IMMERSE 2008: Assignment 5

5.1) Verify that lex, grlex, and grevlex are monomial orders.

5.2) Prove that in \([x, y]\) grlex and grevlex define the same monomial order.

5.3) Rewrite the following polynomials, ordering their terms with respect to lex, grlex, and grevlex, and give \(\text{LM}(f)\) and \(\text{LT}(f)\) in each case.

(a) \(f = 2x^2y^8 - 3x^5yz^4 + xyz^3 - xy^4\)
(b) \(f = 4xy^2z + 4z^2 - 5x^3 + 7x^2z^2\)

5.4) Let \(I = (x^2y - z, xy - 1)\) and \(f = x^3 - x^2y - x^2z + x\).

(a) Compute the remainder when \(f\) is divided by \(\{x^2y - z, xy - 1\}\) with respect to lex and grlex.
(b) Compute the remainder when \(f\) is divided by \(\{xy - 1, x^2y - z\}\) with respect to lex and grlex.

5.5) Let \(I\) be an ideal generated by monomials \(g_1, \ldots, g_n\). Show that a monomial \(f\) is contained in \(I\) if and only if there exists a monomial generator \(g_i\) such that \(g_i | f\).

5.6) Let \(I\) be an ideal of \([x_1, \ldots, x_n]\). Show that \(G = \{g_1, \ldots, g_t\} \subseteq I\) is a Groebner basis of \(I\) if and only if the leading term of any element of \(I\) is divisible by one of the \(\text{LT}(g_i)\).

5.7) Compute the S-polynomials for \(x^4y - z^2\) and \(3xz^2 - y\) with respect to lex with \(x > y > z\), and do it again with \(z > y > x\).

5.8) Compute a Gröbner bases for the ideal \(I = (x^5 + y^4 + z^3 - 1, x^3 + y^2 + z^2 - 1) \subseteq \mathbb{Q}[x, y, z]\) with respect to lex, grlex and grevlex. Do the same with \(I = (x^5 + y^4 + z^3 - 1, x^3 + y^3 + z^2 - 1)\).

5.9) Compute a Gröbner bases for the ideal \(I = (xyz - 1, x^2 + z^3, y^2 - x^3) \subseteq \mathbb{Q}[x, y, z]\) with respect to lex, grlex and grevlex.

5.10) Let \(R = K[x_1, \ldots, x_N]\) with \(K\) a field, and let \(I\) be an ideal of \(R\).

(a) Prove that \(I\) is a binomial ideal if and only if \(I\) has a Gröbner basis consisting of binomials.
(b) Prove that \(I\) is a homogeneous ideal if and only if \(I\) has a Gröbner basis consisting of homogeneous elements.

5.11) Let \(R = K[x_1, \ldots, x_N]\) with \(K\) a field, and let \(F = (f_1, \ldots, f_s)\) be an ordered tuple of binomials in \(R\). Prove that a monomial term \(c\underline{x}^\alpha = cx_1^{\alpha_1}x_2^{\alpha_2} \cdots x_N^{\alpha_N}\) reduces to a monomial term with respect to division by \(F\).

5.12) Find an example of two binomial ideals whose intersection is not a binomial ideal.