IMMERSE 2008: Assignment 4

4.1) Let A be a ring and set $R = A[x_1, \ldots, x_n]$. For each

$$\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{N}^N$$

let $R_{\mathbf{a}} = A \cdot x_1^{a_1} \ldots x_N^{a_N}$. Prove that

$$R = \bigoplus_{\mathbf{a} \in \mathbb{N}^N} R_{\mathbf{a}}$$

is an $\mathbb{N}^N$-graded ring.

4.2) Let R be a graded ring. Prove that if I is a homogeneous ideal of R, then R/I is a homogeneous R-module. That is, show that R/I is generated by homogeneous elements and is hence graded with the inherited grading.

4.3) Let K be a field and $R = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ where we set $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$. Let I be an ideal generated by finitely many monomials. By the previous exercise, $A = R/I$ is a graded R-module. Prove that the monomials of degree (λ, ν) form a basis for $A_{(\lambda,\nu)}$ over K.

4.4) Let $S = k[x_1, \ldots, x_n]$ be a standard graded ring and $f_1, \ldots, f_d$ be homogeneous elements of S of degrees $\alpha_1, \ldots, \alpha_d$ respectively. Prove that $R = S_0[f_1, \ldots, f_d]$ is an $\mathbb{N}$-graded ring where

$$R_n = \left\{ \sum_{m \in \mathbb{N}^d} r_m f_1^{m_1} \cdots f_d^{m_d} : r_m \in S_0 \text{ and } \alpha_1 m_1 + \ldots \alpha_d m_d = n \right\}.$$

4.5) Let k be a field and $R = k[x]$. Set

$$R_n = \{c(x-1)^n : c \in k\}$$

for all $n \in \mathbb{N}$.

(a) Prove that R is an $\mathbb{N}$-graded ring.

(b) Prove that $I = (x)$ is not an homogeneous ideal of R.
Note: This looks like a monomial ideal; however, it is not with this grading.

4.6) Assuming that all units in a $\mathbb{Z}$-graded domain are homogeneous, prove that if R is a $\mathbb{Z}$-graded field, then R is concentrated in degree 0, meaning $R_0 = R$ and $R_n = 0$ for all $|n| \geqslant 1$.

4.7) Let R be a $\mathbb{Z}$-graded ring and I be an ideal of R_0. Prove that $IR \cap R_0 = I$.

4.8) Let R be a nonnegatively graded ring and I_0 an ideal of R_0. Prove that

$$I = I_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

is an ideal of R. Also, show that $\mathfrak{M}$ is a homogeneous maximal ideal of R if and only if

$$\mathfrak{M} = \mathfrak{m} \oplus R_1 \oplus R_2 \oplus \cdots$$

for some maximal ideal $\mathfrak{m}$ of R_0.

4.9) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the integer function defined by

$$f(n) = n!$$

for $n > 1$ and $f(n) = 0$ for $n \leqslant 0$. Show that f is not of polynomial type.

4.10) Let k be a field. Suppose the following rings have the standard grading.

(a) If $R = k[x, y, z]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

(b) If $R = k[x, y, z, w]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

(c) If $R = k[x_1, \ldots, x_i]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

For each of the cases above, what is the respective Hilbert polynomial and Hilbert series?

4.11) Let k be a field. Suppose the following rings have the standard grading.

(a) If $R = k[x^3]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

(b) If $R = k[x^3, x^5]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

(c) If $R = k[x, y^2]$, compute $\mathrm{HF}_R(n)$ for all $n \geqslant 0$.

For each of the cases above, what is the respective Hilbert series?

4.12) Let R be a graded ring and $M = \bigoplus_{i=1}^{\infty} M_n$ a finitely generated graded R-module. Prove $\mathrm{Ann}(M)$ is a homogeneous ideal.

4.13) Let $H(t) = \sum_{n=0}^{\infty} a_n t^n$ be an infinite series with nonnegative integer coefficients, and assume that $H(t) = \frac{L(t)}{(1-t)^d}$, where $L(1) \neq 0$ and $L(t) = b_s t^s + b_{s+1} t^{s+1} + \cdots + b_r t^r$, with each $b_i \in \mathbb{Z}$, $b_s \neq 0$, $b_r \neq 0$. Prove that $a_n = 0$ for all $n < s$ and there exists a polynomial $P(t)$ such that $P(n) = a_n$ for all $n \geq r$.

4.14) Let R be an $\mathbb{N}$-graded ring that is generated in degree one. For an ideal I of R, let I^* denote the ideal of R generated by the homogeneous elements of I. Prove that if P is a prime ideal then P^* is a prime ideal.

4.15) Let R be a graded ring and

$$0 \to M_k \to M_{k-1} \to \cdots \to M_0 \to 0$$

an exact sequence of graded R-modules with degree 0 maps. Prove that $\sum_i (-1)^i \mathrm{HS}_{M_i}(t) = 0$.

IMMERSE 2008: Extras 4

4.16) Prove that all units in a $\mathbb{Z}$-graded domain are homogeneous.

4.17) Suppose I is a homogeneous ideal of a $\mathbb{Z}$-graded ring R. Prove that $\sqrt{I}$ is homogeneous.