Borel Summation and Singularities in PDEs

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Background

Asymptotics is a well-recognized tool for linear/nonlinear equations

Asymptotics involve smallness/largeness of parameters or independent variable(s)

Most work to date involve formal expansions

Borel analysis seeks to rigorously establish correspondence between (usually divergent) formal expansions and actual solutions. This includes exponentially small terms, as well as singularities of solutions to differential systems

Pioneering work by Euler, Borel, Stokes, Dingle, Berry, Kruskal, Ecalle,...

Small t expansion for evolution PDEs

Initial Value Problem:

$$u_t = \mathcal{N}[u]\ ,\ u(x,0) = u_0(x)$$

$$u(x,t) = u_0(x) + t u_1(x) + t^2 u_2(x)\ +\ ..,$$

where $u_1(x) = \mathcal{N}[u_0](x)$, $u_2 = \frac{1}{2}\left\{\mathcal{N}_u[u]\mathcal{N}[u]\right\}(x)$,..

Such expansions generally divergent; illustrated for heat equation in Ovidiu Costin's talk. In that case the expansion can be Borel-summed to an actual solution:

$$u(x,t) = \frac{1}{\sqrt{t}} \int_0^\infty e^{-p/t} U(x,p) dp$$

$$U_{pp} - U_{xx} = 0\ \ ;\ \ \text{where}\ U(x,p)\ \text{is the Borel} - \text{Transform of u}$$

Small t Expansion-II

Note: $1/t$ not always correct choice for Borel-Transform

For instance, for Kuramato-Sivishinsky equation

$$u_t + u_{xx} + uu_x + u_{xxxx} = 0 \ , \ \ u(x,0) = u_0(x),$$

correct variable is $T = t^{-1/3}$ for Borel-Transform:

$$-\frac{T^4}{3} u_T + u_{xx} + uu_x + u_{xxxx} = 0$$

Writing $u(x,T) = u_0(x) + v(x,T)$ and Borel-Transforming in T:

$$\frac{1}{3} [pV]_{pppp} + V_{xx} + V_{xxxx} + u_0 V_x + V u_{0x} + V * V_x = R(x)$$

Fourth-order in both p and x; Cauchy-Kowalewski ideas may be applied. Solution $u(x,t) = u_0(x) + \int_0^\infty V(x,p) e^{-pt^{-1/3}} dp$

Borel Summability of Navier-Stokes for t small

$$v_t - \Delta v = -\mathcal{P}\partial_{x_j}[v_j v]\,,\; v(x,0) = v_0\,, \text{where } \mathcal{P} \text{ is the Hodge Projectio}$$

Introduce Norm $\|.\|_{\mu,\beta}$ so that

$$\|v_0\|_{\mu,\beta} = \sup_{k\in\mathbb{R}^3} e^{\beta|k|}(1+|k|)^{\mu}|\hat{v}_0(k)|$$

Theorem (O. Costin, S.T, 2006):

If $\|v_0\|_{3+\mu,\beta} < \infty$ *where* $\beta > 0$ *and* $\mu > 3$*, then there exists* $\gamma = \gamma\left(\|v_0\|_{3+\mu,\beta}\right)$ *so that if* $\mathrm{Re}\,\frac{1}{t} > \gamma$*, solution to Navier-Stokes equation exist with* $\|v(.,t)\|_{\mu,\beta}$ *finite. More over,*

$$v(x,t) = v_0(x) + tv_1(x) + \int_0^\infty e^{-p/t}U(x,p)dp$$

where $U(x,p)$ *is analytic for* $p \in \{0\}\cup\mathbb{R}^+$ *and exponentially bounded on* $\mathbb{R}^+$*.*

Further, for $0 < t << 1$*,* $\;v(x,t) \sim v_0(x) + tv_1(x) + ...$

Modified Harry-Dym

$$H_t + H_z - H^3 H_{zzz} - \frac{H^3}{2} = 0 \ , \ \ H(z,0) = z^{-1/2}$$

Asymptotic condition: $H(z,t) \sim z^{-1/2}$ **as** $|z| \to \infty$**, with** $\arg z \in \left(-\frac{4}{9}\pi \ , \ \frac{4}{9}\pi\right)$

Problem relevant in viscous fingering and dendritic crystal growth in the small surface tension limit

Sectorially unique analytic solution guaranteed for any t **when** $|z-t| >> t$ **by general theory (O. Costin & S.T (CPAM, 2000)).**

Seek to find singularities of $H(z,t)$ **for** $z \in \mathbb{C}$

MHdym: $t << 1$ expansion and scales

$$H_t + H_z - H^3 H_{zzz} - \frac{H^3}{2} = 0 \ , \ \ H(z,0) = z^{-1/2}$$

Asymptotics for $0 < t << 1$, $y = z - t >> t^{2/9}$:

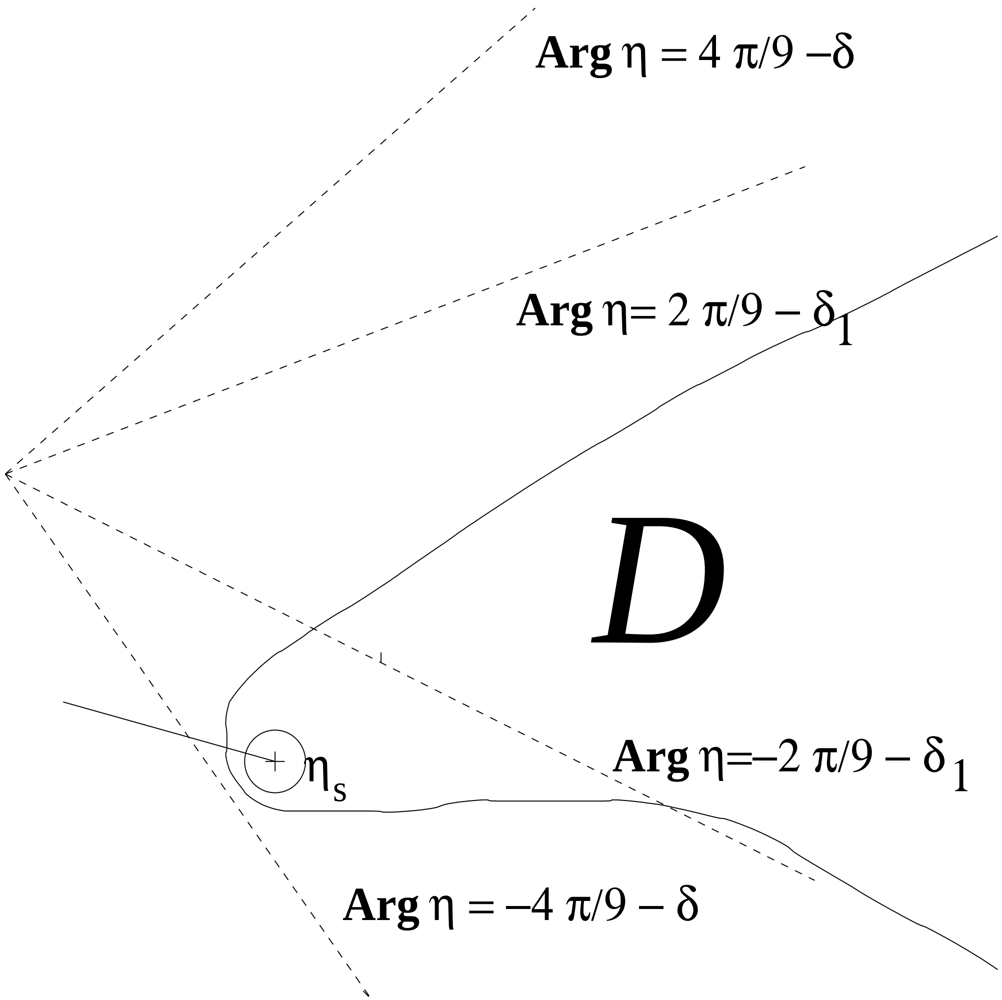
$$H(z,t) = y^{-1/2} - t\left(\frac{1}{2y^{3/2}} + \frac{15}{8y^5}\right) + t^2\left(\frac{195}{32y^6} + \frac{3}{8y^{5/2}} + \frac{25875}{128y^{19/2}}\right) +$$

For $y = z - t = O(t^{2/9})$, $H(z,t) = t^{1/9} G(t^{-2/9}(z-t), t^{7/9})$

$$\frac{7}{9}\tau G_\tau - \frac{2}{9}\eta G_\eta - \frac{G}{9} + \frac{\tau}{2} G^3 - G^3 G_{\eta\eta\eta} = 0$$

Expansion $G(\eta,\tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta)$ convergent for τ small for $\eta \in \mathcal{D}$, where $\mathcal{D}$ encircles a singularity η_s of G_0

Domain $\mathcal{D}$ of τ-series convergence



Leading order similarity ODE for G_0

Leading order G_0 satisfies the nonlinear ODE:

$$\frac{G_0}{9} + \frac{2}{9}\eta G_0' + G_0^3 G_0''' = 0$$

with requirement $G_0(\eta) \sim \eta^{-1/2}$ as $|\eta| \to \infty$, $\eta \in \mathcal{D}$.

Transformation $G_0(\eta) = \eta^{-1/2}\left[1 + g(\eta^{9/4})\right]$ reduces it to a normal form due to O.Costin (Duke J. '98)

$$g''' + \frac{1}{\xi}g'' + \left(\frac{11}{81\xi^2} + \frac{32}{729}\frac{1}{(1+g)^3}\right)g' = \frac{40}{243}\left(\frac{1}{\xi^3} + \frac{g}{\xi^3}\right)$$

With small changes, O.Costin & R. Costin (Inventiones, 2001) analysis holds: a denumerable set of singularities approaching relevant anti-Stokes lines

Illustration of ideas behind Costin-Costin results

$$y' + y = \frac{1}{x^2} + y^2 \ , \ \ y \to 0 \ \ x \to +\infty$$

Primary series: $y \sim \frac{1}{x^2} + \frac{a_{3,0}}{x^3} + \ldots$ **With exponential corrections**

$$y \sim \left\{ \frac{1}{x^2} + \frac{a_{3,0}}{x^3} + \ .. \right\} + Ce^{-x} \left\{ 1 + \frac{a_{1,1}}{x} + \ .. \right\} + C^2 e^{-2x} \{..\} + \ldots$$

As $\arg x \to \pm\frac{\pi}{2}$**, there is domain** $\mathcal{R}$**, where** $e^{-kx} >> \frac{1}{x}$ **for which**

$$y \sim \{ Ce^{-x} + C^2 e^{-2x} a_{0,2} \ + \ ..\} + \frac{1}{x} \{ Ce^{-x} a_{1,1} + C^2 e^{-2x} a_{1,2} \ + \ ..\}$$

$$+ \frac{1}{x^2} \{ 1 + Ce^{-x} a_{2,1} + ..\} \ + \ ..$$

suggests $y = F(Ce^{-x}, x) = F_0(Ce^{-x}) + \frac{1}{x} F_1(Ce^{-x})$**... Plugging in get** $-\chi F_0' + F_0 - F_0^2 = 0$ **, solution** $F_0(\chi) = \frac{\chi}{1+\chi}$

Singularity results for MHDYM for $G(\eta, \tau)$

Straddling the sector $\arg \eta \in \left(-\frac{4}{9}\pi\ ,\ \frac{4}{9}\pi\right)$, there exists singularities for leading order G_0, where $G_0(\eta) \sim\ e^{i\pi/3}\left(\frac{\hat{\eta}_s}{3}\right)^{1/3}(\eta - \hat{\eta}_s)^{2/3}$. $\hat{\eta}_s$ approaches η_s for large $|\hat{\eta}_s|$, where η_s is determined from

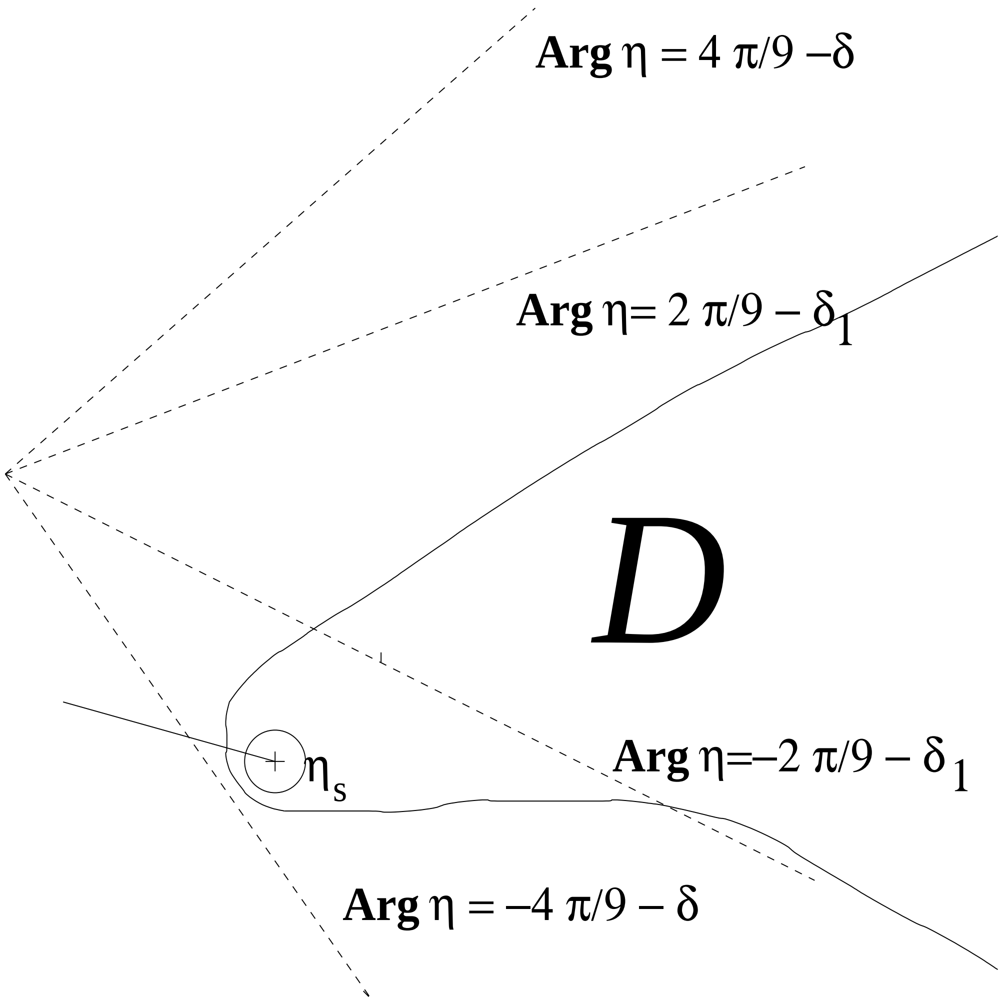
$$\frac{i4\sqrt{2}}{27}\eta_s^{9/4} + \frac{9}{8}\log \eta_s = -2 + \log 4 - 2\hat{n}i\pi + \log C$$

for $\hat{n} \in \mathbb{Z}$

For the full equation $G(\eta, \tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta)$ converges in $\mathcal{D}$.

Theorem *(O. Costin, ST, CPDE, 05) For a singularity η_s of G_0, as $\tau \longrightarrow 0^+$, $G(\eta, \tau)$ (and hence $H(z, t)$) is singular near η_s, to leading order of the same type, $(\eta - \eta_s)^{2/3}$*

Domain $\mathcal{D}$ of τ-series convergence



Equations for $G_k(\eta)$

Recall G satisfies:

$$-\frac{G}{9} - \frac{2}{9}\eta G_\eta + \frac{7}{9}\tau G_\tau + \frac{\tau}{2}G^3 - G^3 G_{\eta\eta\eta} = 0$$

Plugging $G(\eta,\tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta)$, $G_0(\eta)$ satisfies:

$$\frac{1}{9}G_0 + \frac{2}{9}\eta G_0' + G_0^3 G_0''' = 0$$

G_k satisfies linear equation:

$$\mathcal{L}_k G_k \equiv G_k''' + \frac{2}{9G_0^3}\eta G_k' - \left(\frac{7k-1}{9G_0^3} + \frac{3G_0'''}{G_0}\right) G_k = \frac{R_k}{G_0^3}$$

R_k known in terms of G_j, $j = 0, 1..(k-1)$

For $|\eta| \rightarrow \infty, \eta \in \mathcal{D}$:

$$G_0(\eta) \sim \eta^{-1/2} \ , \ G_k(\eta) \sim a_k \, \eta^{-k-1} \ \text{ for known } a_k$$

Crucial Lemma in Proof of Theorem for MHDYM

Lemma: *There exists* A *and* B *independent of integer* k *so that*

$$\|\eta^{3/2} G_k\|_{\infty,\mathcal{D}} \leq \frac{BA^k}{(k+1)^3}$$

Control for large k needed in $\mathcal{L}_k G_k = \frac{R_k}{G_0^3}$. WKB solutions to:

$$\mathcal{L}_k u \equiv u''' + \frac{2}{9G_0^3}\eta u' - \left(\frac{7k-1}{9G_0^3} + \frac{3G_0'''}{G_0}\right) u = 0$$

$$u_j = G_0(\eta) \exp\left[\omega_j k^{1/3} P(\eta)\right] \text{ , where } \omega_j^3 = 1 \text{ , } P(\eta) = \int_{\eta_0}^{\eta} \frac{1}{G_0(\eta')} d$$

Recall $G_0(\eta) \sim \eta^{-1/2}$ for large η only when $\arg \eta \in (-4\pi/9, 4\pi/9)$. Near $\eta = \eta_s >> 1$, $G_0(\eta) = \eta^{-1/2} U(\zeta)$, where $\zeta = \log C - \frac{9}{8}\log \eta + i\frac{4\sqrt{2}}{27}\eta^{9/4}$, $\frac{1}{4}e^{\zeta+2} = e^{2\sqrt{U}}\left(\frac{\sqrt{U}+1}{\sqrt{U}-1}\right)$.

Illustrative Example for large k control

Suppose in $\mathcal{D} \subset \mathbb{C}$, want u satisfying

$$\mathcal{L}_k u = kr \quad , \quad \text{where } e^{\pm k P(\eta)} \text{ solve } \mathcal{L}_k v = 0$$

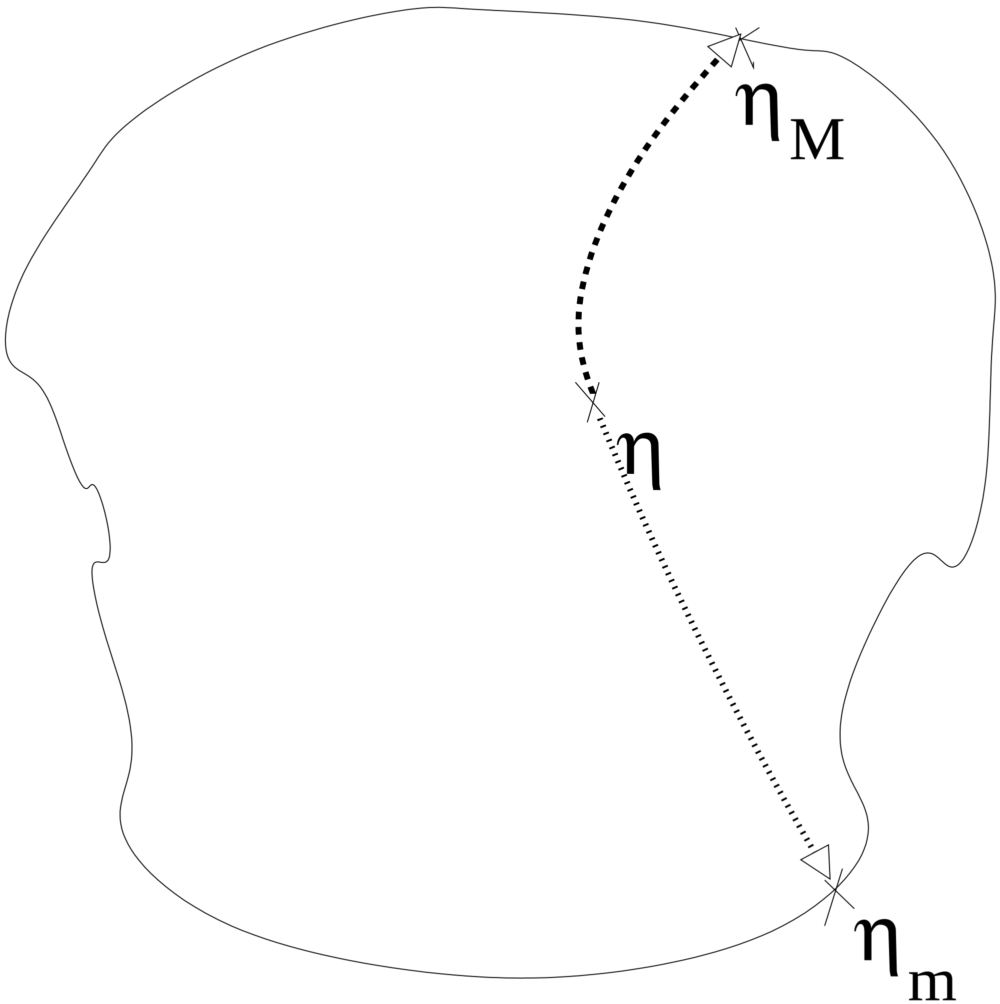
Know that $\mathrm{Re}\, P(\eta_M) = +\infty$ and $\mathrm{Re}\, P(\eta_m) = -\infty$ and at other points in $\partial\mathcal{D}$, P is finite and continuous, while analytic in $\mathcal{D}$. Want u so that

$$\|u\|_{\infty,\mathcal{D}} \leq C \|r\|_{\infty,\mathcal{D}}$$

$$u(\eta) = \int_{\eta_M}^{\eta} e^{k[P(\eta)-P(\eta')]} r(\eta') d\eta' - \int_{\eta_m}^{\eta} e^{-k[P(\eta)-P(\eta')]} r(\eta') d\eta',$$

u above satisfes desired bounds if there exists paths $\mathcal{C}_1$, $\mathcal{C}_2$ in the η'-plane connecting η to η_M, η_m so that $\mathrm{Re}\, P(\eta')$ is monotonic, since $|e^{k[P(\eta)-P(\eta')]}| \leq 1$ on $\mathcal{C}_1$, $|e^{-k[P(\eta)-P(\eta')]}| \leq 1$ on $\mathcal{C}_2$.

Domain $\mathcal{D}$ for illustrative problem



Finding paths $\mathcal{C}_1$, $\mathcal{C}_2$

Problem is to ensure such paths $\mathcal{C}_1$, $\mathcal{C}_2$ can be found. One way is to choose Steepest-descent paths $\mathrm{Im}\, P = c$, when possible.

Note if $P(\eta) = \int_{\eta_0}^{\eta} \frac{1}{G_0(\eta)}$, then, steepest descent paths are generated by:

$$\frac{d\eta}{ds} = G_0(\eta) \quad , \quad \text{where} \quad s \in \mathbb{R}^+$$

since

$$\frac{d}{ds}\,\mathrm{Im}\, P = \mathrm{Im}\left\{ \frac{d\eta}{ds}\frac{dP}{d\eta} \right\} = \mathrm{Im}[1] = 0$$

Implicit Descent Problem as Dynamical System

Steepest descent too restrictive. Instead generate descent or ascent paths for $\mathrm{Re}\, P$, by solving

$$\frac{d\eta}{ds} = \pm e^{i\phi} G_0(\eta) \ , \ \ \phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

For MHDYM problem, $G_0(\eta)$ near singularity known as an ODE solution. Convenient to choose $u(s) \equiv G_0(\eta(s))$ an unknown along with $\eta(s)$ treat $(\eta, u) \in \mathbb{C}^2$ as a function of s.

Other Complications in MHDYM: 3rd order ODE and only WKB solution to $\mathcal{L}_k u = 0$ known. Tackled as an integral equation:

$$\mathcal{L}_k u = kr \ \ \text{written as}\ \mathcal{L}_{WKB} u = [\mathcal{L}_{WKB} - \mathcal{L}_k] u + kr$$

$$u = \mathcal{L}_{WKB}^{-1} \left[[\mathcal{L}_{WKB} - \mathcal{L}_k] u + kr \right]$$

Conclusions

For some PDEs, including Navier-Stokes, Borel summing $t << 1$ expansions gives a rigorous theory for actual solutions

For some evolutionary nonlinear PDEs, singularity generation in $\mathbb{C}$ can be studied rigorously by transforming to scaled variables and proving convergence of $G(\eta, \tau) = \sum_{k=0}^{\infty} \tau^k G_k(\eta)$ in a region encircling singularity of G_0. Singularity study for G_0 relies on a minor adaptation of Costin-Costin (2001) analysis. The divergence in asymptotic series for $t << 1$ (or $|z| >> 1$) is encapsulated in $G_k(\eta)$

Analysis developed useful in similarity blow-up for PDEs:

$$h(x,t) = \frac{1}{(t_s - t)^p} H\left(\frac{x - x_s}{(t_s - t)^q}, (t_s - t)^r\right)$$

Proving $H(\eta, \tau) \sim H_0(\eta)$ demonstrates blow-up.