^{i-1}, \ldots]$ of the polynomial ring $k[x, y]$ is not a Noetherian ring, hence not a finitely generated k-algebra. From this exercise we see that subrings of Noetherian rings need not be Noetherian and subalgebras of finitely generated k-algebras need not be finitely generated.

3.5) Let M be an R-module and N a submodule. Prove that N and M/N are Noetherian if and only if M is Noetherian. Similarly with Artinian in place of Noetherian.

3.6) Consider the following short exact sequence of R-module:

$$0 \to M' \to M \to M'' \to 0$$

Prove that if M has finite length, then $\ell(M) = \ell(M') + \ell(M'')$.

3.7) Prove that submodules, quotient modules, and finite direct sums of Noetherian R-modules are again Noetherian R-modules.

3.8) Suppose I is an ideal in $F[x_1, \ldots, x_n]$ generated by a (possibly infinite) set $\mathcal{S}$ of polynomials. Prove that a finite subset of the polynomials in $\mathcal{S}$ suffice to generate I.

3.9) Let $S^2 = \{(a,b,c) \in \mathbb{R}^3 : a^2 + b^2 + c^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Prove that

$$I = \{f(x,y,z) \in \mathbb{R}[x,y,z] : f(a,b,c) = 0 \text{ for all } (a,b,c) \in S^2\}$$

is a finitely generated ideal in $\mathbb{R}[x,y,z]$.

3.10) Prove that an Artinian integral domain is field. [Use the DCC directly, no major theorems necessary.] From this show that in an Artinian ring, every prime ideal is maximal.

3.11) Let M be an A-module and let N_1, N_2 be submodules of M. If M/N_1 and M/N_2 are Noetherian, so is $M/(N_1 \cap N_2)$. Similarly with Artinian in place of Noetherian.

3.12) Let M be a module and let θ be a module endomorphism of M. Assume that r and s are positive integers which are minimal subject to $\mathrm{Im}(\theta^r) = \mathrm{Im}(\theta^{r+1})$ and $\mathrm{Ker}(\theta^s) = \mathrm{Ker}(\theta^{s+1})$. Prove that $r = s$ and that $M = \mathrm{Im}(\theta^r) \oplus \mathrm{Ker}(\theta^r)$.

3.13) Let M be a Noetherian R-module and $\varphi : M \to M$ be a module homomorphism. Prove that if φ is surjective then it is also injective.

3.14) Let $\alpha : M \to M$ be a surjective homomorphism of modules. Show that α need not be injective if M is Artinian.

3.15) A unique factorization domain (UFD) is an integral domain where every element factors uniquely as a finite product of irreducible elements. Prove that a UFD satisfies the ascending chain condition for principal ideals, but that the ascending chain condition on all ideals need not hold.

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3.16) Let M be a Noetherian R-module and let $I = \mathrm{Ann}_R\, M$. Prove that R/I is a Noetherian ring.

3.17) Let R be a commutative ring, and let $I_1, I_2, \ldots, I_n$ be ideals in R such that $I_1 \cap I_2 \cap \cdots \cap I_n = 0$ and each R/I_i is Noetherian. Prove that R is Noetherian.

3.18) Show that the ring $\mathbb{IP}$ of integer valued polynomials with coefficients in $\mathbb{Q}$ is not Noetherian by exhibiting an explicit infinite increasing chain of ideals. Note, this question seems quite tricky!