We study the Segal–Bargmann transform on a symmetric space \( X \) of compact type, mapping \( L^2(X) \) into holomorphic functions on the complexification \( X_C \). We invert this transform by integrating against a "dual" heat kernel measure in the fibers of a natural fibration of \( X_C \) over \( X \). We prove that the Segal–Bargmann transform is an isometry from \( L^2(X) \) onto the space of holomorphic functions on \( X_C \) which are square integrable with respect to a natural measure. These results extend those of B. Hall in the compact group case.

1. INTRODUCTION

It is well known that a solution of the heat equation on \( \mathbb{R}^n \) can be analytically continued to an entire function on \( \mathbb{C}^n \). The classical Segal–Bargmann transform, in the form we will be interested in, is the map

\[
 f \in L^2(\mathbb{R}^n, dx) \rightarrow e^{-t} f \in \mathcal{O}(\mathbb{C}^n) \cap L^2(\mathbb{C}^n, (2\pi t)^{-n/2} e^{-|\text{Im} z|^2/2t} dz),
\]

where \( \mathcal{O}(\mathbb{C}^n) \) denotes the holomorphic functions on \( \mathbb{C}^n \). This is an isometric isomorphism of Hilbert spaces, identifying \( L^2(\mathbb{R}^n) \) with a natural \( L^2 \) space of entire functions on \( \mathbb{C}^n \). B. Hall has obtained an interesting generalization of the Segal–Bargmann transform by replacing \( \mathbb{R}^n \) with a compact Lie group \( H \) equipped with a bi-invariant metric [5]. Hall proved that solutions of the heat equations on \( H \) can be analytically continued to entire functions on the complexified Lie group \( H_C \), and that the Segal–Bargmann transform

\[
 f \in L^2(H, dh) \rightarrow e^{-t} f \in \mathcal{O}(H_C) \cap L^2(H_C, dh \sigma^{1/2}(p) dp)
\]
is an isometric isomorphism (we consider the “$K$-averaged” version of the transform). Here $P = \exp (-1)\mathfrak{h}$. $H_C$ is identified with $H \times P$, $dh$ is Riemannian measure, and $\sigma_{t}(p) \, dp$ is the heat kernel measure for the heat operator $\partial_t + \Delta_P$, $\Delta_P \geq 0$, on $P$ identified with the symmetric space $H_C/H$ by the quotient map. In a subsequent paper [6], Hall proved the following “fiberwise” inversion formula for the Segal–Bargmann transform: for $f \in C^\infty(H)$,

$$f(h) = \int_{P} e^{-1/4H} f(hp^2) \sigma_{t}(p) \, dp,$$

(1)

and used this to give an elegant proof of its isometricity (our conventions differ slightly from Hall’s; see Section 5).

Independently, Golse, Leichtnam, and the author studied a Segal–Bargmann type transform called the Fourier–Bros–Iagolnitzer (F.B.I.) transform [3]. This transform can be defined on any compact real analytic Riemannian manifold $(X, g)$. We showed that solutions of the heat equation can be analytically continued to a small tube $X_R \subset X$ about $X$ in its complexification. The radius $R$ of this tube, measured by an exhaustion function canonically determined by $g$, does not depend on the initial heat data. The F.B.I. transform is the map

$$f \in C^\infty(X) \mapsto e^{-1/4f} \in C(X_R^\mathbb{C}).$$

The F.B.I. transform is injective, and we proved an inversion formula for it similar to Hall’s. Our inversion formula involves integrating over the fibers $Y_{R}(x)$, $x \in X$, of the fibration of $X^\mathbb{C}$ over $X$ obtained by identifying $X^\mathbb{C}$ with a neighborhood of the zero section in $TX$ (see [4, 9]). We constructed a “pseudo-heat kernel” form $K(\cdot, t) \mu^Y_{R}$ in each fiber $Y_{R}(x)$ for $R$ sufficiently small and gave the following integral prescription for recovering $f \in C^\infty(X)$ from its F.B.I. transform:

$$f(x) = \int_{Y_{R}(x)} e^{-1/4f(\cdot)} K(\cdot, t) \mu^Y_{R}$$

$$\quad + \int_{0 \in Y_{R}(x)} \left[ e^{-1/4f(\cdot)} \nabla_{\mu^Y_{R}} K(\cdot, s) \mu^Y_{R} - K(\cdot, s) \nabla_{\mu^Y_{R}} e^{-1/4f(\cdot)} \mu^Y_{R} \right] ds.
$$

(2)

See [3, Theorem 0.3] for details. Our purpose here is to generalize Hall’s inversion formula for the Segal–Bargmann transform by assuming that a complexification of $X$ admits an unbounded canonical exhaustion function and letting $R \to \infty$ in (2). This assumption imposes severe restrictions on
the geometry of \((X, g)\); for example, the sectional curvatures must be non-negative \([9]\). The list of known examples whose complexifications admit an unbounded canonical exhaustion is quite short: locally symmetric spaces of nonnegative curvature, normal homogeneous spaces of compact Lie groups and their discrete quotients, and certain surfaces of revolution (see \([11, 13, 12]\), respectively). (Added in proof. Recently R. Aguilar has found many more examples.)

We will assume that \(X\) is a connected symmetric space \(U/K\) of Helgason's compact type. For simplicity we will assume that \(U/K\) is simply connected; the main results of this paper hold after passing to quotients by a freely acting finite group of isometries. It follows from the results of Hall \([5, \text{Sect. 11}]\) that solutions of the heat equation on \(X\) can be analytically continued to entire functions on \(X_\mathbb{C}\). We study the Segal–Bargmann transform in this setting. Under these assumptions each (maximal) fiber \(Y(x)\) can be identified with a dual symmetric space of the noncompact type. To prove our inversion formula we will use the corresponding fiber heat kernel measure in (2) and let \(R \to \infty\). Let \(K_{gY}(\cdot, t)\) denote the heat kernel in the fiber \(Y(x)\) evaluated at the base point \(x \in X\), let \(v_{gY}\) be the Riemannian measure on \(Y(x)\), and let \(Y_R(x)\) be the distance ball of radius \(R\) in \(Y(x)\). Use the Riemannian measure on \(X\) to form \(L^2(X)\). Our main results are the following generalizations of Hall's fiberwise inversion formulae for compact Lie groups to the symmetric spaces of compact type.

**Theorem 1.** For all \(f \in L^2(X)\), \(t > 0\),
\[
f(x) = L^2 - \lim_{R \to \infty} \int_{Y_R(x)} e^{-td}(\cdot) K_{gY}(\cdot, t) v_{gY}.
\]

**Theorem 2.** For all \(f \in C_{\infty}(X)\), \(t > 0\), \(x \in X\),
\[
f(x) = \int_{Y(x)} e^{-td}(\cdot) K_{gY}(\cdot, t) v_{gY}
\]
and the integral is absolutely convergent.

In Section 4 we consider the measure \(\omega_t\) on \(X_\mathbb{C}\) induced by the Riemannian measure on \(X\) and the heat kernel measure in the fibers at time \(2t\), rescaled by a factor of 2 in the fibers. Let \(\mathcal{H}_t\) denote the Hilbert space of holomorphic functions in \(L^2(X_\mathbb{C}, \omega_t)\).

**Theorem 3.** The Segal–Bargmann transform is an isometric isomorphism from \(L^2(X)\) onto \(\mathcal{H}_t\).

The organisation of this paper is as follows. In Section 2 we provide some background material and prove Theorem 2 for joint eigenfunctions of
the $U$-invariant differential operators. Our strategy is to prove an analog of (2) for any $R > 0$, then estimate the growth of the joint eigenfunctions in the complex domain and pass to the limit as $R \to \infty$. Section 3 contains the proofs of Theorems 1 and 2, and in Section 4 we define the measure $\omega$, and prove Theorem 3. In Section 5 we show that our inversion formula reduces to (1) when $H$ is a compact, connected semisimple Lie group identified with a symmetric space of compact type.

2. INVERSION OF THE SEGAL–BARGMANN TRANSFORM ON JOINT EIGENFUNCTIONS

We assume that $X = U/K$ is a simply connected Riemannian (globally) symmetric space of Helgason's compact type. The Riemannian symmetric pair $(U, K)$ is associated with an orthogonal symmetric Lie algebra $(\mathfrak{u}, \theta)$ of compact type (see [7, pp. 230, 244]). In particular $\mathfrak{u}$ is semisimple and $U, K$ are compact. By the proof of [7, Proposition 4.2, Chap. V] we may assume that $U$ is simply connected and $K$ is connected. The Riemannian symmetric metric $g$ determines an $\text{Ad}(K)$-invariant inner product on $\mathfrak{p}_*$, where $\mathfrak{p}_*$ is the $-1$ eigenspace of $\theta$ in the eigenspace decomposition $\mathfrak{u} = \mathfrak{t} + \mathfrak{p}_*$. We assume that $U$ acts effectively on $U/K$; then $\mathfrak{t}$ contains no nonzero ideal of $\mathfrak{u}$. Using [7, Proof of Proposition 5.2, Chap. VIII] and the remarks preceding Proposition 5.1, we may assume that the inner product on $\mathfrak{p}_*$ is the restriction of an $\text{Ad}(U)$-invariant inner product on $\mathfrak{u}$ (we thank the referee for pointing this out). We will normalize the metric so that the total volume is one.

Let $U_C$ be the Lie group complexification of $U$ and let $K_C$ be the connected subgroup of $U_C$ with Lie algebra $\mathfrak{t}_C$. Then $K_C$ is a closed subgroup and the natural map, $gK \to gK_C$, embeds $X = U/K$ as a closed, totally real submanifold of $X_C = U_C/K_C$ [3, Proposition 8.1]. We will identify $U/K$ with its image in $U_C/K_C$. Let $\Delta$ be the (nonnegative) Laplacian on $(U/K, g)$. By [5, Lemma 13] solutions of the heat equation $(\partial_t + \Delta) u = 0$ can be analytically continued to entire functions on $U_C/K_C$.

A representation $\delta$ of $U$ on a vector space $V$ is said to be spherical if there is a nonzero vector in $V$ fixed by $\delta(K)$. Let $\hat{U}_K$ denote the set of equivalence classes of irreducible unitary finite dimensional spherical representations of $U$. For each $\delta \in \hat{U}_K$ let $V_\delta$ be a representation space for $\delta$ with inner product $\langle \cdot, \cdot \rangle$. Since $U/K$ is a symmetric space, the subspace of $K$-fixed vectors is spanned by a single unit vector $\mathbf{e}$. There is an orthogonal Hilbert space decomposition of $L^2(U/K)$ into irreducible subspaces under the $U$ action,

$$L^2(U/K) = \bigoplus_{\delta \in \hat{U}_K} C_\delta(U/K),$$

(3)
where the \( C_\delta(U/K) \) are equivalent irreducible representations of type \( \delta \). Explicitly, \( C_\delta(U/K) \) consists of the functions \( uK \rightarrow \langle v, \delta(u) e \rangle \), \( v \in V_\phi \). \( C_\delta(U/K) \) is a joint eigenspace for the \( U \)-invariant differential operators \( D(U/K) \), i.e., for each \( \delta \in \hat{U}_K \), there is a homorphism \( c_\delta: D(U/K) \rightarrow \mathbb{C} \) such that for all \( f \in C_\delta(U/K) \), \( Df = c_\delta(D)f \). See [8, Theorem 4.3, Chap. V]. Since \( U \) acts on \( U/K \) by isometries, \( 2 \) is a \( U \)-invariant differential operator and the elements of \( C_\delta(U/K) \) are eigenfunctions for \( \Delta \). Since solutions of the heat equation can be analytically continued to \( U_{C/K} \), it follows that each \( f \in C_\delta(U/K) \) can be analytically continued to \( U_{C/K} \) (see also the proof of Lemma 1).

We consider the fibration \( \pi: U_{C/K} \rightarrow U/K \) obtained by identifying \( U_{C/K} \) with \( T_{UK} \) as in [11, 12]. This identification is given explicitly by the map

\[
TU/K \ni dt(u) dp_e(V) \rightarrow u \exp -1/VK_{\mathbb{C}} \in U_{C/K}.
\]

where \( V \in \mathfrak{p}_* \), \( \tau \) is the action of \( U \) on \( U/K \), and \( p: U \rightarrow U/K \) is the coset projection (see [3, Proposition 8.2 and Remark 8.4]). We will denote by \( Y \), or \( Y(x) \) if necessary, the fiber \( \pi^{-1}(x) \) in \( U_{C/K} \) over \( x \in U/K \). An essential ingredient in our inversion formula for the Segal-Bargmann transform is a symmetric space metric on each fiber \( Y \), which we now define.

Let \( G \) be the connected subgroup of \( U_C \) with Lie algebra \( \mathfrak{g} = \mathfrak{t} + \mathfrak{p} \) where \( \mathfrak{p} = \sqrt{-1} \mathfrak{p}_* \). Then \( K \) (which we may assume is connected) is a closed subgroup of \( G \). The metric \( g \) on \( U/K \) determines in an obvious way a positive definite \( \text{Ad}(K) \)-invariant inner product on \( \mathfrak{p} \). The homogeneous manifold \( G/K \) with the corresponding \( G \)-invariant metric is a symmetric space of the non-compact type dual to \( U/K \). The map (4) and the fact that \( G/K \) is diffeomorphic with \( \mathfrak{p} \) show that the natural inclusion

\[
\mathfrak{g}_K \in G/K \rightarrow \mathfrak{g}_K \in U_{C/K}
\]

is a diffeomorphism from \( G/K \) onto the fiber \( Y(o) \), where \( o \) is the identity coset. We equip \( Y(o) \) with the symmetric space metric given by this identification and use the transitive \( U \)-action among the fibers to give each \( Y(x) \) a symmetric space metric; since \( K \) acts on \( Y(o) \) by isometries this is well-defined. We denote this metric by \( g_Y \) and its Riemannian measure by \( v_{g_Y} \).

Let \( \phi: U_{C/K} \rightarrow [0, \infty) \) be the function corresponding to the Riemannian length-squared function on \( TU/K \) under the identification (4), and let

\[
Y_\phi(x) := \{ \zeta \in Y(x) : \phi(\zeta) \leq R^2 \}.
\]

Let \( K_{\phi}(\cdot, t) \) denote the heat kernel for \( (Y, g_Y) \) evaluated at the base point \( x \).
Proposition 1. For all $f \in \mathcal{C}_d(U/K)$, $t > 0$, $R > 0$, and $x \in U/K$,

$$f(x) = \int_{Y_R(x)} e^{-td} f(\cdot) \, K_{\mathcal{E}_R}(\cdot, t) \, v_{Y_R} \, ds$$

$$+ \int_0^t \left[ K_{\mathcal{E}_R}(\cdot, s)(\hat{n} e^{-sd} f_\hat{n}) - (\hat{n} K_{\mathcal{E}_R}(\cdot, s)) e^{-sd} f_\hat{n} \right] v_{\partial Y_R} \, ds,$$

where $\hat{n}$ is the outward unit normal vector field on $\partial Y_R$ and $v_{\partial Y_R}$ is the Riemannian measure of the induced metric on $\partial Y_R$.

Proof. This can be proved in the same way as [3, Theorem 0.3] using the estimate (6) below. The starting point is the following relationship between the Laplacians $\mathcal{L}$ on $U/K$ and $\mathcal{L}_{\mathbb{R}}$ on $\mathbb{R}$.

Proposition 2 [3]. For every holomorphic function $F$ on $U/K$, there is a neighborhood $W$ of $U/K$ in $U/K$ such that $\mathcal{L}(F|_{U/K})$ and $\mathcal{L}_{\mathbb{R}}(F|_{\mathbb{R}})$ have the same analytic continuation to $W$.

Proof. We use the notation of [3, Definition 1.13]. By [3, Theorem 8.5.1], $g_\mathcal{E} = -g_{\mathbb{R}}$. By [3, Theorem 1.19.ii], $\mathcal{L}_{\mathbb{R}} = -\mathcal{L}_{\mathbb{R}}$. The proposition now follows from [3, Theorem 1.16].

By [5, Lemma 13], $e^{-td} f_\hat{n}$, $\mathcal{L}_{\mathbb{R}} e^{-td} f_\hat{n}$, and $\partial_s e^{-td} f_\hat{n}$ can be analytically continued to $U/K$. Analytically continuing the equation $(\mathcal{L} + \partial_s) e^{-td} f_\hat{n} = 0$ and restricting to $Y_R$, we obtain

$$(-\mathcal{L}_{\mathbb{R}} + \partial_s)(e^{-td} f_\hat{n})|_{Y_R} = 0$$

for all $t$, $R > 0$. As in the proof of [3, Theorem 0.3], we multiply by $K_{\mathcal{E}_R}$ and integrate: for $0 < \varepsilon < t$,

$$\int_\varepsilon^t \int_{Y_R(x)} K_{\mathcal{E}_R}(\cdot, s)(-\mathcal{L}_{\mathbb{R}} + \partial_s) e^{-sd} f_\hat{n} v_{\mathcal{E}_R} \, ds = 0.$$

The integral is absolutely convergent for $0 < \varepsilon < t < \infty$ and $R < \infty$. Integrating by parts in $s$, using $(\mathcal{L}_{\mathbb{R}} + \partial_s) K_{\mathcal{E}_R} = 0$, and applying Green’s second identity on $Y_R$ gives

$$\int_{Y_R(x)} K_{\mathcal{E}_R}(\cdot, \varepsilon) e^{-td} f_\hat{n} v_{\mathcal{E}_R} = \int_{Y_R(x)} K_{\mathcal{E}_R}(\cdot, t) e^{-td} f_\hat{n} v_{\mathcal{E}_R}$$

$$+ \int_0^t \int_{\partial Y_R(x)} K_{\mathcal{E}_R}(\cdot, s)(\hat{n} e^{-sd} f_\hat{n})$$

$$- (\hat{n} K_{\mathcal{E}_R}(\cdot, s)) e^{-sd} f_\hat{n} \right] v_{\partial Y_R} \, ds. \quad (5)$$
To prove the proposition we let $\varepsilon \to 0^+$. To estimate the integral over $\partial Y_R$ we note that since $f_\delta$ is an eigenfunction of $A$, it is clear that $|e^{-\varepsilon A}f_\delta|$, resp. $|\varepsilon e^{-\varepsilon A}f_\delta|$, is uniformly bounded on $(0, \infty) \times Y_R$, resp. $(0, \infty) \times \partial Y_R$. For the heat kernel $K_{s\gamma}$ we have the estimate

$$\|d_t^s d_t^{s\gamma} K_{s\gamma}(\zeta, s)\| \leq C(t) s^{-n/2 - i - 1/2} e^{-d_{s\gamma}(s, \zeta)/4s}$$

(6)

([1]: here $x$ is the base point of $Y$, $d_{s\gamma}$ is the Riemannian distance in $Y$, and $\|\cdot\|$ is the $g_Y$ norm). This shows that the integral over $\partial Y_R$ is absolutely convergent as $\varepsilon \to 0^+$. The left side of (5) approaches $f_\delta(x)$ as $\varepsilon \to 0^+$ because $f_\delta$ is an eigenfunction and $K_{s\gamma}(\cdot, s) v_{s\gamma}$ approaches the Dirac measure at the base point $x$ as $\varepsilon \to 0$.

The following lemma gives the control over the growth of the joint eigenfunctions in the complex domain needed to prove the inversion formula for joint eigenfunctions. Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{u}_C$ and an ordering of the roots, and let $\lambda$ denote the highest weight of $\delta$ (extended to a representation of $\mathfrak{u}_C$).

**Lemma 1.** There is an $M > 0$ (depending only on the metric $g$) such that for all $\delta \in U_K$, $f_\delta \in C_c(U/K)$ analytically continued to $U_C/K_C$, $\zeta \in Y$,

$$|f_\delta(\zeta)| \leq d(\delta)^{1/2} \|f_\delta\|_{L^2(U/K)} e^{1/2 d_M d_{s\gamma}(x, \zeta)}$$

$$|\delta f_\delta(\zeta)| \leq M |\lambda|_K d(\delta)^{1/2} \|f_\delta\|_{L^1(U/K)} e^{1/2 d_M d_{s\gamma}(x, \zeta)},$$

where $d_{s\gamma}$ is the distance function on $Y$, $d(\delta)$ is the dimension of $V_{s\gamma}$, and $|\cdot|_K$ is the Killing form norm.

**Proof.** Since $U$ is simply connected, each representation $\delta$ extends to a holomorphic representation of $U_C$, still denoted by $\delta$, and $\delta(K_C) e = e$ (since this holds on the real part $K$). The analytic continuation of $f_\delta(uK) = \langle \varphi, \delta(u) e \rangle$ to $U_C/K_C$ is given explicitly by $f_\delta(gK_C) = \langle \varphi, \delta(g) e \rangle$, where the bar denotes complex conjugation of $U_C$ about $U$ (note that $\langle \cdot, \cdot \rangle$ is conjugate linear in the second factor). By (4), given a coset $\zeta = gK_C$ in $Y(x)$ there is a representative $g$ such that $g = u \exp V$ with $u \in U$, $V \in \mathfrak{p}$. Let $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$. Since $\text{Ad}(K) \mathfrak{a} = \mathfrak{p}$, we may assume $g = u \exp H$ with $u \in U$, $H \in \mathfrak{a}$. Since $|e| = 1$ and $\delta$ is unitary, we have

$$|f_\delta(gK_C)| = |\langle \varphi, \delta(u \exp H) e \rangle| \leq |\varphi| \max_{\lambda, \text{weights}} e^{1/2 |\lambda|}$$

$$\leq |\varphi| e^{1/2 |\lambda|_K},$$

where $\lambda$ is a weight of $U_C$. Therefore, the integral over $U/K$ is bounded by

$$\int_{U/K} |f_\delta(gK_C)| dg \leq C \int_{U/K} e^{1/2 d_M d_{s\gamma}(x, \zeta)} dg.$$
where \( \lambda \) is the highest weight, the last inequality by [8, Eq. (7), p. 498]. By the Schur orthogonality relations, \( \| f_\delta \|^2_{L^2(U/K)} = |v|^2 d(\delta)^{-1} \), where \( d(\delta) \) is the dimension of \( V_\delta \) (here we have used the normalization \( vol(U/K) = 1 \)). Since the Killing form norm on \( p \) is comparable to the \( Ad(K) \)-invariant inner product induced by \( g_{Y(\delta)} \), there is a positive constant \( M \) (depending only on \( g \)) such that \( |H|_K \leq M |H|_{\sigma_{\delta,\gamma}} \). The unique unit speed geodesic from \( x \) to \( \zeta \) is \( \gamma(t) = u \exp(-tH|H|_{\sigma_{\delta,\gamma}}) \), so \( d_{\sigma_{\delta,\gamma}}(x, \zeta) = |H|_{\sigma_{\delta,\gamma}} \) (note also this shows \( \phi(\zeta) = d_{\sigma_{\delta,\gamma}}^2(x, \zeta) \)). This gives the estimate for \( f_\delta \).

The unit normal to \( Y_R \) at \( \zeta = u \exp(-HK) \) is \( \hat{\nu} \), \( R = |H|_{\sigma_{\delta,\gamma}} \). Then

\[
\hat{\nu}_\delta f_\delta(\zeta) = \frac{d}{dt} \left|_{t=R} \langle \nu, \delta(u \exp(-tH) e^e) \rangle = \langle \nu, \delta(u \exp(-H) e^e) \rangle / R. \right.
\]

The estimate for \( |\hat{\nu}_\delta f_\delta(\zeta)| \) follows similarly.

We can now prove the inversion formula for the Segal–Bargmann transform acting on the joint eigenfunctions.

**Proposition 3.** For all \( f_\delta \in C_c(U/K), \ t > 0, \ x \in U/K, \)

\[
f_\delta(x) = \int_{Y(x)} e^{-s^2 f_\delta(\cdot)} K_{\sigma_{\delta,t}}(\cdot, t) v_{\sigma_t}
\]

and the integral is absolutely convergent.

**Proof.** We let \( R \to \infty \) in Proposition 1. In the proof of Lemma 1 we showed that \( d^2_{\sigma_{\delta,t}}(x, \zeta) = \phi(\zeta) \). So for \( \zeta \in \partial Y_R, \ d_{\sigma_{\delta,t}}(x, \zeta) = R \). By Lemma 1 and (6) there are positive constants \( C_i \) (depending only on \( \delta, f_\delta, \ t, \) and \( g \)) such that for all \( (s, \zeta) \in (0, t] \times \partial Y_R, \)

\[
\begin{align*}
|e^{-s^2 f_\delta(\zeta)}| &\leq C_1 e^{-s^2 C_2 R} + C_2 R, \\
|\hat{\nu} e^{-s^2 f_\delta(\zeta)}| &\leq C_3 e^{-s^2 C_4 R} + C_4 R \\
|K_{\sigma_{\delta,t}}(\zeta, s)| &\leq C_5 e^{-s^2 C_6 R - R^4/4s} \\
|\hat{\nu} K_{\sigma_{\delta,t}}(\zeta, s)| &\leq C_7 e^{-s^2 C_8 R - R^4/4s}.
\end{align*}
\]

Then we can estimate the boundary integral in Proposition 1 as \( R \to \infty \) by

\[
\int_0^t \int_{\partial Y R} |K_{\sigma_{\delta,t}}(\cdot, s)(\hat{\nu} e^{-s^2 f_\delta}) - (\hat{\nu} K_{\sigma_{\delta,t}}(\cdot, s))| e^{-s^2 f_\delta} v_{\partial Y_R} ds
\leq C_5 vol(\partial Y_R) e^{C_4 R} \int_0^t e^{-s^2 C_6 R - R^4/4s} ds
\leq C_6 vol(\partial Y_R) e^{C_4 R - R^4/4s}.
\]
The Ricci curvature of $g_Y$ is bounded below by a negative constant (since a symmetric space has a transitive group of isometries). Following the proof of [2, Theorem 6], we have $\text{vol}(\partial Y_R) \leq C_2 e^{C_3 R}$. This shows that the boundary integral in Proposition 1 goes to zero as $R \to \infty$. Similarly one can show that the integral over $Y$ converges absolutely.

3. PROOF OF THEOREMS 1 AND 2

Proof of Theorem 1. It is possible to prove the inversion formula for functions in the range of $e^{-td}$ for some $t > 0$ by brute force using the estimates above. But to prove Theorems 1 and 2 in this fashion would require unreasonably sharp estimates on $e^{-td}f$ and $K_{\mathfrak{g}}$. We will instead take the following approach, which was suggested to us by B. Hall. For fixed $t > 0$, consider the linear operator

$$M_R f(x) = \int_{Y(x)} e^{-td} f(\cdot) K_{\mathfrak{g}}(\cdot, t) \nu_{\mathfrak{g}}.$$  \hspace{1cm} (7)

The integral is absolutely convergent for $R < \infty$ and maps $L^2(U/K)$ into smooth functions. $M_R$ commutes with the action of $U$ on $L^2(U/K)$, since the action commutes with the heat operator and with analytic continuation and acts by isometries among the fibers $(Y, g_Y)$. The kernel of $M_R$ is a $U$-invariant subspace, so the restriction of $M_R$ to an irreducible representation space is either a linear isomorphism onto its image or identically zero. Since the representations $C(\mathfrak{g}(U/K))$ in the decomposition (3) are irreducible and inequivalent, there is a constant $m_{R, \delta}$ such that

$$m_{R, \delta} = m_{R, \delta}I.$$  \hspace{1cm} (8)

We claim that the $m_{R, \delta}$ are positive and increase with $R$. Applying $M_R$ to $f_{\delta}(\cdot) = \langle e, \delta(\cdot) e \rangle$ and evaluating at $o$ gives

$$m_{R, \delta} = e^{-td(\delta)} \int_{gK_C \subset Y(o)} \langle e, \delta(g) e \rangle K_{\mathfrak{g}}(gK, t) \nu_{\mathfrak{g}}.$$  \hspace{1cm} (9)

Each coset $gK_C \in Y(o)$ has a representative $g$ with $\delta = \exp \sqrt{-I} V$, $V \in \mathfrak{p}_+$. Since $\delta(u)$ consists of skew-symmetric matrices, $\delta(\sqrt{-I} V)$ is symmetric with real eigenvalues and so $\delta(g) = e^{\delta(\sqrt{-I} V)}$ is positive definite. Since $K_{\mathfrak{g}}$ is positive, the claim follows. Proposition 3 shows that $\lim_{R \to \infty} m_{R, \delta} = 1$.

To prove the theorem we argue as in the proof of [6, Theorem 1]. If $F$ is a holomorphic function on $U_C/K_C$ and $F|_{U/K} = \sum \delta \in \mathcal{U}_k (F|_{U/K})_\delta$ is the decomposition (3), then the termwise analytic continuation of this series converges uniformly and absolutely on compacta in $U_C/K_C$ to $F$ (cf. [5,
Lemma 9). It follows that \( e^{-df} = \sum_{\mu \in \hat{G}} e^{-\mu(d)} f_{\mu} \) with uniform and absolute convergence on compacts in \( U_c/K \). Inserting this into (7) gives \( M_x f = \sum_{\mu \in \hat{G}} m_{\mu} f_{\mu} \). Since the series is orthogonal, its convergence in \( L^2(U/K) \) to \( f \) follows from monotone convergence.

**Proof of Theorem 2.** It suffices to show that the integral is absolutely convergent, for then the pointwise limit \( \lim_{R \to \infty} \int_{Y(x)} e^{-t\lambda f} df_{\xi} = \int_{Y(x)} e^{-t\lambda f} df_{\xi} \) is equal to \( f \) in \( L^2(U/K) \) by Theorem 1. Since the integral depends continuously on parameters they must be equal everywhere.

In the proof of Lemma 1 we observed that the analytic continuation of \( f_{\xi} \) is \( f_{\xi}(gK_C) = \langle \psi, \delta(g) \psi \rangle \). Write \( e^{-t\lambda f} = \sum_{\mu \in \hat{G}} e^{-\mu(d)} \langle \psi, \delta(g) \psi \rangle \). We will fix \( x \) and estimate \( \langle \psi, \delta(\zeta) \psi \rangle \) for \( \zeta \in Y(x) \). Define \( w \in V_{\lambda} \) by \( w = \delta(u) e \) where \( x = uK \), and note that \( w \) does not depend on the choice of representative \( u \). Writing \( \zeta \in Y(x) \) as \( \zeta = u \exp \sqrt{-1} FK_C \) with \( \sqrt{-1} V \in p \), we see that \( \langle w, \delta(\zeta) \psi \rangle \) is strictly positive on \( Y(x) \). We claim that for all \( \zeta \in Y(x) \) and \( v \in V_{\lambda} \),

\[
|\langle \psi, \delta(\zeta) \psi \rangle| \leq |\psi| |\delta(\zeta)| e^{-\lambda(H)} \leq |\psi| e^{\lambda(H)}
\]

where \( e \) is the Harish–Chandra e-function, \( \lambda \) is the restriction of the highest weight of \( \delta \) to \( a \), and \( \rho \) is one-half the sum of the positive restricted roots (with multiplicities; we have fixed a Weyl chamber \( a^+ \)). To prove this we write \( \zeta = u' \exp HK_C \) with \( H \) in the closure of \( a^+, u' \in U \), and \( u'K = x \). Then

\[
|\langle \psi, \delta(\zeta) \psi \rangle| \leq |\psi| \max_{\lambda \in \text{restricted weights}} e^{\lambda(H)} \leq |\psi| e^{\lambda(H)}
\]

(the last inequality follows from the expression \( \lambda = \lambda - \sum m_k \tilde{r}_k \) where \( m_k \in \mathbb{Z} \) and \( \tilde{r}_k \) are positive restricted roots). Let \( u_1, \ldots, u_d \) be an orthonormal basis of \( V_{\lambda} \) consisting of weight vectors with corresponding weights \( \lambda_i \), and let \( \lambda_d = \lambda \) be the highest weight. Then \( |\langle e, u_d \rangle|^2 = e(-\sqrt{-1}(\lambda + \rho)) > 0 \) (see \([8, \text{Chap. V, Sect. 4, Eqs. (7), (8)})\]. Then

\[
|\langle \psi, \delta(\zeta) \psi \rangle| \leq |\psi| |e(-\sqrt{-1}(\lambda + \rho))|^{-1} |\langle e, u_d \rangle|^2 e^{\lambda(H)}
\]

\[
\leq |\psi| |e(-\sqrt{-1}(\lambda + \rho))|^{-1} \sum_{k=1}^d |\langle e, u_k \rangle|^2 e^{\lambda(H)}
\]

\[
= |\psi| |e(-\sqrt{-1}(\lambda + \rho))|^{-1} \langle \delta(u') e, \delta(u' \exp(H)) e \rangle
\]

\[
= |\psi| e(-\sqrt{-1}(\lambda + \rho))^{-1} |\langle w, \delta(\zeta) e \rangle|
\]

which proves the claim.
Keeping \(x\) and \(w\) fixed, we apply Proposition 3 to the joint eigenfunction \(y \in U/K \rightarrow \langle w, \delta(y) e \rangle\) to obtain

\[ e^{-nu(x)} \int_{\gamma(x)} \langle w, \delta(\gamma) e \rangle K_{g_{\gamma}}(\cdot, t) v_{g_{\gamma}} = \langle w, \delta(x) e \rangle = 1, \]

and so, by the claim above,

\[ \int_{\gamma(x)} |e^{-nu(x)}| K_{g_{\gamma}}(\cdot, t) v_{g_{\gamma}} \preceq \sum_{\delta \in U_K} |\delta| \cdot e(-\sqrt{-1}(\lambda + \rho))^{-1}. \tag{8} \]

There are positive constants \(C_1, C_2, p\) such that

\[ e(-\sqrt{-1}(\lambda + \rho))^{-1} \leq C_1 + C_2 |\lambda + \rho|_K^p \]

([8, Proposition 7.2, Chap. IV]; since \(\lambda\) is a highest weight, \(\lambda + \rho\) is in \(a_+^K\)). If \(f \in C^\infty(U/K)\), then the coefficients \(|v|\) decrease faster than any polynomial in \(|\lambda|_K\), and the sum in (8) converges.

4. PROOF OF THEOREM 3

Let \(\eta; U_{C/KC} \rightarrow U_{C/KC}\) be the diffeomorphism corresponding to multiplication by \(2\) in the fibers of \(TU/K\) under the identification (4); i.e., \(\eta(uyK) = uy^2K\) for \(u \in U, y \in \exp p\). By the Riesz Representation Theorem there is a unique measure \(\omega_i\) on \(U_{C/KC}\) such that for \(F \in C_0(U_{C/KC})\),

\[ F_{U_{C/KC}} = \int_{x \in U/K} \int_{\gamma(x)} F(\cdot) \eta^* (K_{\gamma})(\cdot, 2t) v_{\gamma}(t) v_{U/K}. \]

where \(v_{U/K}\) is the Riemannian measure on \(U/K\) (we normalize the metric so \(\text{vol}(U/K) = 1\)). Let \(\pi_{C}; U_{C} \rightarrow U_{C/KC}\) be (the holomorphic) coset projection. Since \(\pi_{C}\) restricted to \(\exp p\) is a diffeomorphism onto \(Y(o)\), this can be written as

\[ F_{U_{C/KC}} = \int_{y \in \exp p} \int_{u \in U} F \cdot \pi_{C}(uy) du(\pi_{C}(uy)| \exp p) (K_{Y(o)}(\cdot, 2t) v_{Y(o)}). \]

Here \(du\) is the Haar measure on \(U\) normalized by \(\text{vol}(U) = 1\), and we have used \([8, Theorem 1.9, Chap. I]\), the fact that \(\tau(u); Y(o) \rightarrow Y(uK)\) is an isometry, and \(\tau(u)\) commutes with \(\eta\) for \(u \in U\). Let \(H\) denote the subspace of holomorphic functions in \(L^2(U_{C/KC}, \omega_i)\).

Proof of Theorem 3. Let \(V\) be the algebraic span of the joint eigenspaces \(C_0(U/K), \delta \in U_K\). We argue as in the idea of the proof of \([6, \S4]\).
Theorem 2], to show that the Segal–Bargmann transform is an isometry from $V$ into $H$: for $f \in V$,

\[
\|e^{-td}f\|^2_{HS} = \int_{\gamma \in \exp \mathfrak{g} \cdot u \in U} |e^{-td}f \cdot \pi_C(uy)\, du \\
\times (\eta \cdot \pi_C |_{\exp \mathfrak{g}}^* (K_{\gamma}(\cdot, 2t) v_{\gamma})) \\
= \int_{\gamma \in \exp \mathfrak{g} \cdot u \in U} e^{-td}f \cdot \pi_C(uy) e^{-td} \tilde{f} \cdot \pi_C(uy^{-1})\, du \\
\times (\eta \cdot \pi_C |_{\exp \mathfrak{g}}^* (K_{\gamma}(\cdot, 2t) v_{\gamma})).
\]

Here $\tilde{f}(\cdot)$ denotes the holomorphic function on $U_C/K_C$ whose restriction to $UK$ is $f$. Using [6, Lemma 9], to make the “holomorphic change of variables” $w = uy^{-1}$ and Fubini’s Theorem, we find that the above is equal to

\[
\int_{u \in U} e^{-td}f(uK) \cdot \int_{y \in \exp \mathfrak{g}} e^{-td}f \cdot \pi_C(uy^2) (\eta \cdot \pi_C |_{\exp \mathfrak{g}}^* (K_{\gamma}(\cdot, 2t) v_{\gamma}))\, du.
\]

Since $f \in V$, there is a $g \in V$ such that $e^{-td}g = f$. Changing variables in the inner integral by $\tau(u) \cdot \eta \cdot \pi_C |_{\exp \mathfrak{g}}^* \exp \mathfrak{g} \to Y(uK)$ gives

\[
\|e^{-td}f\|^2_{HS} = \int_{u \in U} e^{-td}f(uK) \cdot \int_{\gamma(uK)} e^{-td}g(\cdot) K_{\gamma}(uK)(\cdot, 2t) v_{\gamma(uK)}\, du.
\]

Proposition 3 and the fact that $e^{-td}$ is formally self-adjoint give

\[
\|e^{-td}f\|^2_{HS} = \int_{u \in U} e^{-td}f(uK) g(uK)\, du = \int_{U/K} \tilde{f} f_{U/K}.
\]

To show that the Segal–Bargmann transform is an isometry from $L^2(U/K)$ onto $\mathcal{H}$, it suffices to show that the image of $V$ is dense in $\mathcal{H}$. We will briefly indicate how the arguments in [5, Lemmas 9, 10], can be adapted to our situation. The projection of $f \in L^2(U/K)$ onto $C_\delta(U/K)$ can be written as

\[
f_\delta(x) = d(\delta) \text{Tr}(\delta(x) A_{\delta,f}), \quad \text{where} \quad A_{\delta,f} = \int U \delta(u^{-1}) f(u)\, du.
\]
Here $\delta$ is the representation contragredient to $\delta$; see [8, Lemma 1.7, Corollary 1.8, Chap. IV]). Given a holomorphic function $F$ on $U_C/U_C$, consider the “holomorphic Fourier series” (HFS)

$$F(gK_C) = \sum_{\delta \in \hat{U}_C} d(\delta) \text{Tr}(\delta(g) A_{\delta, F|U_C}).$$

The proof of [5, Lemma 9] shows that the HFS converges uniformly and absolutely on compact subsets of $U_C/K_C$. Using the $\tau(U)$-invariance of $\omega$, one shows exactly as in [5, Lemma 10] that the terms in the HFS are orthogonal in $L^2(U_C/K_C, \omega)$ and remain orthogonal when integrated over any $\tau(U)$-invariant subset of $U_C/K_C$. The rest of the proof of [5, Lemma 10] goes through unchanged to show that the HFS for $F$ converges to $F$ in $L^2(U_C/K_C)$. Since each term in the HFS is in the image of $V$, the image is dense in $H$.

5. COMPARISON WITH B. HALL’S INVERSION FORMULA

Let $H$ be a compact, simply connected semisimple Lie group with a bi-invariant metric $g_H$ (normalized to unit volume) and Laplacian $A_H \geq 0$. Let

$$P = \{ \exp \sqrt{-1} V : V \in \mathfrak{h} \} \subset H_C$$

with the metric $g_P$ obtained by identifying $P$ with the symmetric space $H_C/H$ by the quotient map $q: H_C \to H_C/H$. Let $A_P$ be the nonnegative Laplacian on $P$, let $\sigma_i(p)$ be the fundamental solution at the identity of the equation $(\partial_t + A_P) u = 0$, and let $dp$ be the Riemannian measure on $P$. With these conventions, B. Hall's inversion formula for the Segal-Bargmann transform is

$$f(h) = \int_P e^{-tA_P (hp^2)} \sigma(t, p) \, dp \quad (f \in C^\infty(H)).$$

Put $U = H \times H$, let $K$ be the diagonal in $H \times H$, and let $\theta$ be the involutive automorphism of $\mathfrak{u}$ given by $\theta(X, Y) = (Y, X)$. Then $(\mathfrak{u}, \theta)$ is an orthogonal symmetric Lie algebra, and $U/K$ is a symmetric space of the compact type with respect to any $U$-invariant metric. We identify $U/K$ with $H$ by the $U$-equivariant map $g((g, h) K) = gh^{-1}$ and give $U/K$ the metric that makes $\alpha$ an isometry. There is a unique biholomorphic map $\alpha_C: U_C/K_C \to H_C$ extending $\alpha$, and it is not hard to see that $\alpha_C(Y(x))$ is $L_n x P$, the left translate of $P$ by $\alpha(x)$. We reconcile our inversion formula with Hall’s in the following proposition.
Proposition 4. For all \( f \in C^\infty(H) \), \( x \in U/K \), \( t > 0 \),
\[
\int_{Y(x)} e^{-it\mathfrak{a}_x^* f(\cdot) K_{x_t}(\cdot, t)} v_{x_t} = \int_P e^{-it\mathfrak{a}_x f(\mathfrak{a}(x) p^2)} \mathfrak{a}_x dp.
\]

Proof. Changing variables by \( \tau(x, e) : Y(o) \to Y(x) \) we may assume that \( x = o \). Let \( \sqrt{t} : P \to P \) be the diffeomorphism \( \sqrt{t}(p) = p^2 \) and let \( \sqrt{t} \) be its inverse. We will use the change of variables \( \sqrt{t}^* \) to \( Y(o) \). Since \( \mathfrak{a}_x \) is the biholomorphic extension of the isometry \( \mathfrak{a}_x \), we have for the analytic continuations \( (\mathfrak{a}_x^{-1})^* (e^{-it\mathfrak{a}_x^* f}) = e^{-it\mathfrak{a}_x f} \). To prove the proposition it suffices to show that \( (\sqrt{t} \cdot \mathfrak{a}_x)^* g = \frac{1}{4} g_{Y(o)} \). The main point is that \( \sqrt{t} \cdot \mathfrak{a}_x \) is equivariant with respect to the group actions on \( P \) and \( Y(o) \), as we now explain.

As above let \( G \) be the connected subgroup of \( U_C \) with Lie algebra \( g = \mathfrak{f} + \mathfrak{p} \). Here \( \mathfrak{p} = \{(\sqrt{-1} V, -\sqrt{-1} V) : V \in \mathfrak{h}\} \), so
\[
G = \{(z, \bar{z}) : z \in H_C\}, \quad \text{and} \quad Y(o) = \{(z, \bar{z}) K_C : z \in H_C\}.
\]

We will identify \( G \) with \( H_C \) by projection onto the first factor. \( G \) acts on \( Y(o) \) as a subgroup of \( U_C \), and \( H_C \) acts on \( P \) through its identification with \( H_C \) (not through the left action of \( H_C \) on \( H_C \)). The action of \( H_C \) on \( P \) is given explicitly by \( \tau_p(z)(p) = \sqrt{t} z p \bar{z}^{-1} \) (note that complex conjugation of \( H_C \) about \( H \) is a group automorphism). To show that \( \sqrt{t} \cdot \mathfrak{a}_x \) is equivariant with respect to the \( G_C \otimes H_C \) action, let \( (z, \bar{z}) \in G \equiv H_C \cdot (w, \bar{w}) K_C \in Y(o) \). Then
\[
\sqrt{t} \cdot \mathfrak{a}_x \cdot \tau((z, \bar{z}))((w, \bar{w})) = \sqrt{t} \mathfrak{a}_x \cdot \tau((z w \bar{w}^{-1} z^{-1}))
\]
\[
= \tau_p(z)((\sqrt{t} w \bar{w}^{-1}) = \mathfrak{a}_x \cdot \sqrt{t} \cdot \mathfrak{a}_x ((w, \bar{w}) K_C),
\]
which is the desired equivariance property.

Since \( g_{Y(o)} \) and \( g \) are \( G \equiv H_C \) invariant, it suffices to check that \( (\sqrt{t} \cdot \mathfrak{a}_x)^* g = \frac{1}{4} g_{Y(o)} \) at the identity coset. The identification of \( U/K \) with \( H \) determines an inner product on \( \mathfrak{p} ; \); the metric \( g_Y \) on \( Y(o) \) at the identity coset is given by (plus once times) this inner product under the usual identification of \( T_o Y(o) \) with \( \mathfrak{p} \). One computes
\[
g_{Y(o)}((\sqrt{-1} V, -\sqrt{-1} V), (\sqrt{-1} W, -\sqrt{-1} W)) = 4(V, W)_{\mathfrak{h}},
\]
where \( (\cdot, \cdot)_{\mathfrak{h}} \) is the \( \text{Ad}(H) \)-invariant inner product on \( \mathfrak{h} \) determined by \( g_H \).

The metric on \( P \) at the identity coset is given by \( -\cdot, \cdot \) under the identification of \( T_o P \) with \( \sqrt{-1} \mathfrak{h} \). One computes easily that the tangent map
at the identity coset in $Y(o)$ to $\sqrt{b}\cdot x_C$ is simply projection onto the first factor, so

$$(\sqrt{b}\cdot x_C)^* g_P((\sqrt{-1} V, -\sqrt{-1} V), (\sqrt{-1} W, -\sqrt{-1} W))^o = (V, W)_b.$$  

This proves that $(\sqrt{b}\cdot x_C)^* g_P = \frac{1}{4} g_{Y(o)}$.  

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\section*{References}