A computation with local cohomology

Bart Snapp

Department of Mathematics and Statistics
Coastal Carolina University

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Goal

- The goal of this presentation is to show you some homological techniques in commutative algebra.
- The example discussed in this talk is a famous example due to Hartshorne. It is discussed in depth in:

  *Lectures in Local Cohomology* by Craig Huneke with Appendix 1 by Amelia Taylor.

which can be downloaded from:

http://www.math.ku.edu/~huneke/Vita/Preprints.html
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The problem

Complexes and cohomology

Local cohomology

Saving the day
Consider the ring $A = k[x, y, u, v]$ and the ideals:

$$I = (x, y)$$

$$J = (u, v)$$

We can take the sum of the ideals

$$I + J = (x, y, u, v)$$

and the intersection of the ideals

$$I \cap J = (xu, xv, yu, yv)$$
Radical of an ideal

Recall the definition of the radical of an ideal:

\[ \sqrt{I} = \{ a \in A : a^t \in I \text{ for some } t > 0 \} \]

Question

Recalling \( I = (x, y) \), what is \( \sqrt{I} \)?

Answer

It is pretty clear that \( \sqrt{I} = I \).

The same is true for \( J \) and \( I + J \).
A question and answer

Question

We see that we can find two elements

$$\sqrt{(x, y)} = I$$

Why? Can you find fewer elements that will generate $I$ up to radical?

No. Same is true for $J$ and $I + J$. 
Free Radicals

\[ \sqrt{I} = \{ a \in A : a^t \in I \text{ for some } t > 0 \} \]

**Question**

Recalling \( I \cap J = (xu, xv, yu, yv) \), what is \( \sqrt{I \cap J} \)?

**Answer**

\[ \sqrt{I \cap J} = I \cap J. \]

Why?
A question and partial answer

Question

We see that we can find four elements

\[ \sqrt{(xu, xv, yu, yv)} = I \cap J \]

Can you find fewer elements that will generate \( I \cap J \) up to radical?

Answer

Yes!

\[ \sqrt{(xu, yv, xv + yu)} = I \cap J \]
Some details

Why is it that $\sqrt{(xu, yv, xv + yu)} = (xu, xv, yu, yv)$?

$$(xv)^2 = (xv)^2 + xvyu - xvyu$$

$$= xv(xv + yu) - (xu)(yu)$$

Hence $(xv) \in \sqrt{(xu, yv, xv + yu)}$.

Hence $\sqrt{(xu, yv, xv + yu)} = \sqrt{(xu, xv, yu, yv)} = I \cap J$. 
The question

Question

*Ok we can generate $I \cap J$ with three elements up to radical. Can we generate $I \cap J$ with two elements up to radical? What about one element?*

We will use “homological methods” to solve this problem.
Definition
A chain complex is a sequence of $A$-modules and $A$-module homomorphisms

$$
\cdots \longrightarrow E_{i-1} \xrightarrow{d_{i-1}} E_i \xrightarrow{d_i} E_{i+1} \longrightarrow \cdots
$$

such that $d_i \circ d_{i-1} = 0$ for all $i \in \mathbb{Z}$. We denote a chain complex by $E^\bullet$. 
Cohomology

The upshot is that when given a chain complex \((E^\bullet, d^\bullet)\), one has

\[\text{Im}(d^{i-1}) \subseteq \text{Ker}(d^i) \subseteq E^i\]

we can make a new module:

\[H^i(E^\bullet) = \frac{\text{Ker}(d^i)}{\text{Im}(d^{i-1})}\]

called the **ith cohomology** of \(E^\bullet\).
How do we make these things?

**Question**

*But where do we get our complexes from?*

**Answer**

*This will take some explaining.*
Injective modules

If $A$ is noetherian and $M$ is any $A$-module, then there exists a special module with nice properties which we can inject $M$ into. The type of module which we desire is called an injective module. Specifically, we are looking for the injective hull of $M$.

Aside

*How does this relate to free modules?*
An injective resolution

Time to build a complex: Start with

\[
\begin{array}{ccccccc}
0 & \rightarrow & M & \xrightarrow{\iota} & E^0 & \rightarrow & C^1 & \rightarrow & 0 \\
0 & \rightarrow & C^1 & \rightarrow & E^1 & \rightarrow & C^2 & \rightarrow & 0 \\
0 & \rightarrow & C^2 & \rightarrow & E^2 & \rightarrow & C^3 & \rightarrow & 0 \\
\end{array}
\]

and so on. Put it all together and it sounds like this:

\[
\begin{array}{ccccccc}
0 & \rightarrow & M & \xrightarrow{\iota} & E^0 & \xrightarrow{d^0} & E^1 & \xrightarrow{d^1} & \cdots \\
0 & \rightarrow & C^1 & \xrightarrow{} & C^2 & \xrightarrow{} & \cdots \\
\end{array}
\]
**Boring cohomology**

Now lose the extraneous parts to get

$$
0 \longrightarrow M \xrightarrow{\iota} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow \cdots
$$

Note by the construction of our complex, it is necessarily exact.

**Question**

*What is the cohomology?*

That’s not very interesting.
Functors

Roughly speaking, a **functor** is a mapping of both objects and morphisms. Whatever that means. Consider

\[ \Gamma_I(M) = \{ a \in M : I^t a = 0 \text{ for some } t > 0 \} . \]

So if we have

\[ M \xrightarrow{\varphi} N \]

we may write

\[ \Gamma_I(M) \xrightarrow{\Gamma_I(\varphi)} \Gamma_I(N) \]
We define **local cohomology** as follows:

1. Take an injective resolution $E^\bullet$ of $M$.
2. Apply $\Gamma_I(-)$ to the resolution above.
3. Take cohomology.

Explicitly:

$$H^i_I(M) = \frac{\ker \Gamma_I(d^i)}{\text{Im} \Gamma_I(d^{i-1})}$$

What does that mean?
Here be dragons: Mayer-Vietoris

If \( A \) is a noetherian ring, \( I \) and \( J \) are two ideals, and \( M \) is an \( A \)-module, then we have a long exact sequence of local cohomology modules:

\[
0 \to H^0_{I+J}(M) \to H^0_I(M) \oplus H^0_J(M) \to H^0_{I \cap J}(M) \to \]
\[
\to H^1_{I+J}(M) \to H^1_I(M) \oplus H^1_J(M) \to H^1_{I \cap J}(M) \to \]
\[
\to \cdots \]
\[
\to H^i_{I+J}(M) \to H^i_I(M) \oplus H^i_J(M) \to H^i_{I \cap J}(M) \to \cdots
\]
Big theorems

Theorem (Invariance up to radical)

Given an ideal $I$

$$H^i_I(A) \cong H^i_{\sqrt{I}}(A)$$

Theorem (Grothendieck)

An ideal $I$ can be generated by no fewer than $n$ elements up to radical if and only if

$$H^n_I(A) \neq 0$$

and

$$H^i_I(A) = 0 \quad \text{for all } i > n.$$
Just remember

Remember

\[ I = (x, y) \]
\[ J = (u, v) \]
\[ I + J = (x, y, u, v) \]
\[ I \cap J = (xu, xv, yu, yv) \]
Don’t forget

Remember our Mayer-Vietoris sequence:

\[
\cdots \to H^3_i(A) \oplus H^3_j(A) \to H^3_{i \cap J}(A) \to H^4_{i+J}(A) \to H^4_i(A) \oplus H^4_j(A) \to \cdots
\]

Remember what Grothendieck said:

\[
\cdots \to 0 \to H^3_{i \cap J}(A) \to H^4_{i+J}(A) \to 0 \to \cdots
\]

and so \( H^3_{i \cap J}(A) \cong H^4_{i+J}(A) \neq 0 \). Hence we are done! Why?
The end

THE END ?