A computation with local cohomology

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Goal

- The goal of this presentation is to show you some homological techniques in commutative algebra.
- The example discussed in this talk is a famous example due to Hartshorne. It is discussed in depth in:

  Lectures in Local Cohomology by Craig Huneke with Appendix 1 by Amelia Taylor.

  which can be downloaded from:

  `http://www.math.ku.edu/~huneke/Vita/Preprints.html`

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Saving the day

Setup

Consider the ring $A = k[x, y, u, v]$ and the ideals:

$$I = (x, y)$$
$$J = (u, v)$$

We can take the sum of the ideals

$$I + J = (x, y, u, v)$$

and the intersection of the ideals

$$I \cap J = (xu, xv, yu, yv)$$

Radical of an ideal

Recall the definition of the *radical* of an ideal:

$$\sqrt{I} = \{a \in A : a^t \in I \text{ for some } t > 0\}$$

Question

Recalling $I = (x, y)$, what is $\sqrt{I}$?

Answer

It is pretty clear that $\sqrt{I} = I$.

The same is true for J and $I + J$.

A question and answer

Question

We see that we can find two elements

$$\sqrt{(x,y)} = I$$

Why? Can you find fewer elements that will generate I up to radical?

No. Same is true for J and $I + J$.

Free Radicals

$$\sqrt{I} = \{a \in A : a^t \in I \text{ for some } t > 0\}$$

Question

Recalling $I \cap J = (xu, xv, yu, yv)$, what is $\sqrt{I \cap J}$?

Answer

$\sqrt{I \cap J} = I \cap J$.

Why?

A question and partial answer

Question

We see that we can find four elements

$$\sqrt{(xu, xv, yu, yv)} = I \cap J$$

Can you find fewer elements that will generate $I \cap J$ up to radical?

Answer

Yes!

$$\sqrt{(xu, yv, xv + yu)} = I \cap J$$

Some details

Why is it that $\sqrt{(xu, yv, xv + yu)} = (xu, xv, yu, yv)$?

$$\begin{aligned}(xv)^2 &= (xv)^2 + xvyu - xvyu \\ &= xv(xv + yu) - (xu)(yv)\end{aligned}$$

Hence $(xv) \in \sqrt{(xu, yv, xv + yu)}$.
Hence $\sqrt{(xu, yv, xv + yu)} = \sqrt{(xu, xv, yu, yv)} = I \cap J$.

The question

Question

Ok we can generate $I \cap J$ with three elements up to radical. Can we generate $I \cap J$ with two elements up to radical? What about one element?

We will use “homological methods” to solve this problem.

Complexes

Definition

A **chain complex** is a sequence of A-modules and A-module homomorphisms

$$\cdots \longrightarrow E^{i-1} \xrightarrow{d^{i-1}} E^{i} \xrightarrow{d^{i}} E^{i+1} \longrightarrow \cdots$$

such that $d^i \circ d^{i-1} = 0$ for all $i \in \mathbb{Z}$. We denote a chain complex by $E^\bullet$.

Cohomology

The upshot is that when given a chain complex $(E^\bullet, d^\bullet)$, one has

$$\mathrm{Im}(d^{i-1}) \subseteq \mathrm{Ker}(d^i) \subseteq E^i$$

we can make a new module:

$$H^i(E^\bullet) = \frac{\mathrm{Ker}(d^i)}{\mathrm{Im}(d^{i-1})}$$

called the **ith cohomology** of $E^\bullet$.

How do we make these things?

Question

But where do we get our complexes from?

Answer

This will take some explaining.

Injective modules

If A is noetherian and M is any A-module, then there exists a special module with nice proprieties which we can *inject* M into. The type of module which we desire is called an *injective* module. Specifically, we are looking for the *injective hull* of M.

Aside

How does this relate to free modules?

An injective resolution

Time to build a complex: Start with

$$0 \longrightarrow M \xrightarrow{\iota} E^0 \longrightarrow C^1 \longrightarrow 0$$
$$0 \longrightarrow C^1 \longrightarrow E^1 \longrightarrow C^2 \longrightarrow 0$$
$$0 \longrightarrow C^2 \longrightarrow E^2 \longrightarrow C^3 \longrightarrow 0$$

and so on. Put it all together and it sounds like this:

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 \\
 & & & & & C^2 \arrow[ur] \arrow[dr] & \\
0 \arrow[r] & M \arrow[r, "\iota"] & E^0 \arrow[rr, dashed, "d^0"] \arrow[dr] & & E^1 \arrow[rr, dashed, "d^1"] \arrow[ur] & & \cdots \\
 & & & C^1 \arrow[ur] \arrow[dr] & & & \\
 & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Boring cohomology

Now lose the extraneous parts to get

$$0 \longrightarrow M \xrightarrow{\iota} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow \cdots$$

Note by the construction of our complex, it is necessarily exact.

Question

What is the cohomology?

That's not very interesting.

Functors

Roughly speaking, a **functor** is a mapping of both objects and morphisms. Whatever that means. Consider

$$\Gamma_I(M) = \{a \in M : I^t a = 0 \text{ for some } t > 0\}.$$

So if we have

$$M \xrightarrow{\varphi} N$$

we may write

$$\Gamma_I(M) \xrightarrow{\Gamma_I(\varphi)} \Gamma_I(N)$$

Enter cohomology

We define **local cohomology** as follows:

1. Take an injective resolution $E^\bullet$ of M.
2. Apply $\Gamma_I(-)$ to the resolution above.
3. Take cohomology.

Explicitly:

$$\mathrm{H}_I^i(M) = \frac{\operatorname{Ker} \Gamma_I(d^i)}{\operatorname{Im} \Gamma_I(d^{i-1})}$$

What does that mean?

Here be dragons: Mayer-Vietoris

If A is a noetherian ring, I and J are two ideals, and M is an A-module, then we have a long exact sequence of local cohomology modules:

$$
\begin{array}{l}
0 \longrightarrow \mathrm{H}^0_{I+J}(M) \longrightarrow \mathrm{H}^0_I(M) \oplus \mathrm{H}^0_J(M) \longrightarrow \mathrm{H}^0_{I\cap J}(M) \longrightarrow \\
\longrightarrow \mathrm{H}^1_{I+J}(M) \longrightarrow \mathrm{H}^1_I(M) \oplus \mathrm{H}^1_J(M) \longrightarrow \mathrm{H}^1_{I\cap J}(M) \longrightarrow \\
\longrightarrow \cdots \qquad \cdots \longrightarrow \\
\longrightarrow \mathrm{H}^i_{I+J}(M) \longrightarrow \mathrm{H}^i_I(M) \oplus \mathrm{H}^i_J(M) \longrightarrow \mathrm{H}^i_{I\cap J}(M) \longrightarrow \cdots
\end{array}
$$

Big theorems

Theorem (Invariance up to radical)

Given an ideal I

$$\mathrm{H}^i_I(A) \simeq \mathrm{H}^i_{\sqrt{I}}(A)$$

Theorem (Grothendieck)

An ideal I can be generated by no fewer than n elements up to radical if and only if

$$\mathrm{H}^n_I(A) \neq 0$$

and

$$\mathrm{H}^i_I(A) = 0 \qquad \textit{for all } i > n.$$

Just remember

Remember

$$
\begin{aligned}
I &= (x, y) \\
J &= (u, v) \\
I + J &= (x, y, u, v) \\
I \cap J &= (xu, xv, yu, yv)
\end{aligned}
$$

Don't forget

Remember our Mayer-Vietoris sequence:

$$\cdots \to \mathrm{H}^3_I(A) \oplus \mathrm{H}^3_J(A) \to \mathrm{H}^3_{I\cap J}(A) \to \mathrm{H}^4_{I+J}(A) \to \mathrm{H}^4_I(A) \oplus \mathrm{H}^4_J(A) \to \cdots$$

Remember what Grothendieck said:

$$\cdots \to 0 \to \mathrm{H}^3_{I\cap J}(A) \to \mathrm{H}^4_{I+J}(A) \to 0 \to \cdots$$

and so $\mathrm{H}^3_{I\cap J}(A) \simeq \mathrm{H}^4_{I+J}(A) \neq 0$. Hence we are done! Why?

The end

THE END ?