IMMERSE 2008: Assignment 5

5.1) Verify that *lex*, *grlex*, and *grevlex* are monomial orders.

5.2) Prove that in $k[x, y]$ *grlex* and *grevlex* define the same monomial order.

5.3) Rewrite the following polynomials, ordering their terms with respect to *lex*, *grlex*, and *grevlex*, and give $\mathrm{LM}(f)$ and $\mathrm{LT}(f)$ in each case.

(a) $f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4$

(b) $f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2$

5.4) Let $I = (x^2y - z, xy - 1)$ and $f = x^3 - x^2y - x^2z + x$.

(a) Compute the remainder when f is divided by $\{x^2y - z, xy - 1\}$ with respect to *lex* and *grlex*.

(b) Compute the remainder when f is divided by $\{xy - 1, x^2y - z\}$ with respect to *lex* and *grlex*.

5.5) Let I be an ideal generated by monomials $g_1, \dots, g_n$. Show that a monomial f is contained in I if and only if there exists a monomial generator g_i such that $g_i | f$.

5.6) Let I be an ideal of $k[x_1, \dots, x_n]$. Show that $G = \{g_1, \dots, g_t\} \subseteq I$ is a Groebner basis of I if and only if the leading term of any element of I is divisible by one of the $\mathrm{LT}(g_i)$.

5.7) Compute the S-polynomials for $x^4y - z^2$ and $3xz^2 - y$ with respect to *lex* with $x > y > z$, and do it again with $z > y > x$.

5.8) Compute a Gröbner bases for the ideal $I = (x^5+y^4+z^3-1, x^3+y^2+z^2-1) \subseteq \mathbb{Q}[x, y, z]$ with respect to *lex*, *grlex* and *grevlex*. Do the same with $I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)$.

5.9) Compute a Gröbner bases for the ideal $I = (xyz - 1, x^2 + z^3, y^2 - x^3) \subseteq \mathbb{Q}[x, y, z]$ with respect to *lex*, *grlex* and *grevlex*.

5.10) Let $R = K[x_1, \dots, x_N]$ with K a field, and let I be an ideal of R.

(a) Prove that I is a binomial ideal if and only if I has a Gröbner basis consisting of binomials.

(b) Prove that I is a homogeneous ideal if and only if I has a Gröbner basis consisting of homogeneous elements.

5.11) Let $R = K[x_1, \dots, x_N]$ with K a field, and let $F = (f_1, \dots, f_s)$ be an ordered tuple of binomials in R. Prove that a monomial term $c\underline{x}^\alpha = cx_1^{\alpha_1}x_2^{\alpha_2}\dots x_N^{\alpha_N}$ reduces to a monomial term with respect to division by F.

5.12) Find an example of two binomial ideals whose intersection is not a binomial ideal.