Section 82, #4. \( F(s) = \frac{s^2 - a^2}{(s+i\alpha)^2} \) has poles of order 2 at \( s = ia \) and \(-ia\). By Equation (2) in section 82 (see example 1),

\[
f(t) = 2\Re(e^{iat}(b_1 + b_2 t))
\]

where

\[
F(s) = \frac{b_1}{s-ia} + \frac{b_2}{(s-ia)^2} + a_0 + a_1(s-ia) + a_2(s-ia)^2 + \cdots .
\]

Write

\[
F(s) = \frac{\phi(s)}{(s-ia)^2}
\]

where \( \phi(s) = \frac{s^2 - a^2}{(s+ia)^2} \).

Then

\[
F(s) = \frac{1}{(s-ia)^2} \cdot \phi(s) = \frac{1}{(s-ia)^2} \cdot (\phi(ia) + \phi'(ia)(s-ia) + \cdots )
\]

\[
= \frac{1}{(s-ia)^2} \left( \frac{1}{2} + 0 + \cdots \right).
\]

So \( b_2 = 1/2 \) and \( b_1 = 0 \). Then

\[
f(t) = 2\Re\left(e^{iat}\frac{1}{2}t\right) = t \cos(at).
\]

Verifying the decay condition \( M_R \rightarrow 0 \) as \( R \rightarrow \infty \) is straightforward.

Section 82, #7. \( F(s) = \frac{1}{s \cosh(s^{1/2})} \). Note \( \cosh(z) \) is a power series in \( x^2 \) so \( \cosh(s^{1/2}) \) is analytic on a neighborhood of 0 and you don’t need to choose a branch for the square root. There are poles at \( s = 0 \) and \( s_k = -(\frac{\pi}{2} + k\pi)^2, \ k = 0, \pm 1, \pm 2, \ldots \) Note this lists each pole twice since \( s_k = s_{-(k+1)} \). Thus the distinct poles are \( s = 0 \) and \( s_k = -(\frac{\pi}{2} + k\pi)^2, \ k = 0, 1, 2, \ldots \)

The pole at \( s = 0 \) is a simple pole since \( 1/\cosh(s^{1/2}) \) is non-zero and analytic at \( s = 0 \). The poles at \( s_k = -(\frac{\pi}{2} + k\pi)^2 \) are also simple poles since \( 1/s \cosh(s^{1/2}) \) has the form \( p/q \) with \( p(s) = 1, \ q(s) = s \cosh(s^{1/2}), \ p, q \) analytic near \( s_k, \ p(s_k) \neq 0, \ q(s_k) = 0, \ q'(s_k) = \frac{1}{2} s_k^{1/2} \sinh(s_k^{1/2}) \neq 0 \). Using Equation (3), Section 82,

\[
f(t) = e^{0t} \text{Res}_{s=0} F(s) + \sum_{s_k} e^{s_kt} \text{Res}_{s=s_k} F(s) \quad (t > 0).
\]

It is easy to see from the first term of the Maclaurin series for \( 1/\cosh(s^{1/2}) \) that \( \text{Res}_{s=0} F(s) = 1 \). The residues at \( s = s_k \) are

\[
\text{Res}_{s=s_k} F(s) = \frac{p(s_k)}{q'(s_k)} = \frac{2}{s_k^{1/2} \sinh(s_k^{1/2})} = \frac{2(-1)^{k+1}}{(\frac{\pi}{2} + k\pi)}
\]

since \( \sinh(i(\frac{\pi}{2} + k\pi)) = i \sin(\frac{\pi}{2} + k\pi) = (-1)^k \) (using either branch of the square root). Plugging this into (1) above gives formally

\[
f(t) = 1 + 2 \sum_{k=0}^{\infty} e^{-(\frac{\pi}{2} + k\pi)^2 t} \frac{(-1)^{k+1}}{(\frac{\pi}{2} + k\pi)}
\]

\[
= 1 + \frac{4}{\pi} \sum_{k=0}^{\infty} e^{-(1+2k)^2 \pi^2 t/4} \frac{(-1)^{k+1}}{1+2k}
\]

\[
= 1 + \frac{4}{\pi} \sum_{n=1}^{\infty} e^{-(2n-1)^2 \pi^2 t/4} \frac{(-1)^n}{2n-1} \quad (t > 0).
\]
It is easy to see this series converges for all \( t > 0 \) (in fact it converges for \( t = 0 \) also by the alternating series test). It diverges for \( t < 0 \) so certainly this cannot be valid for \( t < 0 \).

This computation is only formal in that we have not verified that

\[
\lim_{R \to \infty} \int_{C_R} e^{st} F(s) \, ds = 0
\]

where \( C_R \) is as in Section 81. Since there are infinitely many poles on the negative real axis this condition is difficult to verify. Following the textbook’s suggestion we will accept the result as it stands.