Analyticity and Nonexistence of Classical Steady Hele-Shaw Fingers

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Abstract

This paper concerns analyticity of a classical, steadily translating finger in a Hele-Shaw cell and nonexistence of solutions when the relative finger width λ is smaller than $\frac{1}{2}$. It is proven that any classical solution to the finger problem, if it exists for sufficiently small but nonzero surface tension, is close to some Saffman-Taylor zero-surface-tension solution, and satisfies some algebraic decay conditions at ∞, must belong to the analytic function space $\mathbf{A}_0$, as defined in Section 1, and chosen in a previous study [34] of the existence of finger solutions. Further, it is proven that for any fixed $\lambda \in (0, \frac{1}{2})$, there can be no classical steady-finger solution when surface tension is sufficiently small, which contradicts a previous conclusion based on numerical simulation.

1 Introduction

1.1 Background

The problem of a less viscous fluid displacing a more viscous fluid in a Hele-Shaw cell has been the subject of numerous investigations since the 1950s. Reviews of the subject from various perspectives can be found in [1, 10, 12, 19, 22, 31]. In a seminal paper, Saffman and Taylor [23] found experimentally that an unstable planar interface evolves through finger competition to a steady translating finger, with relative finger width λ close to a half at large displacement rates. Theoretical calculations [23, 35] ignoring surface tension revealed a one-parameter family of exact steady solutions, parametrized by width λ. When the experimentally determined λ was used, the theoretical shape (usually referred to in the literature as the Saffman-Taylor finger) agreed well with experiments for relatively large displacement rates, or, equivalently, for small surface tension. However, in the zero-surface-tension steady-state theory, λ remained undetermined in the (0, 1) interval.

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The selection of λ remained unresolved until the mid 1980s. Numerical calculations [14, 17, 32] supported by formal asymptotic calculations in the steady finger [3, 4, 5, 9, 11, 25, 29] and the closely related steady Hele-Shaw bubble problem [6, 27] suggested that a discrete family of solutions exists for which the limiting shape, as surface tension tends to zero, approaches the Saffman-Taylor with $\lambda = \frac{1}{2}$. Subsequent numerical [13] and formal asymptotic [28] calculations suggest that only a branch of the solutions is stable. However, the conclusion about the existence of steady states is not universally accepted. Based on numerical simulation of a time-evolving interface for small but nonzero surface tension, and with the same model equations used in [5, 17], it was suggested [8] that the limiting steady shape was a Saffman-Taylor solution with $\lambda < \frac{1}{2}$. In this paper, we conclude otherwise through rigorous mathematical analysis. It is to be noted that selection of the Saffman-Taylor finger with $\lambda < \frac{1}{2}$ is possible for a more mathematically complicated model that incorporates thin-film effects [18, 19, 21], as shown in [20, 30]. The same is true when anisotropy [9] in surface tension or other perturbations near the tip are introduced.

There has been a rigorous study [26] for a problem mathematically similar, though not identical, to the steady viscous fingering problem considered here. In that case, it was proven that at least one finger solution exists for fixed surface tension, though the relative finger width and shape remains unknown. On the other hand, our primary focus is the selection of finger width as surface tension tends to zero. A mathematically rigorous study of selection is difficult in this limit since exponentially small terms in surface tension play a critical role. While a rigorous theory of exponential asymptotics for nonlinear ordinary differential equations is by now well developed ([7], for instance), this is not the case for integro-differential equations, even though such problems have arisen in a number of other physical contexts like dendritic crystal growth and water waves; see, for instance, [24]. Formal calculations rely on the assumption that integral terms do not contribute to exponentially small terms, at least to the leading order. With this assumption, integro-differential equations are simplified to essentially nonlinear ordinary differential equations, where variants of the procedure due to Kruskal and Segur [15, 16] have been used. We have recently shown [33, 34] how integral terms can be controlled and a rigorous theory was developed for the integro-differential equation presented here.

1.2 Conditions and Definitions

Following [31], a steady symmetric finger is equivalent to finding function F analytic in the upper-half ξ-plane ($\mathbb{C}^+$) and twice differentiable in its closure, i.e., in $\mathcal{C}^2(\bar{\mathbb{C}}^+)$, such that the following conditions are satisfied:

CONDITION 1: On the real ξ-axis, F satisfies

$$\operatorname{Re} F = \frac{\epsilon^2}{|F' + H|} \operatorname{Im} \left[\frac{F'' + H'}{F' + H} \right], \tag{1.1}$$

where

$$H(\xi) = \frac{\xi + i\gamma}{\xi^2 + 1} \quad \text{with} \quad \gamma = \frac{\lambda}{1-\lambda}, \ \epsilon^2 = \frac{\pi^2 \lambda \mathcal{B}}{4(1-\lambda)^2}, \tag{1.2}$$

where λ is the relative finger width and $\mathcal{B}$ is a nondimensional surface tension parameter.

CONDITION 2:

$$F(\xi), \xi F'(\xi) \to 0 \quad \text{as } \xi \to \pm\infty . \tag{1.3}$$

CONDITION 3 (SYMMETRY CONDITION):

$$\operatorname{Re} F(-\xi) = \operatorname{Re} F(\xi), \ \operatorname{Im} F(-\xi) = -\operatorname{Im} F(\xi) \quad \text{for real } \xi . \tag{1.4}$$

DEFINITION 1.1 Let $\mathcal{R}$ be the open connected set between $\operatorname{Im}\xi = 0$ and $\ell^+ \cup \ell^-$ where

$$\ell^+ = \left\{\xi : \xi = -ib + re^{-i\varphi_0}, \ 0 < r < \infty, \ b > 0, \ \tfrac{\pi}{2} > \varphi_0 > 0 \text{ fixed}\right\},$$

$$\ell^- = \left\{\xi : \xi = -ib - re^{i\varphi_0}, \ 0 < r < \infty\right\}.$$

We also define $\mathcal{R}^- = \mathcal{R} \cap \{\xi : \operatorname{Re}\xi < 0\}$ and $\mathcal{R}^+ = \mathcal{R} \cap \{\xi : \operatorname{Re}\xi > 0\}$.

DEFINITION 1.2 For fixed $\tau \in (0, 1)$,

$$\mathbf{A}_j = \Big\{F : F(\xi) \text{ is analytic in } \{\operatorname{Im}\xi \geq 0\} \cup \mathcal{R} \text{ with } \|F\|_j \equiv \sup_{\xi \in \overline{\mathcal{R}}} |(\xi - 2i)^{j+\tau} F(\xi)| < \infty\Big\}, \quad j = 0, 1, 2,$$

$$\mathbf{A}_{0,\hat{\delta}} = \left\{F : F \in \mathbf{A}_0, \ \|F\|_0 \leq \hat{\delta}\right\}, \quad \mathbf{A}_{1,\hat{\delta}_1} = \left\{F : F \in \mathbf{A}_1, \ \|F\|_1 \leq \hat{\delta}_1\right\}.$$

1.3 Findings

Our previous result [34] on the existence of a solution satisfying Conditions 1 through 3 involved $F \in \mathbf{A}_{0,\hat{\delta}}$ and $F' \in \mathbf{A}_{1,\hat{\delta}_1}$, where $\hat{\delta}$ and $\hat{\delta}_1$ are assumed a priori to be small but independent of ϵ. In this function space, for $\lambda \in [\frac{1}{2}, \lambda_m)$, with $\lambda_m - \frac{1}{2}$ small enough (though independent of ϵ), it was shown that a solution existed if and only if

$$\frac{2\lambda - 1}{1 - \lambda} = \epsilon^{4/3} \beta_n(\epsilon^{2/3}),$$

where $\{\beta_n\}_{n=1}^{\infty}$ is a sequence of functions that are analytic at the origin.

However, there are two limitations of this result: The first is the choice of the function space. Nonexistence in this function space need not mean nonexistence of a classical solution F, analytic in $\mathbb{C}^+$ and $\mathcal{C}^2$ in its closure $\bar{\mathbb{C}}^+$. The second limitation is the restriction on λ. In this paper, we prove two theorems (Theorems 1.3 and 1.8) to relax these restrictions to a great degree.

THEOREM 1.3 *For small enough ϵ, any analytic function F in the upper-half ξ-plane $\mathbb{C}^+$, which is C^2 on its closure and satisfies Conditions* 1 *through* 3 *belongs to function space $\mathbf{A}_{0,\hat{\delta}}$, with $F' \in \mathbf{A}_{1,\hat{\delta}_1}$, where $\hat{\delta} = O(\epsilon^2)$ and $\hat{\delta}_1 = O(\epsilon)$, provided the following two assumptions are also satisfied*:

ASSUMPTION 1: *There exists τ independent of ϵ, $0 < \tau < 1$, so that*

$$\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau} |F(\xi)| \equiv \delta < \infty \,. \tag{1.5}$$

ASSUMPTION 2: *We assume that each of δ_1 and $\epsilon \ln \frac{1}{\epsilon} \delta_2$ are sufficiently small, where*

$$\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{1+\tau} |F'(\xi)| = \delta_1, \quad \sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{2+\tau} |F''(\xi)| = \delta_2 \,, \tag{1.6}$$

Remark 1.4. Assumption 1 is consistent with the results from McLean and Saffman's formal procedure [17] near the tail of a finger that in our formulation implies $F \sim a_0 e^{i\pi\hat{\tau}/2} \xi^{-\hat{\tau}}$ as $\xi \to +\infty$, where $\hat{\tau}$ is a positive root of the transcendental equation $\cot(\frac{\pi}{2}\hat{\tau}) = \epsilon^2 \hat{\tau}^2$ and a_0 is real. Condition 3 on symmetry implies similar behavior as $\xi \to -\infty$. This asymptotic relation was also found to be consistent with numerical calculations [17]. While $\hat{\tau}$ depends on ϵ, we can clearly choose $\tau < \hat{\tau}$ independent of ϵ for small ϵ ($\tau = \frac{1}{2}$ would suffice, for instance). The next lemma shows that we need not assume a priori that δ_1 and δ_2 in Assumption 2 exist and are finite, only that δ_1 and $\epsilon \ln \frac{1}{\epsilon} \delta_2$ are small. Note that Assumptions 1 and 2 are stronger than Condition 2. However, these assumptions are mild since the slope deviation from some Saffman and Taylor solution (which scales as δ_1 in the above theory) is observed to be small in experiment for large displacement rates and in all numerical calculations for small ϵ; we are not making any a priori assumption on how this deviation scales with ϵ. Also, the curvature deviation (which scales as F'', and hence δ_2) a priori is allowed to be large, though not as large as $1/(\epsilon \ln 1/\epsilon)$.

LEMMA 1.5 *If F satisfies Assumption* 1 *in addition to being analytic in $\mathbb{C}^+$, C^2 in $\bar{\mathbb{C}}^+$, and satisfying Conditions* 1 *through* 3*, then*

$$\sup_{\xi \in \mathbb{C}^+} |\xi + 2i|^{\tau} |F(\xi)| = \delta < \infty \,, \tag{1.7}$$

$$\sup_{\xi \in \mathbb{C}^+} |\xi + 2i|^{1+\tau} |F'(\xi)| = \delta_1 < \infty \,, \tag{1.8}$$

$$\sup_{\xi \in \mathbb{C}^+} |\xi + 2i|^{2+\tau} |F''| = \delta_2 < \infty \,. \tag{1.9}$$

The proof of this lemma relies on some straightforward properties of the Hilbert transform and the use of Phragmen-Lindelof methods and is relegated to Appendix A.

Remark 1.6. From examining (1.1), $\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau} |\operatorname{Re} F(\xi)| = O(\epsilon^2 \delta_2)$. From the Hilbert transform of $\operatorname{Re} F$ (which gives $\operatorname{Im} F$ on the real axis) and by using Lemma A.1 with $g = \operatorname{Re} F$ and $k = \epsilon$, it follows from Assumption 2 and Lemma 1.5 that $\delta = o(\epsilon)$.

DEFINITION 1.7 F will be called a classical solution if F is analytic in the upper half ξ-plane ($\mathbb{C}^+$), C^2 in its closure $\bar{\mathbb{C}}^+$, satisfies Conditions 1 through 3 and Assumptions 1 and 2.

In Section 4 we will prove the following theorem:

THEOREM 1.8 *For any fixed* $\lambda \in (0, \frac{1}{2})$*, there exists* $\epsilon_0 > 0$ *small so that there can be no classical solution* F *for any* ϵ *in the interval* $(0, \epsilon_0]$.

1.4 Outline of the Paper

The strategy followed in this paper is as follows: We first derive the integro-differential equation for F in the lower half-plane, if indeed F can be analytically continued, as done before. Besides F and its derivatives, this integro-differential equation involves functions $I(\xi)$ and $\bar{F}(\xi)$, each of which are analytic in the lower half-plane for any classical solution F. $I(\xi)$ can be calculated from F on the real axis alone, while $\bar{F}(\xi)$ is defined as the analytic function that equals $F^*(\xi)$ on the the real axis. $\bar{F}$ in the lower half-plane is completely determined by the classical solution F in the upper half-plane. We replace F in this integro-differential equation by f, which we now think of as an unknown, unlike I and $\bar{F}$, which are considered known in terms of a classical solution F (if one exists). Thus, we obtain a second-order nonlinear differential equation for f in the lower half-plane. We prove that a unique solution f exists in the appropriate analytic function space in part of the lower half-plane. The uniqueness argument is repeated for ξ on a real-line segment. Since $f = F$ is a solution on the real-line segment, it follows that any classical solution F, if it exists, must be analytic in some region of the lower half-plane. All the above arguments are detailed in Section 2. In Section 3, we use additional arguments to prove that analyticity of F extends to other regions in the lower half ξ-plane region, including part of the negative imaginary ξ-axis. Thus, we complete the proof that a classical solution, if one exists, must be in the function space $\mathbf{A}_0$ for any fixed $\lambda \in (0, 1)$.

In Section 4, we consider the special case $\lambda \in (0, \frac{1}{2})$. By considering a neighborhood of a turning point $\xi = -i\gamma$ in the lower half-plane, considering the leading-order inner equation, and using continuity arguments, we prove that on an imaginary ξ, just above $-i\gamma$, $\operatorname{Im} F \neq 0$ for sufficiently small ϵ, provided the Stokes constant for a particular nonlinear parameter-free ODE is nonzero. This Stokes constant was computed by other researchers and found to be nonzero. We then invoke analyticity arguments of previous sections to prove that a nonvanishing $\operatorname{Im} F$ on a part of the imaginary ξ-axis above $\xi = -i\gamma$ is inconsistent with symmetry Condition 3. Hence we conclude that no classical solution can exist for any fixed λ in the $(0, \frac{1}{2})$ interval for sufficiently small ϵ.

2 Analytic Continuation to the Lower Half-Plane

DEFINITION 2.1 Let F be analytic in the upper half ξ-plane and $\bar{F}$ is an analytic function in the lower half ξ-plane defined by

$$\bar{F}(\xi) = [F(\xi^*)]^* \,, \tag{2.1}$$

$$\bar{H} = \frac{\xi - i\gamma}{\xi^2 + 1} \,. \tag{2.2}$$

We define operator $\mathcal{G}$ so that

$$\mathcal{G}(f,g)[t] := \tag{2.3}$$
$$\frac{1}{(f'(t)+H(t))^{1/2}(g'(t)+\bar{H}(t))^{1/2}}\left[\frac{f''(t)+H'(t)}{f'(t)+H(t)}-\frac{g''(t)+\bar{H}'(t)}{g'(t)+\bar{H}(t)}\right].$$

LEMMA 2.2 *If F is a classical solution as in Definition* 1.7*, then*

$$\mathcal{G}(F,\bar{F})[\xi] = O(\xi^{-\tau}) \quad \textit{as } \xi \to \pm\infty\,. \tag{2.4}$$

PROOF: Since the right-hand side of (1.1) can be written as $\frac{\epsilon^2}{2i}\mathcal{G}(F,\bar{F})(\xi)$, the lemma follows from (1.5). □

DEFINITION 2.3 We define operator $\mathcal{I}$ so that

$$I(\xi) \equiv \mathcal{I}(F)[\xi] = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathcal{G}(F,\bar{F})[t]dt}{t-\xi} \quad \text{for } \operatorname{Im}\xi < 0\,. \tag{2.5}$$

LEMMA 2.4 *For $I(\xi)$ in the lower half-plane $\mathbb{C}^- = \{\xi : \operatorname{Im}\xi < 0\}$, we have*

$$\sup_{\xi\in\mathbb{C}^-}|\xi-2i|^{\tau}\epsilon^2|I(\xi)| = \sup_{\xi\in(-\infty,\infty)}|\xi-2i|^{\tau}|F(\xi)| = \delta\,. \tag{2.6}$$

PROOF: From (1.1), (2.3), and (2.5), $\lim_{\operatorname{Im}\xi\to 0^-}\epsilon^2 I(\xi) = -\bar{F}(\xi)$ for ξ real. Since $I(\xi)$ is analytic in the lower half-plane, this lemma follows from Lemma A.5. □

LEMMA 2.5 *Let F be a classical solution to the finger problem. If $F(\xi)$ can be analytically continued at least to a part of $\mathbb{C}^-$, then F satisfies*

$$\epsilon^2F''(\xi)+L(\xi)F(\xi) = \mathcal{N}(F,I,\bar{F})[\xi] \quad \textit{for } \{\xi\in\mathbb{C}^-\}\,, \tag{2.7}$$

where

$$\begin{aligned} L(\xi) &= -iH^{3/2}(\xi)\bar{H}^{1/2}(\xi) = -\frac{i\sqrt{\gamma^2+\xi^2}(\xi+i\gamma)}{(\xi^2+1)^2}\,,\\ \bar{F}(\xi) &\equiv [F(\xi^*)]^*\,, \end{aligned} \tag{2.8}$$

and the operator $\mathcal{N}$ is defined as

$$\begin{aligned} \mathcal{N}(F,I,\bar{F}) = {}& \epsilon^2\left(\frac{\bar{H}'H}{\bar{H}}-H'\right) - i\epsilon^2(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2}I\\ &+ iF\big[(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2}-H^{3/2}\bar{H}^{1/2}\big]\\ &+\epsilon^2\left[(\bar{F}''+\bar{H}')\frac{F'+H}{\bar{F}'+\bar{H}}-\frac{\bar{H}'H}{\bar{H}}\right]. \end{aligned} \tag{2.9}$$

PROOF: Since F is analytic in the upper half ξ-plane and satisfies equation (1.1), using Poisson formula, we have in the upper half-plane

$$\begin{aligned} F(\xi) &= \frac{\epsilon^2}{\pi i} \int_{-\infty}^{\infty} \frac{dt}{(t-\xi)} \frac{1}{|F'(t)+H(t)|} \operatorname{Im} \left[\frac{F''(t)+H'(t)}{F'(t)+H(t)} \right] \\ &= -\frac{\epsilon^2}{2\pi} \int_{-\infty}^{\infty} \frac{\mathcal{G}(F,\bar{F})[t]dt}{t-\xi}, \quad \operatorname{Im}\xi > 0 . \end{aligned} \tag{2.10}$$

Using Plemelj formula [2], analytic continuation to the lower half ξ-plane leads to

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i} \mathcal{G}(F,\bar{F})(\xi) \quad \text{for } \operatorname{Im}\xi < 0 . \tag{2.11}$$

Multiplying the above by $i(F'+H)^{3/2}(\bar{F}'+\bar{H})^{1/2}$ results in (2.7). □

DEFINITION 2.6

$$g_1(\xi) = L^{-1/4}(\xi) \exp\left\{ -\frac{P(\xi)}{\epsilon} \right\}, \tag{2.12}$$

$$g_2(\xi) = L^{-1/4}(\xi) \exp\left\{ \frac{P(\xi)}{\epsilon} \right\}, \tag{2.13}$$

where

$$P(\xi) = i \int_{-i\gamma}^{\xi} L^{1/2}(t)dt = i \int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4}(\gamma+it)^{\frac{1}{4}}}{(1+t^2)} \, dt . \tag{2.14}$$

A branch of $L^{1/2}$ in the definition of P is chosen so that, as $\xi \to -\infty$, $P'(\xi) \sim e^{-i\pi/4}/\xi$. The choice of branch for $L^{-1/4}$ in (2.12) is not as important as long as the same branch is consistently chosen.

We will use the following properties of $P(\xi)$, which are shown in Appendix B. Some of these properties were shown in the appendix of Xie and Tanveer [34] for the restricted case $\lambda \in [\frac{1}{2}, \lambda_m)$.

PROPERTY 1: Re $P(\xi)$ decreases along the negative Re ξ-axis $(-\infty, 0)$ with Re $P(-\infty) = \infty$. Re $P(\xi)$ decreases monotonically on the imaginary ξ-axis from $-ib$ to 0 where $0 < b < \min\{1, \gamma\}$.

PROPERTY 2: There exists a constant R independent of ϵ so that for $|\xi| \geq R$, Re $P(t)$ increases with increasing s along any ray

$$r = \left\{ t : t = \xi - se^{i\varphi},\ 0 < s < \infty,\ 0 \leq \varphi \leq \varphi_0 < \tfrac{\pi}{2} \right\}$$

in $\mathcal{R}$ from ξ to $\xi + \infty e^{i\varphi}$ and

$$C_1 |t - 2i|^{-1} \leq \left| \frac{d}{ds} \operatorname{Re} P(t(s)) \right| \leq C_2 |t-2i|^{-1} ,$$

where C_1 and C_2 are constants, independent of ϵ, with $C_1 > 0$.

PROPERTY 3: There exists sufficiently small $\nu > 0$ independent of ϵ so that $\frac{d}{ds}[\operatorname{Re} P(t(s))] \geq C > 0$ on the parametrized straight line $\{t(s) = -\nu + se^{-i\frac{\pi}{4}} : 0 \leq s \leq \sqrt{2}\nu\}$, where C is some constant independent of ϵ and ν.

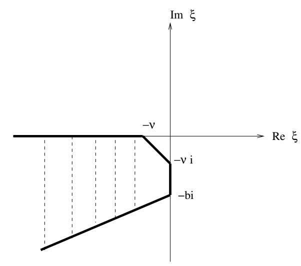

FIGURE 2.1. Region $\mathcal{D}$ in the complex ξ-plane.

PROPERTY 4: There exist b and φ_0, with $\nu < b < \min\{1, \gamma\}$, $0 < \varphi_0 < \frac{\pi}{2}$, each independent of ϵ, so that $\frac{d}{ds}\,\mathrm{Re}\,P(t(s)) \geq \frac{C}{|t(s)-2i|}$ on $t(s) = -bi + se^{i(\pi+\varphi_0)}$, where $C > 0$ is independent of ϵ.

$g_1(\xi)$ and $g_2(\xi)$ are the two WKB solutions of the homogeneous equation corresponding to (2.7). They satisfy the following equation exactly:

$$\epsilon^2 g''(\xi) + \big(L(\xi) + \epsilon^2 L_1(\xi)\big) g(\xi) = 0\,, \tag{2.15}$$

where

$$L_1(\xi) = \frac{L''(\xi)}{4L(\xi)} - \frac{5L'^2(\xi)}{16L^2(\xi)}\,. \tag{2.16}$$

Remark 2.7. By (2.8) and (2.16), $L_1(\xi) \sim O(\xi^{-2})$ as $|\xi| \to \infty$.

The Wronskian of g_1 and g_2 is

$$W(\xi) = g_1(\xi) g_2'(\xi) - g_2(\xi) g_1'(\xi) = \frac{2i}{\epsilon}\,. \tag{2.17}$$

DEFINITION 2.8 We define the operator $\mathcal{V}$ so that

$$\mathcal{V}F(\xi) \equiv \epsilon^2 F''(\xi) + \big(L(\xi) + \epsilon^2 L_1(\xi)\big) F(\xi)\,. \tag{2.18}$$

Remark 2.9. Equation (2.7) implies

$$\mathcal{V}F(\xi) = N_1(\xi) \equiv \mathcal{N}(F, I, \bar{F})[\xi] + \epsilon^2 L_1(\xi) F(\xi)\,. \tag{2.19}$$

DEFINITION 2.10 Let $\mathcal{D}$ be an open, connected domain (see Figure 2.1) in the lower left complex ξ-plane bounded by lines

$$R_1 = \big\{\xi : \mathrm{Im}\,\xi = 0,\ -\infty < \mathrm{Re}\,\xi < -\nu\big\}\,,$$
$$R_2 = \big\{\xi : \xi = -\nu + se^{-\pi i/4},\ 0 \leq s \leq \sqrt{2}\nu\big\}\,,$$
$$R_3 = \big\{\xi : \mathrm{Re}\,\xi = 0,\ -b < \mathrm{Im}\,\xi < -\sqrt{2}\nu\big\}\,,$$
$$R_4 = \big\{\xi : \xi = -bi + se^{i(\pi+\varphi_0)},\ 0 \leq s < \infty\big\}\,.$$

where ν, ϕ_0, and b are chosen so that Properties 3 and 4 are satisfied.

In addition to Properties 1 through 4 above, we show in Appendix B two other properties:

PROPERTY 5: For any $\xi \in \mathcal{D}$, there is a path $\mathcal{P}(-\nu, \xi) = \{t : t = t(s)\}$, parametrized by arc length s, going from $-\nu$ to ξ, entirely contained in $\overline{\mathcal{D}}$, so that $\frac{d}{ds} \operatorname{Re} P(t(s)) \geq C > 0$ for a constant C independent of ϵ.

PROPERTY 6: For any $\xi \in \mathcal{D}$, there is a path $\mathcal{P}(\xi, -\infty) = \{t : t = t(s)\}$ parametrized by arc length s, going from ξ to $-\infty$, entirely contained in $\overline{\mathcal{D}}$ so that $\frac{d}{ds}[\operatorname{Re} P(t(s))] \geq \frac{C}{|t-2i|} > 0$, where $C > 0$ is independent of ϵ.

We now introduce spaces of functions.

DEFINITION 2.11

$$\mathbf{B}_j = \Big\{ F(\xi) : F(\xi) \text{ is analytic in } \mathcal{D} \text{ and continuous in } \overline{\mathcal{D}}, \text{ with } \sup_{\xi \in \overline{\mathcal{D}}} |(\xi - 2i)^{j+\tau} F(\xi)| < \infty \Big\}, \quad j = 0, 1, 2,$$

$$\|F\|_j := \sup_{\xi \in \overline{\mathcal{D}}} |(\xi - 2i)^{j+\tau} F(\xi)| .$$

Remark 2.12. $\mathbf{B}_j$ are Banach spaces and $\mathbf{B}_0 \supset \mathbf{B}_1 \supset \mathbf{B}_2$.

DEFINITION 2.13 Let $\mathcal{Q}$ be any connected set in the complex ξ-plane. We introduce norms $\|F(\xi)\|_{j,\mathcal{Q}} := \sup_{\xi \in \mathcal{Q}} |(\xi - 2i)^{j+\tau} F(\xi)|$, $j = 0, 1, 2$.

DEFINITION 2.14 Let $\tilde{\delta} > 0$ and $\tilde{\delta}_1 > 0$ be two constants; define balls

$$\mathbf{B}_{0,\tilde{\delta}} = \left\{ f : f \in \mathbf{B}_0, \|f\|_0 \leq \tilde{\delta} \right\} \quad \text{and} \quad \mathbf{B}_{1,\tilde{\delta}_1} = \left\{ g : g \in \mathbf{B}_1, \|g\|_1 \leq \tilde{\delta}_1 \right\} .$$

Remark 2.15. In order to avoid a proliferation of constants, we have used C (and sometimes C_1 and C_2) as a generic constant whose value differs from lemma to lemma, and sometimes even from step to step within a lemma. However, C does not depend on ϵ. For more specific constants, we have reserved the symbol K, K_1, K_2, etc.

LEMMA 2.16 *Let $N \in \mathbf{B}_2$; then*

$$f_1(\xi) := \frac{1}{\epsilon^2} g_2(\xi) \int_\xi^{-\infty} \frac{N(t)}{W(t)} g_1(t) dt \in \mathbf{B}_0 \quad \textit{and} \quad \|f_1\|_0 \leq K_1 \|N\|_2 ,$$

where K_1 is a constant independent of ϵ.

PROOF: *Case* 1. $|\xi| \geq R$, where R is large enough for Property 2 to hold but independent of ϵ. On path $\mathcal{P}(\xi, -\infty) = \{t : t = \xi - s, 0 < s < \infty\}$,

$\mathrm{Re}(P(t) - P(\xi))$ increases monotonically from 0 to ∞ as s increases.

$$|f_1(\xi)| = \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int_{\mathcal{P}(\xi,-\infty)} N(t) L^{-1/4}(t) \exp\left\{ \frac{1}{\epsilon}(P(\xi) - P(t)) \right\} dt \right| \tag{2.20}$$

$$\leq \|N\|_2 |L^{-1/4}(\xi)|$$

$$\times \int_0^1 \frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\frac{d}{ds}\,\mathrm{Re}\, P(t(s))} d\left[\exp\left\{ \frac{1}{\epsilon}(\mathrm{Re}\, P(\xi) - \mathrm{Re}\, P(t(s))) \right\} \right].$$

Since $|\xi - 2i| \leq |t(s) - 2i|$ for any s, we have $|L^{-1/4}(\xi)| \leq C|\xi - 2i|^{1/2}$ and

$$\frac{d}{ds}\,\mathrm{Re}\, P(t(s)) = \mathrm{Re}(P'(t) t'(s)) \geq C|L^{1/2}(t)| \geq C|t(s) - 2i|^{-1},$$

$$|L^{-1/4}(\xi)| \frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{\frac{d}{ds}\,\mathrm{Re}\, P(t(s))} \leq C|\xi - 2i|^{-\tau},$$

so $\|f_1\|_0 \leq K_1 \|N\|_2$ and the lemma follows.

Case 2. For $\xi \in \mathcal{D} \cap \{|\xi| \leq R\}$, by Property 6, there exists a path $\mathcal{P}(\xi, -\infty)$ so that $\mathrm{Re}(P'(t(s))t'(s)) \geq \frac{C}{|t(s)-2i|}$. Then in (2.20)

$$|L^{-1/4}(\xi)| \frac{|(t(s) - 2i)^{-2-\tau}||L^{-1/4}(t(s))|}{|\frac{d}{ds}\,\mathrm{Re}\, P(t(s))|} \leq C, \tag{2.21}$$

and therefore the lemma follows since ξ is bounded in this region. □

LEMMA 2.17 *Let* $N \in \mathbf{B}_2$; *then for sufficiently small* $\epsilon_0 > 0$, *we have for all* $\epsilon \in (0, \epsilon_0]$,

$$f_2(\xi) := \frac{1}{\epsilon^2} g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t) dt \in \mathbf{A_0} \quad \textit{and} \quad \|f_2\|_0 \leq K_2 \|N\|_2,$$

where K_2 *is independent of* ϵ.

PROOF: *Case* 1. For $|\xi| \leq 4R^2$, by Property 5, there is a path $\mathcal{P}(-\nu, \xi)$ entirely in $\mathcal{D}$ so that $\frac{d}{ds}\,\mathrm{Re}\, P(t(s)) \geq C > 0$ for $t(s)$ going from $-\nu$ to ξ. So

$$|f_2(\xi)| \leq C\|N\|_2 |L^{-1/4}(\xi)|$$
$$\times \int_0^1 \left|(t - 2i)^{-2-\tau}\right| \left|L^{-1/4}(t)\right| \frac{\left|d\left[\exp\left\{ -\frac{1}{\epsilon}(\mathrm{Re}\, P(\xi) - \mathrm{Re}\, P(t))\right\}\right]\right|}{\frac{d}{ds}\,\mathrm{Re}\, P(t(s))}.$$

Since (2.21) holds here, too, the result follows since $|\xi - 2i|^\tau$ is bounded in this case as well.

Case 2. For the case where $\xi \in \mathcal{D}, |\xi| \geq 4R^2$, we choose path $\mathcal{P}(-\nu,\xi) = \mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3$, where

$$\mathcal{P}_1 = \left\{t : t = \rho e^{i \arg \xi},\ |\xi| \geq \rho \geq \sqrt{|\xi|}\right\},$$
$$\mathcal{P}_2 = \left\{t : t = \rho e^{i \arg \xi},\ \sqrt{|\xi|} \geq \rho \geq 2R\right\},$$
$$\mathcal{P}_3 = \mathcal{P}(-\nu, \xi^0) \quad \text{where } \xi_0 = 2\,\mathrm{R}\mathrm{e}^{i \arg \xi}\ .$$

We break up integral $\int_{\mathcal{P}} = \int_{\mathcal{P}_3} + \int_{\mathcal{P}_2} + \int_{\mathcal{P}_1}$ and accordingly write $f_2 = f_{2,1} + f_{2,2} + f_{2,3}$.

Now from (2.14), from the asymptotics for large $|\xi|$ and $|t|$, it follows that

$$\mathrm{Re}(P(t) - P(\xi)) \leq C_1 \int_{|\xi|}^{|t|} \frac{1}{r}\, dr \leq C_1 \ln\left(\frac{|t|}{|\xi|}\right) \tag{2.22}$$
$$\text{where } C_1 \text{ is independent of } \epsilon,$$

$$\begin{aligned} |f_{2,1}(\xi)| &= \left| \frac{2}{\epsilon} L^{-1/4}(\xi) \int\limits_{\mathcal{P}_1} N(t) L^{-1/4}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\ &\leq C \frac{2}{\epsilon} \|N\|_2 \left|L^{-1/4}(\xi)\right| \int_{\sqrt{|\xi|}}^{|\xi|} \rho^{-\frac{3}{2}-\tau} \exp\left\{\frac{C_1}{\epsilon} \ln(\frac{\rho}{|\xi|})\right\} d\rho \\ &\leq C \|N\|_2 |\xi|^{-\tau}\ . \end{aligned} \tag{2.23}$$

Also, (2.22) and (2.23) are still valid on $\mathcal{P}_2$; hence

$$\begin{aligned} |f_{2,2}(\xi) &= \left| \frac{2}{\epsilon} L^{-\frac{1}{4}}(\xi) \int\limits_{\mathcal{P}_2} N(t) L^{-\frac{1}{4}}(t) \exp\left\{-\frac{1}{\epsilon}(P(\xi) - P(t))\right\} dt \right| \\ &\leq C \frac{2}{\epsilon} \|N\|_2 \left|L^{-\frac{1}{4}}(\xi)\right| \int_{|\xi^0|}^{\sqrt{|\xi|}} \rho^{-\frac{3}{2}-\tau} \exp\left\{\frac{C_1}{\epsilon} \ln\left(\frac{\rho}{|\xi|}\right)\right\} d\rho \\ &\leq C \|N\|_2 \left|L^{-\frac{1}{4}}(\xi)\right| |\xi|^{-\frac{C_1}{2\epsilon} - \frac{1}{4} - \frac{\tau}{2}} \\ &\leq C \|N\|_2\, |\xi|^{-\tau} \quad \text{for } \epsilon < C_1\ . \end{aligned}$$

On $\mathcal{P}_3$,

$$
\begin{aligned}
|f_{2,3}(\xi)| &\le \|N\|_2 \left|L^{-\frac{1}{4}}(\xi)\right| \\
&\quad\times \int_0^{\exp[-\frac{1}{\epsilon}(\operatorname{Re} P(\xi)-\operatorname{Re} P(\xi_0))]} \frac{|(t(s)-2i)^{-2-\tau}||L^{-\frac{1}{4}}(t(s))|}{\frac{d}{ds}\operatorname{Re} P(t(s))} \\
&\quad\times d\left[\exp\left\{\frac{1}{\epsilon}\Big(\operatorname{Re} P(\xi)-\operatorname{Re} P(t(s))\Big)\right\}\right] \\
&\le C\|N\|_2 \left|L^{-\frac{1}{4}}(\xi)\right| \exp\left[-\frac{1}{\epsilon}\Big(\operatorname{Re} P(\xi)-\operatorname{Re} P(\xi_0)\Big)\right] \\
&\le C\|N\|_2 |\xi|^{-\frac{C_1}{\epsilon}+\frac{1}{2}} \le C\|N\|_2 |\xi|^{-\tau} \quad \text{when } \epsilon \le \frac{C_1}{2}\,.
\end{aligned} \tag{2.24}
$$

Combining bounds for $f_{2,1}$, $f_{2,2}$, and $f_{2,3}$, the proof of the lemma follows. □

DEFINITION 2.18 Define operator $\mathcal{U} : \mathbf{B}_2 \to \mathbf{B}_0$ and $\mathcal{U}_1 : \mathbf{B}_2 \to \mathbf{B}_1$ so that

$$
\mathcal{U}N(\xi) := -\frac{1}{\epsilon^2} g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t)dt + \frac{1}{\epsilon^2} g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t)dt\,, \tag{2.25}
$$

$$
\begin{aligned}
\mathcal{U}_1 N(\xi) := &-\frac{1}{\epsilon^2} h_1(\xi) g_1(\xi) \int_{-\nu}^{\xi} \frac{N(t)}{W(t)} g_2(t)dt \\
&+ \frac{1}{\epsilon^2} h_2(\xi) g_2(\xi) \int_{-\infty}^{\xi} \frac{N(t)}{W(t)} g_1(t)dt,\,,
\end{aligned} \tag{2.26}
$$

where

$$
h_1(\xi) = -\frac{L'(\xi)}{4L(\xi)} - \frac{1}{\epsilon} P'(\xi)\,, \quad h_2(\xi) = -\frac{L'(\xi)}{4L(\xi)} + \frac{1}{\epsilon} P'(\xi)\,. \tag{2.27}
$$

LEMMA 2.19

$$
\sup_{\mathcal{D}} |(\xi - 2i) h_j(\xi)| \le \frac{K_3}{\epsilon}\,, \quad j = 1, 2, \tag{2.28}
$$

where K_3 is a constant independent of ϵ.

PROOF: The lemma follows from the fact that $P'(\xi) = iL^{1/2}(\xi)$ and equations (2.8), (2.14), and (2.27). □

DEFINITION 2.20 Let $\mathbb{R}^- = \{\xi : \operatorname{Im}\xi = 0,\ \operatorname{Re}\xi < -\nu\}$.

LEMMA 2.21 $\|N_1\|_{2,\mathbb{R}^-} < \infty$.

PROOF: From (1.2), (2.2), (2.16), and Lemmas 1.5 and 2.4, it follows that as $\xi \to -\infty$,

$$
\begin{gathered}
\frac{\bar{H}'H}{\bar{H}} - H' = O(\xi^{-3})\,, \quad (F' + H)^{3/2} (\bar{F}' + \bar{H}')^{1/2} I = O(\xi^{-2-\tau})\,, \\
F\Big[(F' + H)^{3/2} (\bar{F}' + \bar{H}')^{1/2} - H^{3/2} \bar{H}^{1/2}\Big] = O(\xi^{-2-2\tau})\,,
\end{gathered}
$$

and

$$\frac{\bar{F}'' + \bar{H}'}{\bar{F} + \bar{H}}(F' + H) - \frac{\bar{H}' H}{\bar{H}} = O(\xi^{-2-\tau}), \quad L_1 F = O(\xi^{-2-\tau}).$$

Using these relations in (2.9) and (2.19) in the expression for N_1, the lemma follows. □

LEMMA 2.22 *Let $F(\xi)$ be a classical solution as in Definition* 1.7. *If F can be analytically extended to $\mathcal{D}$, then F satisfies the following equation for $\xi \in \mathcal{D}$*:

$$F(\xi) = \beta g_1(\xi) + \mathcal{U} N_1(\xi), \tag{2.29}$$

where β is given by

$$\beta = g_1^{-1}(-\nu)\left(F(-\nu) - \frac{1}{\epsilon^2} g_2(-\nu) \int_{-\infty}^{-\nu} \frac{N_1(t)}{W(t)} g_1(t) dt\right). \tag{2.30}$$

PROOF: First, we consider $\xi \in \mathbb{R}^-$ on the boundary of $\mathcal{D}$. From continuity, (2.29) holds, where $I(\xi)$ occurring in $N_1(\xi)$ is understood as $\lim_{\operatorname{Im}\xi \to 0^-} I(\xi)$. Using the method of variation of parameters for $\xi \in \mathbb{R}^-$, we have

$$F(\xi) = C_1 g_1 + C_2 g_2 + \mathcal{U} N_1(\xi). \tag{2.31}$$

Since $\|N_1\|_{2,\mathbb{R}^-} < \infty$, it follows on using Lemmas 2.16 and 2.17, restricted to $\mathbb{R}^-$ instead of $\mathcal{D}$, that $\|\mathcal{U} N_1\|_{0,\mathbb{R}^-} < \infty$. Since $g_1(-\infty) = 0$ and $g_2(-\infty) = +\infty$, it follows from $\sup_{\xi \in \mathbb{R}^-} |\xi - 2i|^\tau |F| < \infty$ that $C_2 = 0$. Using $C_2 = 0$ in (2.31) and evaluating it at $\xi = -\nu$, we obtain $F(-\nu) = C_1 g_1(-\nu) + \mathcal{U} N_1(-\nu)$. Hence $C_1 = \beta$ as given by (2.30). So (2.29) holds for $\xi \in \mathbb{R}^-$. By analytic continuation of each side of the equation, it follows that it must be valid in $\mathcal{D}$ as well. □

DEFINITION 2.23 $n_1(\xi) = \mathcal{N}(f, I, \bar{F})[\xi] + \epsilon^2 L_1(\xi) f(\xi)$.

We consider the integral equation

$$f(\xi) = \beta g_1(\xi) + \mathcal{U} n_1(\xi), \tag{2.32}$$

where β is still given as before by (2.30).

LEMMA 2.24 $\bar{F}' \in \mathbf{B}_1$, $\bar{F}'' \in \mathbf{B}_2$, *with* $\|\bar{F}'\|_1 \le \delta_1$ *and* $\|\bar{F}''\|_2 \le \delta_2$.

PROOF: The lemma follows from Definition 2.1 and Lemma 1.5. □

DEFINITION 2.25 $H_m \equiv \inf_{\xi \in \mathcal{D}} \left\{ |\xi - 2i||H(\xi)|,\ |\xi - 2i||\bar{H}(\xi)| \right\}$.

LEMMA 2.26 *Define operator $\mathcal{G}_1$ so that*

$$\mathcal{G}_1(f')[t] = (f'(t) + H(t))^{3/2} (\bar{F}'(t) + \bar{H}(t))^{1/2}.$$

Let $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$ and $\tilde{\delta}_1, \delta_1 < H_m/2$, where δ_1 is as defined in (1.8). *Then, for $\xi \in \mathcal{D}$,*

$$|\mathcal{G}_1(f')[\xi]| \le C|\xi - 2i|^{-2}, \tag{2.33}$$

where C is independent of ϵ.

PROOF: From (1.2) and (2.2),

$$H_m|\xi - 2i|^{-1} \le |H| \le C_2|\xi - 2i|^{-1}, \tag{2.34}$$

$$H_m|\xi - 2i|^{-1} \le |\bar{H}| \le C_2|\xi - 2i|^{-1}, \tag{2.35}$$

where C_1 and H_m are independent of ϵ.

$$|\mathcal{G}_1(f')| = \left|H^{3/2}\bar{H}^{1/2}\right|\left|\frac{f'}{H} + 1\right|^{3/2}\left|\frac{\bar{F}'}{\bar{H}} + 1\right|^{1/2} \le C|\xi - 2i|^{-2}.$$

□

LEMMA 2.27 *Let $\mathcal{G}_2$ be an operator so that*

$$\mathcal{G}_2(f')[\xi] = \left[(\bar{F}'' + \bar{H}')\frac{f' + H}{\bar{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}}\right](\xi). \tag{2.36}$$

If $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$ and $\delta_1, \tilde{\delta}_1 < H_m/2$, then for $\xi \in \mathcal{D}$,

$$|\mathcal{G}_2(f')[\xi]| \le C|\xi - 2i|^{-2-\tau}\left[\delta_1 + \tilde{\delta}_1 + \delta_2\right], \tag{2.37}$$

where C is independent of ϵ, and δ_1 and δ_2 are as defined in (1.6).

PROOF: Note from (2.2), for large $|\xi|$ we have

$$\bar{H}' = -\frac{(\xi - i\gamma)^2 + (\gamma^2 - 1)}{(\xi^2 + 1)^2} = O(\xi - 2i)^{-2}, \tag{2.38}$$

$$\frac{\bar{H}'H}{\bar{H}} = -\frac{[(\xi - i\gamma)^2 + (\gamma^2 - 1)](\xi + i\gamma)}{(\xi^2 + 1)^2(\xi - i\gamma)} = O((\xi - 2i)^{-2}), \tag{2.39}$$

and

$$\begin{aligned}|\mathcal{G}_2(f')| &= \left|f'\frac{\bar{H}'}{\bar{F}' + \bar{H}} - \frac{\bar{H}'H}{\bar{H}}\frac{\bar{F}'}{\bar{F}' + \bar{H}} + \bar{F}''\frac{f' + H}{\bar{F}' + \bar{H}}\right| \\ &\le C|\xi - 2i|^{-2-\tau}[\delta_1 + \tilde{\delta}_1 + \delta_2].\end{aligned}$$

□

LEMMA 2.28 *We define operator $\mathcal{G}_3$ so that*

$$\mathcal{G}_3(f') = (f' + H)^{3/2}(\bar{F}' + \bar{H})^{1/2} - H^{3/2}\bar{H}^{1/2}. \tag{2.40}$$

Assume that $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$ with $\delta_1, \tilde{\delta}_1 < H_m/2$; then for $\xi \in \mathcal{D}$,

$$|\mathcal{G}_3(f')[\xi]| \le C|\xi - 2i|^{-2-\tau}(\delta_1 + \tilde{\delta}_1), \tag{2.41}$$

where C is independent of ϵ.

PROOF: Using (2.40) we obtain

$$
\begin{aligned}
|\mathcal{G}_3(f)| &\le |H^{3/2}\bar{H}^{1/2}| \left| \left(\frac{f'}{H}+1\right)^{3/2} \left(\frac{\bar{F}'}{\bar{H}}+1\right)^{1/2} - 1 \right| \\
&\le C|\xi - 2i|^{-2} \left\{ \left(\frac{|f'|}{|H|}+1\right)^{3/2} \left(\frac{|\bar{F}|}{|\bar{H}|}+1\right)^{1/2} - 1 \right\} \\
&\le C|\xi - 2i|^{-2-\tau}(\delta_1 + \tilde{\delta}_1)\,.
\end{aligned}
$$

□

LEMMA 2.29 *Let* $f \in \mathbf{B}_{0,\tilde{\delta}}$ *and* $f' \in \mathbf{B}_{1,\tilde{\delta}_1}$; *then* $n_1 \in \mathbf{B}_2$ *for* $\tilde{\delta}_1, \delta_1 < H_m/2$, *and*

$$
\|n_1\|_2 \le K_4\big(\epsilon^2(1+\delta_2) + \delta + \tilde{\delta}(\epsilon^2 + \tilde{\delta}_1 + \delta_1)\big)\,,
$$

where K_4 *is independent of* ϵ.

PROOF: Note that

$$
\begin{aligned}
n_1 &= \mathcal{N}(f, \mathcal{I}(F), \bar{F}) \\
&= \epsilon^2\left(\frac{\bar{H}'H}{\bar{H}} - H'\right) \\
&\quad - i\epsilon^2\mathcal{G}_1(f')I(F) + if\mathcal{G}_3(f') + \epsilon^2\mathcal{G}_2(f') + \epsilon^2 L_1 f\,,
\end{aligned}
\tag{2.42}
$$

$$
\left|\epsilon^2\left(\frac{\bar{H}'H}{\bar{H}} - H'\right)\right| = \epsilon^2 \frac{2\gamma}{|(\xi^2+1)(\xi - i\gamma)|} \le C\epsilon^2|\xi - 2i|^{-3}\,;
\tag{2.43}
$$

from Lemmas 2.4 and 2.26,

$$
|\epsilon^2\mathcal{G}_1(f')(\xi)\mathcal{I}(F)[\xi]| \le C\delta|\xi - 2i|^{-2-\tau}\,.
\tag{2.44}
$$

Applying Lemmas 2.26, 2.27, and 2.28, we obtain

$$
\begin{aligned}
|f||\mathcal{G}_3(f')| &\le C|\xi - 2i|^{-2-\tau}\tilde{\delta}\big(\delta_1 + \tilde{\delta}_1\big)\,, \\
\epsilon^2|\mathcal{G}_2(f')| &\le C\epsilon^2|\xi - 2i|^{-2-\tau}\big(\tilde{\delta}_1 + \delta_1 + \delta_2\big)\,.
\end{aligned}
$$

From the expression of $L_1(\xi)$, we have

$$
|\epsilon^2 f||L_1(\xi)| \le C\epsilon^2|\xi - 2i|^{-2-\tau}\tilde{\delta}\,.
\tag{2.45}
$$

On using the expression for n_1 in (2.42), we have the proof by combining the above inequalities. It is to be noted that terms like $\epsilon^2\tilde{\delta}$, $\epsilon^2\delta_1$, etc., do not appear because they are smaller than terms explicitly appearing on the right-hand side of the lemma statement. Clearly, for suitable choice of K_4, such terms can be estimated away. □

LEMMA 2.30 *Let* $\mathcal{G}_1$ *be as defined in Lemma* 2.26. *Let* $f_j' \in \mathbf{B}_{1,\tilde{\delta}_1}$, $j = 1, 2$; *then for* $\delta_1, \tilde{\delta}_1 < H_m/2$,

$$
\big|\mathcal{G}_1(f_1')(\xi) - \mathcal{G}_1(f_2')(\xi)\big| \le C|\xi - 2i|^{-2-\tau}\big\|f_1' - f_2'\big\|_1\,,
\tag{2.46}
$$

where C is independent of ϵ.

PROOF: By straightforward algebra,

$$\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') = \\ \frac{(f_1' - f_2')(\bar{F}' + \bar{H})^{1/2}[(f_1' + H)^2 + (f_1' + H)(f_2' + H) + (f_2' + H)^2]}{(f_1' + H)^{3/2} + (f_2' + H)^{3/2}} .$$

The lemma follows from the equation above, on using upper and lower estimates for $|f_i' + H|$ and $|\bar{F}' + \bar{H}|$ as in preceding lemmas. □

LEMMA 2.31 *Let $f_j' \in \mathbf{B}_{1,\tilde{\delta}_1}$, $j = 1, 2$. Let $\mathcal{G}_2(f')$ be defined as in Lemma 2.27; then for $\delta_1 < H_m/2$,*

$$\left\| \mathcal{G}_2(f_1') - \mathcal{G}_2(f_2') \right\|_2 \le C(\delta_2 + 1) \left\| f_1' - f_2' \right\|_1 . \tag{2.47}$$

PROOF: We note

$$\left| (\bar{F}'' + \bar{H}') \right| \le \frac{C}{|\xi - 2i|^2} + \frac{\delta_2}{|\xi - 2i|^{2+\tau}} \le C|\xi - 2i|^{-2}(1 + \delta_2)$$

and

$$\left| (\bar{H} + \bar{F}')^{-1} \right| \le \frac{4}{H_m} |\xi - 2i| .$$

By straightforward algebra,

$$\mathcal{G}_2(f_1') - \mathcal{G}_2(f_2') = \frac{(\bar{F}_2'' + \bar{H}')}{(\bar{F}_1' + \bar{H})} (f_1' - f_2') .$$

Using inequalities as above, we obtain the proof of the lemma. □

LEMMA 2.32 *Let $f_j \in \mathbf{B}_{0,\tilde{\delta}}$ and $f_j' \in \mathbf{B}_{1,\tilde{\delta}_1}$, $j = 1, 2$; then for $\tilde{\delta}_1, \delta_1 < H_m/2$,*

$$\begin{aligned} &\left\| \mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_2 - f_1) \right\|_2 \le \\ &\quad K_5\big((\epsilon^2 + \delta_1 + \tilde{\delta}_1) \| f_1 - f_2 \|_0 + (\epsilon^2 + \delta + \tilde{\delta} + \epsilon^2 \delta_2) \| f_1' - f_2' \|_1 \big) , \end{aligned} \tag{2.48}$$

where K_5 is independent of ϵ.

PROOF: From (2.42),

$$\begin{aligned} &\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) = \\ &\quad - i\epsilon^2 \mathcal{I}(F) \big(\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') \big) + i(f_1 - f_2) \mathcal{G}_3(f_1') \\ &\quad + i f_2 \big(\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') \big) + \epsilon^2 \big(\mathcal{G}_2(f_1') - \mathcal{G}_2(f_2') \big) . \end{aligned} \tag{2.49}$$

On using Lemmas 2.4, 2.28, 2.30, and 2.31 and the expression for $L_1(\xi)$, we obtain

$$\begin{aligned} \left\| \epsilon^2 I \big(\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') \big) \right\|_2 &\le C\delta \| f_1' - f_2' \|_1 , \\ \left\| (f_1 - f_2) \mathcal{G}_3(f_1') \right\|_2 &\le C \big(\delta_1 + \tilde{\delta}_1 \big) \left\| f_1 - f_2 \right\|_0 , \\ \left\| f_2 \big(\mathcal{G}_1(f_1') - \mathcal{G}_1(f_2') \big) \right\|_2 &\le C\tilde{\delta} \left\| f_1' - f_2' \right\|_1 , \\ \left\| \epsilon^2 \big(\mathcal{G}_2(f_1') - \mathcal{G}_2(f_2') \big) \right\|_2 &\le C\epsilon^2 (1 + \delta_2) \left\| f_1' - f_2' \right\|_1 , \\ \left\| \epsilon^2 L_1(f_1 - f_2) \right\|_2 &\le C\epsilon^2 \| f_1 - f_2 \|_0 . \end{aligned}$$

Combining all these inequalities, we get the proof. $\square$

LEMMA 2.33 *For sufficiently small ϵ, we have*

$$\|\beta g_1\|_0 \leq K_6\big(\epsilon^2 + \delta + \epsilon^2\delta_2\big)\,, \tag{2.50}$$

where K_6 is independent of ϵ.

PROOF: For $|\xi| \leq R$, from (2.30),

$$|\beta g_1(\xi)| \leq |\beta g_1(-\nu)| \leq |F(-\nu)| + |\mathcal{U}N_1(-\nu)| \leq \delta + \|\mathcal{U}N_1\|_{2,\mathbb{R}^-}\,,$$

but from (2.9) and (2.19) and by using Lemma 2.29 with domain $\mathcal{D}$ replaced by $\mathcal{R}^-$, and f replaced by F (and hence $\tilde{\delta}$ by δ and $\tilde{\delta}_1$ by δ_1), we get

$$\|N_1\|_{2,\mathbb{R}^-} \leq C\big(\epsilon^2 + \delta + \epsilon^2\delta_2\big)\,.$$

So $\|\mathcal{U}N_1\|_0 \leq C(K_1 + K_2)(\epsilon^2 + \delta + \epsilon^2\delta_2)$ from Lemmas 2.16 and 2.17. Therefore $|\beta g_1(-\nu)| < \tilde{K}_6(\epsilon^2 + \delta + \epsilon^2\delta_2)$ for some $\tilde{K}_6$ independent of ϵ. For $|\xi| \geq R$, on using equations (2.12) and (2.22),

$$\left|g_1(\xi) g_1^{-1}(-\nu)\right| < C\Big(|\xi|^{\frac{1}{2} - \frac{C_1}{\epsilon}}\Big)\,,$$

where C and C_1 are independent of ϵ. For $\epsilon \leq C_1/2$, the above expression is $< C|\xi - 2i|^{-\tau}$, and the lemma follows. $\square$

Remark 2.34. The estimates in each of the Lemmas 2.16 through 2.33 generally depend on γ and therefore λ, as quantities such as H_m and upper bounds for $(\xi - 2i)H$ or $(\xi - 2i)\bar{H}$ are dependent on γ. If we consider λ in any fixed compact subset of the interval (0, 1), i.e., for $\gamma = \lambda/(1-\lambda)$ in a compact subset of $(0, \infty)$, such dependence can be removed since H and $\bar{H}$ are continuous functions of γ in this interval.

DEFINITION 2.35 We define the space $\mathbf{E} := \mathbf{B} \oplus \mathbf{B}_1$. For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}$,

$$\|\mathbf{e}\|_{\mathbf{E}} := \|u(\xi)\|_0 + \epsilon\|v(\xi)\|_1\,.$$

It is easy to see that $\mathbf{E}$ is Banach space. We replace (f, f') by (u, v). Also, we denote operator $\mathbf{n}$ so that $\mathbf{n}(u, v)(\xi) = n_1(\xi)$.

DEFINITION 2.36 Let $\mathbf{O} : \mathbf{E} \longmapsto \mathbf{E}$,

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathbf{e}) = (\mathbf{O}_1(\mathbf{e}), \mathbf{O}_2(\mathbf{e}))\,,$$

where

$$\mathbf{O}_1(\mathbf{e}) = \beta g_1 + \mathcal{U}\mathbf{n}(u, v)\,, \tag{2.51}$$

$$\mathbf{O}_2(\mathbf{e}) = \beta h_1 g_1 + \mathcal{U}_1\mathbf{n}(u, v)\,. \tag{2.52}$$

DEFINITION 2.37 Let

$$\Delta = 8K\big(\delta + \epsilon^2(1 + \delta_2)\big) \tag{2.53}$$

where

$$K = \max\big\{K_6, (K_1 + K_2)K_4, K_3K_6, K_3K_4(K_1 + K_2)\big\}\,. \tag{2.54}$$

We define space $\mathbf{E}_\Delta = \{\mathbf{e} \in \mathbf{E} : \|\mathbf{e}\|_{\mathbf{E}} \leq \Delta\}$.

LEMMA 2.38 *If* $\mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E}_\Delta$, *then for* ϵ, δ_1, *and* $\epsilon \ln \frac{1}{\epsilon}\delta_2$, *each sufficiently small* (*the latter two are part of Assumption* 2, $\mathbf{O}(\mathbf{e}) \in \mathbf{E}_\Delta$.

PROOF: If $\mathbf{e} \in \mathbf{E}_\Delta$, it follows from the expression for Δ that

$$\frac{\Delta}{\epsilon} \leq 8K\left[\epsilon + \frac{\delta}{\epsilon} + \epsilon\delta_2\right], \tag{2.55}$$

and this is small by assumption and Remark 1.6. Thus, both $\|v\|_1$ (and therefore $\tilde{\delta}_1$) and δ_1 can be taken smaller than $H_m/2$ so as to apply Lemmas 2.29 and 2.33, which together with Lemmas 2.16 and 2.17, give the following:

$$\begin{aligned}\|\mathbf{O}_1(\mathbf{e})\|_0 &\leq \|\beta g_1\|_0 + \|\mathcal{U}\mathbf{n}(u, v)\|_0 \\ &\leq K_6\big(\epsilon^2 + \delta + \epsilon^2\delta_2\big) \\ &\quad + (K_1 + K_2)K_4\big[\epsilon^2(1 + \delta_2) + \delta + \|u\|_0(\epsilon^2 + \delta_1 + \|v\|_1)\big] \\ &\leq 2K\big[\epsilon^2(1 + \delta_2) + \delta\big] + K\|u\|_0\big(\epsilon^2 + \delta_1 + \|v\|_1\big)\,.\end{aligned}$$

Using $\|u\|_0 \leq \Delta, \epsilon\|v\|_1 \leq \Delta$, and (2.55), we get

$$K\|u\|_0\big(\epsilon^2 + \delta_1 + \|v\|_1\big) \leq \Delta\left[K(\epsilon^2 + \delta_1) + K\frac{\Delta}{\epsilon}\right],$$

so

$$\|\mathbf{O}_1(\mathbf{e})\|_0 \leq \Delta\left[\frac{1}{4} + K(\epsilon^2 + \delta_1) + K\frac{\Delta}{\epsilon}\right].$$

From Lemma 2.19

$$\epsilon\|\mathbf{O}_2(\mathbf{e})\|_1 \leq K_3\big[\|\beta g_1\|_0 + \|\mathcal{U}\mathbf{n}(u, v)\|_0\big] \leq \Delta\left[\frac{1}{4} + K(\epsilon^2 + \delta_1) + K\frac{\Delta}{\epsilon}\right];$$

then, for sufficiently small ϵ, δ_1, and $\epsilon \ln \frac{1}{\epsilon}\delta_2$,

$$\|\mathbf{O}(\mathbf{e})\| = \|\mathbf{O}_1(\mathbf{e})\|_0 + \epsilon\|\mathbf{O}_2(\mathbf{e})\|_1 \leq \Delta\left[\frac{1}{2} + 2K(\epsilon^2 + \delta_1) + 2K\frac{\Delta}{\epsilon}\right] \leq \Delta\,.$$

□

LEMMA 2.39 *If* $\mathbf{e}_j = (u(\xi), v(\xi)) \in \mathbf{E}_\Delta$, $j = 1, 2$, *then for* ϵ, δ_1, *and* $\epsilon \ln \frac{1}{\epsilon}\delta_2$ *small enough,*

$$\|\mathbf{O}(\mathbf{e}_1) - \mathbf{O}(\mathbf{e}_2)\| \leq \Delta_1\|\mathbf{e}_1 - \mathbf{e}_2\|\,,$$

where

$$\Delta_1 = \tilde{K}\left[2\epsilon + \delta_1 + \epsilon\delta_2 + \frac{\Delta}{\epsilon}\right], \tag{2.56}$$

where $\tilde{K} = 2\max\{K_5(K_1 + K_2), K_3K_5(K_1 + K_2)\}$.

PROOF: Since $(u_1, v_1), (u_2, v_2) \in \mathbf{E}_\Delta$, it follows that each of $\|u_1\|_0$, $\|u_2\|_0$, $\epsilon\|v_1\|_1$, and $\epsilon\|v_2\|_1$ are bounded by Δ and that we can assume each of $\|v_1\|_1$ and $\|v_2\|_1 < H_m/2$ so as to apply Lemmas 2.32, 2.16, and 2.17, which, on using $\tilde{\delta} \leq \Delta$ and $\tilde{\delta}_1 \leq \frac{\Delta}{\epsilon}$, gives the following:

$$\begin{aligned} \|\mathbf{O}_1(\mathbf{e}_1) - \mathbf{O}_1(\mathbf{e}_2)\|_0 &\leq (K_1 + K_2)K_5 \Bigg\{ \left(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon}\right)\|u_1 - u_2\|_0 \\ &\qquad + \left(\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon}\right)\epsilon\|v_1 - v_2\|_1 \Bigg\} \\ &\leq \frac{\Delta_1}{2}\|\mathbf{e}_1 - \mathbf{e}_2\| , \\ \epsilon\|\mathbf{O}_2(\mathbf{e}_1 - \mathbf{O}_1(\mathbf{e}_2)\|_1 &\leq K_3(K_1 + K_2)K_5 \Bigg\{ \left(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon}\right)\|u_1 - u_2\|_0 \\ &\qquad + \left(\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon}\right)\epsilon\|v_1 - v_2\|_1 \Bigg\} \\ &\leq \frac{\Delta_1}{2}\|\mathbf{e}_1 - \mathbf{e}_2\| . \end{aligned}$$

So, the proof of the lemma follows from combining the above. □

THEOREM 2.40 *For sufficiently small δ_1, $\epsilon \ln \frac{1}{\epsilon}\delta_2$, and ϵ, the operator $\mathbf{O}$ is a contraction mapping from $\mathbf{E}_\Delta$ to $\mathbf{E}_\Delta$. Therefore, a unique solution $(u(\xi), v(\xi)) \in \mathbf{E}_\Delta$ to $\mathbf{e} = \mathbf{O}(\mathbf{e})$ exists and hence a unique solution to the integral equation* (2.32) *exists, where $f = u$ and $f' = v$.*

PROOF: From assumptions and Remark 1.6, we know that $\Delta_1 < 1$. The theorem follows from Lemmas 2.38 and 2.39. □

LEMMA 2.41 *If f is the solution in Theorem* 2.40 *and F is a classical solution as defined earlier, then $f(\xi) \equiv F(\xi)$ for $\xi \in (-\infty, -\nu]$ for small enough ϵ, δ_1, and $\epsilon \ln \frac{1}{\epsilon}\delta_2$.*

PROOF: Let $u = f - F$ and $v = f' - F'$. From (2.29) and (2.32), u and v satisfy the following equations:

$$u = \mathcal{U}(n_1 - N_1) , \qquad v = \mathcal{U}_1(n_1 - N_1) .$$

By Lemma 2.32 restricted to domain $\mathbb{R}^-$, with $f_1 = f$ and $f_2 = F$ and using $\|\epsilon^2 L_1 u\|_{0,\mathbb{R}^-} \leq C\epsilon^2\|u\|_{0,\mathbb{R}^-}$,

$$\|n_1 - N_1\|_{2,\mathbb{R}^-} \leq C\left[\left(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon}\right)\|u\|_{0,\mathbb{R}^-} + \left(\epsilon^2 + \epsilon^2\delta_2 + \delta + \Delta\right)\|v\|_{1,\mathbb{R}^-}\right].$$

So, from using Lemmas 2.16 and 2.17, restricted to domain $\mathbb{R}^-$,

$$\|u\|_{0,\mathbb{R}^-} \leq C\left[\left(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon}\right)\|u\|_{0,\mathbb{R}^-} + \left(\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon}\right)\epsilon\|v\|_{1,\mathbb{R}^-}\right],$$

$$\epsilon\|v\|_{1,\mathbb{R}^-} \leq C\left[\left(\epsilon^2 + \delta_1 + \frac{\Delta}{\epsilon}\right)\|u\|_{0,\mathbb{R}^-} + \left(\epsilon + \epsilon\delta_2 + \frac{\delta}{\epsilon} + \frac{\Delta}{\epsilon}\right)\epsilon\|v\|_{1,\mathbb{R}^-}\right],$$

where C is a constant independent of ϵ. So, combining the above,

$$\|u\|_{0,\mathbb{R}^-} + \epsilon\|v\|_{1,\mathbb{R}^-} \leq C\left(\epsilon + \delta_1 + \frac{\Delta}{\epsilon} + \epsilon\delta_2 + \frac{\delta}{\epsilon}\right)\left(\|u\|_{0,\mathbb{R}^-} + \epsilon\|v\|_{1,\mathbb{R}^-}\right).$$

Since the constant C is independent of ϵ in the estimate on the right side of the above equation, it follows that for small ϵ, $\epsilon \ln \frac{1}{\epsilon}\delta_2$, and δ_1 (and hence small Δ/ϵ because of Remark 1.6), $(u, v) \equiv \mathbf{0}$. Hence, the lemma follows. □

THEOREM 2.42 *If F is a classical solution satisfying Assumptions* 1 *and* 2, *then for small enough ϵ, $F \in \mathbf{B}_{0,\Delta}$ and $F' \in \mathbf{B}_{1,\Delta/\epsilon}$.*

PROOF: The theorem follows from Theorem 2.40 and Lemma 2.41. □

3 Analyticity in the Triangular Region

Let $\mathcal{S} = \{\xi : \operatorname{Re}\xi = -a, -\nu + a \leq \operatorname{Im}\xi \leq 0\}$ where $0 \leq a < \nu$ be a vertical straight-line segment in the triangular region $\mathcal{T}$ bounded by negative real axis, negative imaginary axis, and line segment $\{\xi : \xi = -\nu + se^{-\pi i/4}, 0 \leq s \leq \sqrt{2}\nu\}$. This is the triangular region (see Figure 2.1) in the third quadrant in the complement of $\mathcal{D}$. It is to be noted that in the triangular region , $P(\xi) = P(0) + i\gamma\xi + O(\nu^2)$ and so on $\mathcal{S}$ when $\xi = -a - is$, Re P increases monotonically with s such that $\frac{d}{ds}\operatorname{Re} P(\xi(s)) > C > 0$, where C is independent of ϵ and ν for sufficiently small ν.

We consider the following boundary value problem on the line segment $\mathcal{S}$:

$$\begin{gathered}\epsilon^2 f'' + \big(L(\xi) + \epsilon^2 L_1(\xi)\big) f = \mathcal{N}(f, I(F), \bar{F})[\xi] + \epsilon^2 L_1 f(\xi) \equiv n_1(\xi)\,, \\ f(-a) = F(-a)\,, \quad f(-a_1) = F(-a_1)\,,\end{gathered} \tag{3.1}$$

where $a_1 = a + i(\nu - a)$.

LEMMA 3.1 *$f \in C^2(\mathcal{S})$ is a solution of boundary value problem* (3.1) *if and only if f is a solution of the following integral equation*:

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \mathcal{U}_3 n_1\,, \tag{3.2}$$

where

$$\mathcal{U}_3 n_1 = -\frac{1}{\epsilon^2} g_1 \int_{-a}^{\xi} \frac{n_1(t)}{W(t)} g_2(t)dt + \frac{1}{\epsilon^2} g_2 \int_{-a_1}^{\xi} \frac{n_1(t)}{W(t)} g_1(t)dt\,, \tag{3.3}$$

$$\alpha_1 = \frac{\gamma_1 g_2(-a_1) - \gamma_2 g_2(-a)}{g_1(-a)g_2(-a_1) - g_1(-a_1)g_2(-a)}\,, \tag{3.4}$$

$$\alpha_2 = \frac{\gamma_1 g_1(-a_1) - \gamma_2 g_1(-a)}{g_1(-a)g_2(-a_1) - g_1(-a_1)g_2(-a)}\,, \tag{3.5}$$

where

$$\gamma_1 = F(-a) - \frac{1}{\epsilon^2} g_2(-a) \int_{-a_1}^{-a} \frac{n_1(t)}{W(t)} g_1(t)dt\,, \tag{3.6}$$

$$\gamma_2 = F(-a_1) + \frac{1}{\epsilon^2} g_1(-a_1) \int_{-a}^{-a_1} \frac{n_1(t)}{W(t)} g_2(t)dt\,. \tag{3.7}$$

PROOF: If $f \in C^2(\mathcal{S})$ is a solution of boundary value problem (3.1), then by variation of parameters, we have

$$f = \alpha_1 g_1 + \alpha_2 g_2 + \mathcal{U}_3 n_1 \tag{3.8}$$

for some α_1 and α_2. Plugging the boundary conditions in (3.1) and solving for α_1 and α_2, we have (3.4) and (3.5). By straightforward computation, we get that a solution of (3.2) is a solution of the boundary problem (3.1). We note that the denominator D in the expressions for α_1 and α_2 is given by

$$D = g_1(-a)g_2(-a_1) - g_2(-a)g_1(-a_1)\,, \tag{3.9}$$

and using (2.12), we have

$$\begin{aligned} D = {} & L^{-\frac{1}{4}}(-a)L^{-\frac{1}{4}}(-a_1)\exp\left\{\frac{1}{\epsilon}(P(-a_1) - P(-a))\right\} \\ & \times \left[1 - \exp\left\{\frac{2}{\epsilon}(P(-a) - P(-a_1))\right\}\right], \end{aligned} \tag{3.10}$$

which is nonzero because $\operatorname{Re} P(-a_1) > \operatorname{Re} P(-a)$. □

Remark 3.2. γ_1 and γ_2 depend on f and f' through n_1, γ_1 and γ_2 are functionals of f and f', and so are α_1 and α_2. We use the notation $\alpha_j(f, f')$ to indicate the dependence on f and f'. The norm $\|\cdot\|$ means the maximum norm $\|\cdot\|_\infty$ in this section.

LEMMA 3.3 *If* $\tilde{n} \in C(\mathcal{S})$, *let*

$$\tilde{f}_1(\xi) = \frac{1}{\epsilon^2} g_2(\xi) \int_{-a_1}^{\xi} \frac{\tilde{n}(t)}{W(t)} g_1(t)dt\,;$$

then $\tilde{f}_1 \in C(\mathcal{S})$ *and* $\|\tilde{f}_1\| \le K_1\|\tilde{n}\|$ *for constant* K_1 *independent of* ϵ.

PROOF: Using monotonicity of Re P on $\mathcal{S}$ with s, as noted before,

$$
\begin{aligned}
|\tilde{f}_1(\xi)| &= \left| \frac{1}{2i\epsilon} \int_{-a_1}^{\xi} L^{-\frac{1}{4}}(\xi) L^{-\frac{1}{4}}(t) \tilde{n}(t) \exp\left\{ -\frac{1}{\epsilon}(P(t) - P(\xi)) \right\} dt \right| \\
&\leq C \int_{\exp\{-\frac{1}{\epsilon}(P(-a_1)-P(\xi))\}}^{1} \frac{|L^{-\frac{1}{4}}(\xi) L^{-\frac{1}{4}}(t) \tilde{n}(t)|}{\frac{d}{ds} \operatorname{Re} P(t(s))} \\
&\quad \times d\left[\exp\left\{ -\frac{1}{\epsilon}(P(t) - P(\xi)) \right\} \right] \\
&\leq K_1 \|\tilde{n}\| .
\end{aligned}
$$

□

LEMMA 3.4 *If* $\tilde{n} \in C(\mathcal{S})$, *let*

$$
\tilde{f}_2 = \frac{1}{\epsilon^2} g_1(\xi) \int_{-a}^{\xi} \frac{\tilde{n}(t)}{W(t)} g_2(t) dt \,,
$$

then $\tilde{f}_2 \in C(\mathcal{S})$ *and* $\|\tilde{f}_2\| \leq K_2 \|f\|$.

PROOF: The proof is very similar to Lemma 3.3. □

LEMMA 3.5 *Let* $f_j \in C(\mathcal{S})$ *and* $f_j' \in C(\mathcal{S})$, $j = 1, 2$; *then*

$$
\begin{aligned}
&\left\| \mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2) \right\| \leq \\
&\quad K_5\Big(\big(\epsilon^2 + \delta_1 + \|f_1'\| \big) \|f_1 - f_2\| + \big(\epsilon^2 + \delta + \|f_2\| + \epsilon^2 \delta_2 \big) \|f_1' - f_2'\| \Big) .
\end{aligned} \tag{3.11}
$$

PROOF: The proof parallels that of Lemma 2.32 except that the domain is $\mathcal{S}$ instead of $\mathcal{D}$ and the norm is the max norm. □

LEMMA 3.6 *If* $f' \in C(\mathcal{S})$, *then* $\alpha_j g_j \in C(\mathcal{S})$ *for* $j = 1, 2$, *and*

$$
\|\alpha_j g_j\| \leq k_1\big(|F(-a)| + |F(-a_1)| + \|n_1\| \big) \quad \text{where } k_1 \text{ is independent of } \epsilon. \tag{3.12}
$$

PROOF: If we define D as in (3.9), it follows from (3.10) that D^{-1} is exponentially small in ϵ, Re $P(\xi) \geq$ Re $P(-a)$ for $\xi \in \mathcal{S}$, since Re $P(-a_1) >$ Re $P(-a)$. We also have

$$
\begin{aligned}
\left| \frac{g_2(-a) g_1(\xi)}{D} \right| &= \left| \frac{L^{-\frac{1}{4}}(\xi)}{L^{-\frac{1}{4}}(-a_1)} \right| \left| \frac{\exp\{\frac{1}{\epsilon}(2P(-a) - P(\xi) - P(-a_1 i))\}}{1 - \exp\{\frac{2}{\epsilon}(P(-a) - P(-a_1))\}} \right| \\
&\leq C
\end{aligned} \tag{3.13}
$$

with C independent of ϵ, and

$$
\left| \frac{g_2(-a_1) g_1(\xi)}{D} \right| = \left| \frac{L^{-\frac{1}{4}}(\xi)}{L^{-\frac{1}{4}}(-a)} \right| \left| \frac{\exp\{-\frac{1}{\epsilon}(P(\xi) - P(-a))\}}{1 - \exp\{\frac{2}{\epsilon}(P(-a) - P(-a_1))\}} \right| \leq C \,. \tag{3.14}
$$

Similarly, we get constant upper bounds for $g_1(-a_1)g_2(\xi)/D$ and $g_1(-a)g_2(\xi)/D$. Using Lemmas 3.3 and 3.4 in (3.6) and (3.7), we have

$$|\gamma_1| \leq (|F(-a)| + K_1\|n_1\|)\,, \tag{3.15}$$

$$|\gamma_2| \leq (|F(-a_1)| + K_2\|n_1\|)\,. \tag{3.16}$$

Using (3.13), (3.14), and similar bounds in (3.4) and (3.5), we get the lemma. □

LEMMA 3.7 *If* $f_j' \in C(\mathcal{S})$, $j = 1, 2$, *then* $(\alpha_j(f_1, f_1') - \alpha_j(f_2, f_2'))g_j \in C(\mathcal{S})$ *and*

$$\begin{aligned} &\left\|\left(\alpha_j(f_1, f_1') - \alpha_j(f_2, f_2')\right)g_j\right\| \leq \\ &\qquad C\left(\epsilon^2 + \delta_1 + \|f_1'\|\right)\|f_1 - f_2\| + \left(\epsilon^2 + \|f_2\| + \delta + \epsilon^2\delta_2\right)\|f_1' - f_2'\|\,. \end{aligned} \tag{3.17}$$

PROOF:

$$\begin{aligned} &\left|\left(\alpha_1(f_1, f_1') - \alpha_1(f_2, f_2')\right)g_1\right| \\ &\quad \leq \left|\gamma_1(f_1, f_1') - \gamma_1(f_2, f_2')\right|\left|\frac{g_2(-a)g_1(\xi)}{D}\right| \\ &\qquad + \left|\gamma_2(f_1, f_1') - \gamma_2(f_2, f_2')\right|\left|\frac{g_2(-a_1)g_1(\xi)}{D}\right|. \end{aligned} \tag{3.18}$$

Using formulae (3.6) and (3.7) and Lemmas 3.3 and 3.4, we obtain

$$\begin{aligned} &\left|\gamma_1(f_1, f_1') - \gamma_1(f_2, f_2')\right| \leq \\ &\qquad C\left\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2)\right\|\,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} &\left|\gamma_2(f_1, f_1') - \gamma_2(f_2, f_2')\right| \leq \\ &\qquad C\left\|\mathcal{N}(f_1, I, \bar{F}) - \mathcal{N}(f_2, I, \bar{F}) + \epsilon^2 L_1(f_1 - f_2)\right\|\,. \end{aligned} \tag{3.20}$$

The lemma follows from (3.13), (3.14), (3.18), and Lemma 3.5. The proof is similar for $j = 2$. □

We consider the following integral equations:

$$f(\xi) = \mathbf{o}_3(f, f') := \alpha_1 g_1(\xi) + \alpha_2 g_2(\xi) + \mathcal{U}_3 n_1(\xi)\,, \tag{3.21}$$

$$f'(\xi) = \mathbf{o}_4(f, f') := \alpha_1 h_1(\xi)g_1(\xi) + \alpha_2 h_2(\xi)g_2(\xi) + \mathcal{U}_4 n_1(\xi)\,, \tag{3.22}$$

where

$$\mathcal{U}_4 n_1 = -\frac{1}{\epsilon^2}h_1(\xi)g_1\int_{-a}^{\xi}\frac{n_1(t)}{W(t)}g_2(t)dt + \frac{1}{\epsilon^2}h_2(\xi)g_2\int_{-a_1}^{\xi}\frac{n_1(t)}{W(t)}g_1(t)dt\,. \tag{3.23}$$

We define the following spaces:

DEFINITION 3.8 $\mathbf{E}(\mathcal{S}) := C(\mathcal{S}) \oplus C(\mathcal{S})$. For $\mathbf{e}(\xi) = (u(\xi), v(\xi)) \in \mathbf{E}(\mathcal{S})$,

$$\|\mathbf{e}\|_{\mathbf{E}(\mathcal{S})} := \|u(\xi)\|_\infty + \epsilon\|v(\xi)\|_\infty\,.$$

It is easy to see that $\mathbf{E}(\mathcal{S})$ is a Banach space.

DEFINITION 3.9 We define k_3 independent of ϵ so that

$$k_3 \geq \sup_{\xi \in \mathcal{T}} \left\{ \epsilon |h_1(\xi)|, \ \epsilon |h_2(\xi)| \right\},$$

where h_1 and h_2 are as defined by (2.27).

DEFINITION 3.10

$$\mathbf{E}_{\Delta,\mathcal{S}} := \left\{ \mathbf{e} = (u(\xi), v(\xi)) \in \mathbf{E}(\mathcal{S}) : \|u(\xi)\| \leq 8k_1 \Delta \,, \ \|v(\xi)\| \leq 8k_1 k_3 \frac{\Delta}{\epsilon} \right\}. \tag{3.24}$$

where k_1 and k_3 are $O(1)$ constants, as defined in Lemma 3.6 and Definition 3.9 and Δ is as defined in (2.53).

DEFINITION 3.11 Let $\mathbf{O}(\mathcal{S}) : \mathbf{E}(\mathcal{S}) \longmapsto \mathbf{E}(\mathcal{S})$,

$$\mathbf{e}(\xi) = (u(\xi), v(\xi)) \longmapsto \mathbf{O}(\mathcal{S})(\mathbf{e}) = \big(\mathbf{O}_3(\mathbf{e}), \mathbf{O}_4(\mathbf{e})\big).$$

THEOREM 3.12 *For sufficiently small δ_1, $\epsilon \ln \frac{1}{\epsilon} \delta_2$, and ϵ, the operator $\mathbf{O}(\mathcal{S})$ is a contraction mapping from $\mathbf{E}_{\Delta,\mathcal{S}}$ to $\mathbf{E}_{\Delta,\mathcal{S}}$. Therefore, there exists a unique solution $(u(\xi), v(\xi)) \in \mathbf{E}_{\Delta,\mathcal{S}}$ to equations* (3.21) *and* (3.22)*.*

PROOF: Replacing space $\mathbf{B}_j$ with $C(\mathcal{S})$, the proof is parallel to that for Theorem 2.40. □

THEOREM 3.13 *Let F be the classical solution in Theorem* 2.40; *then F is analytic inside the triangular region $\mathcal{T}$.*

PROOF: Let f be the solution in Theorem 3.12; then f satisfies the boundary value problem (3.1). Since all the coefficients in equation (3.1) are analytic in a neighborhood of $\mathcal{S}$, it follows from the classical local theory of ordinary differential equations that f must be analytic in a neighborhood of $\mathcal{S}$. Since a is arbitrary in interval $(0, \nu)$, f is analytic in $\mathcal{T}$ and continuous on the closure of $\mathcal{T}$. From boundary conditions in (3.1), f equals the analytic function F on $(-\nu, 0) \cup \{\xi : \xi = -\nu + s e^{-\pi i/4}, 0 \leq s \leq \sqrt{2}\nu\}$. From properties of analytic continuation, f must be an analytic continuation of F across $(-\nu, 0) \cup \{\xi : \xi = -\nu + s e^{-\pi i/4}\}$ in the region $\mathcal{T}$. Therefore, the theorem follows. □

LEMMA 3.14 *Let F be the classical solution in Theorem* 2.40; *then F is analytic on the line segment on imaginary axis $\mathcal{S}_0 = \{\xi : \operatorname{Re}\xi = 0, -b \leq \operatorname{Im}\xi \leq 0\}$.*

PROOF: Considering the boundary problem for $\xi \in \mathcal{S}_0$:

$$\begin{gathered} \epsilon^2 f'' + \big(L(\xi) + \epsilon^2 L_1(\xi)\big) f = \mathcal{N}(f, I(F), \bar{F})(\xi) + \epsilon^2 L_1 f(\xi) \equiv n_1(\xi)\,, \\ f(-a) = F(-a)\,, \quad f(-bi) = F(-bi)\,. \end{gathered} \tag{3.25}$$

It follows from a variation of the proof of Theorem 3.12 that there exists a unique solution f in $\mathbf{E}_{\Delta,\mathcal{S}_0}$ to the above boundary problem. Since the coefficients of (3.25) are all analytic in a neighborhood of $\mathcal{S}_0$, the solution must be analytic on $\mathcal{S}_0$ from the classical theory of differential equations. On the other hand, from Theorem 3.13, F satisfies equation (3.25) in $\mathcal{D} \cup \mathcal{T}$, since F and F' are continuous up to the closure of $\mathcal{D} \cup \mathcal{T}$. From continuity, F restricted on $\mathcal{S}_0$ satisfies the boundary problem (3.25) and $F \in \mathbf{E}_{\Delta,\mathcal{S}_0}$. By uniqueness, $F \equiv f$; therefore, the theorem follows. $\square$

DEFINITION 3.15

$$k_2 = \sup_{\xi \in \mathcal{T}} \left\{ |\xi - 2i|^{\tau},\ |\xi - 2i|^{\tau+1} \right\} . \tag{3.26}$$

Remark 3.16. It is to be noted that

$$\sup_{\xi \in \mathcal{T}} |\xi - 2i|^{\tau} |F(\xi)| \le k_2 \sup_{\xi \in \mathcal{T}} |F(\xi)| \,,$$

$$\sup_{\xi \in \mathcal{T}} |\xi - 2i|^{\tau+1} |F'(\xi)| \le k_2 \sup_{\xi \in \mathcal{T}} |F'(\xi)| \,.$$

DEFINITION 3.17

$$\hat{\Delta} = \max \left\{ \Delta , 8k_1k_2\Delta , 8k_1k_2k_3\Delta \right\} . \tag{3.27}$$

THEOREM 3.18 *If F is a classical solution as in Definition* 1.7*, then F is analytic in $\mathcal{R} \cup \bar{\mathbb{C}}^+$ and $F \in \mathbf{A}_{0,\hat{\Delta}}$, $F' \in \mathbf{A}_{1,\hat{\Delta}/\epsilon}$.*

PROOF: Combining Theorems 2.42, 3.12, and 3.13, F is analytic in the domain $\mathcal{R}^-$, as defined in Definition 1.1, with

$$\sup_{\xi \in \mathcal{R}^-} |\xi - 2i|^{\tau} |F(\xi)| \le \hat{\Delta} \quad \text{and} \quad \sup_{\xi \in \mathcal{R}^-} |\xi - 2i|^{\tau+1} |F'(\xi)| \le \frac{\hat{\Delta}}{\epsilon} \,.$$

Since F is analytic in $\mathbb{C}^+$ as well as on the line segment $\mathcal{S}_0$ on the imaginary axis, from Condition 3 and successive Taylor expansions of F on the imaginary ξ-axis, starting at $\xi = 0$, this implies that $\operatorname{Im} F = 0$ on $\mathcal{S}_0$. From the Schwartz reflection principle for $\xi \in \mathcal{R}^+$, $F(\xi) = [F(-\xi^*)]^*$ provides the analytic extension to $\operatorname{Re} \xi > 0$. Thus F is analytic in $\mathcal{R}$ and continuous up to its boundary, including the real axis. Thus, F must be analytic in $\mathcal{R} \cup \bar{\mathbb{C}}^+$. Since from reflection, $\|F\|_{0,\mathcal{R}} = \|F\|_{0,\mathcal{R}^-}$ and $\|F'\|_{1,\mathcal{R}} = \|F'\|_{1,\mathcal{R}^-}$, the proof of theorem is complete. $\square$

LEMMA 3.19 *If F is a classical solution as in Definition* 1.7*, then δ, δ_1, and δ_2, as defined in* (1.5) *and* (1.6)*, equals $O(\epsilon^2)$. Therefore, in the domain $\mathcal{R}$, $\|F\|_0 = O(\epsilon^2)$ and $\|F'\|_1 = O(\epsilon)$.*

PROOF: Since F is analytic in $\mathcal{R} \cup \mathbb{C}^+$ and decays algebraically at ∞ in this region, it follows from Cauchy's formula that for real $\xi \in (-\infty, \infty)$,

$$F^{(j)}(\xi) = \frac{j!}{2\pi i} \int_{l_1 \cup l_2} \frac{F(t)}{(t - \xi)^{j+1}} \, dt \,.$$

Using Lemma 2.11 in [34], it follows that

$$\delta_j \equiv \sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{j+\tau} |F^{(j)}(\xi)| \le C_j \|F\|_0 = O(\hat{\Delta}) \,.$$

Therefore, using (1.1), on the real axis, we have

$$\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau} |\operatorname{Re} F(\xi)| = O(\epsilon^2)\,.$$

Further, on taking the derivative of (1.1) with respect to ξ and using $O(\hat{\Delta})$ a priori bounds on F', F'', and F''' on the real axis as above, it follows that

$$\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau+1} |\operatorname{Re} F'(\xi)| = O(\epsilon^2)\,.$$

Using the Hilbert transform property (see Lemma A.1 for $k = \frac{1}{2}$) yields

$$\|(\xi - 2i)^{\tau}\mathcal{H}(g)[\xi]\|_\infty \le C_1\|(\xi - 2i)^{\tau} g\|_\infty + C_2\|(\xi - 2i)^{\tau+1} g'\|_\infty\,;$$

it then follows that for $g(\xi) = \operatorname{Re} F(\xi)$, because $\operatorname{Im} F(\xi) = \mathcal{H}(\operatorname{Re} F)[\xi]$, we have that $\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau} |\operatorname{Im} F(\xi)| = O(\epsilon^2)$. Therefore,

$$\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau} |F(\xi)| = O(\epsilon^2)\,.$$

Hence $\delta = O(\epsilon^2)$. By taking up to the third derivative of (1.1) and using a priori bounds on all derivatives of F for real ξ occurring on the right of (1.1), we get $O(\epsilon^2)$ upper bounds for $|\xi - 2i|^{\tau+1}|g'(\xi)|$, $|\xi - 2i|^{\tau+2}|g''(\xi)|$, and $|\xi - 2i|^{\tau+3}|g'''(\xi)|$ where $g(\xi) = \operatorname{Re} F(\xi)$, as before.

Using properties of the Hilbert transform (Lemmas A.2 and A.3), it follows that $|\xi - 2i|^{\tau+1}|\operatorname{Im} F'(\xi)|$ and $|\xi - 2i|^{\tau+2}|\operatorname{Im} F''(\xi)|$ also have $O(\epsilon^2)$ upper bounds. Hence $\delta_1, \delta_2 = O(\epsilon^2)$. Therefore, $\hat{\Delta} = O(\epsilon^2)$, where $\hat{\Delta}$ is as defined in Definition 3.17. From the previous theorem, in the domain $\mathcal{R}$, $\|F\|_0 = O(\epsilon^2)$ and $\|F'\|_1 = O(\epsilon)$. ☐

PROOF OF THEOREM 1.3: The proof follows from Theorem 3.18 after using Lemma 3.19. ☐

4 Nonexistence of a Solution for $\lambda < \frac{1}{2}$

Rewriting (2.11), we have

$$F(\xi) = \epsilon^2 I(\xi) + \frac{\epsilon^2}{i(F'(\xi)+H)^{1/2}(\bar{F}'(\xi)+\bar{H})^{1/2}}\left[\frac{F''(\xi)+H'}{F'(\xi)+H} - \frac{\bar{F}''(\xi)+\bar{H}'}{\bar{F}'(\xi)+\bar{H}}\right]. \tag{4.1}$$

On multiplying (4.1) by $(F' + H)^{3/2}(\bar{F}' + \bar{H})^{1/2}$ and introducing the change of variable

$$\xi + i\gamma = i\tilde{k}_1\epsilon^{4/7}\chi \quad \text{where } \tilde{k}_1 = (1-\gamma^2)^{3/7}\left[i\bar{F}'(-i\gamma) + i\bar{H}(-i\gamma)\right]^{-1/7}, \tag{4.2}$$

$$F(\xi(\chi)) = \frac{\tilde{k}_1^2\epsilon^{8/7}G(\chi)}{(1-\gamma^2)}\,, \tag{4.3}$$

equation (4.1) becomes

$$
\begin{aligned}
G'' - 1 - \chi^{3/2}\left(1 - \frac{G'}{\chi}\right)^{3/2} G \\
&= \epsilon^{4/7}\chi \tilde{A}_1(\epsilon^{4/7}\chi)\left[1 - \frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right] \\
&\quad + \left[\left(1 - \frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right)^{3/2} - \left(1 - \frac{G'}{\chi}\right)^{3/2}\right] G\chi^{3/2} \\
&\quad + \epsilon^{4/7}\chi\tilde{A}_3(\epsilon^{4/7}\chi)\left[1 - \frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right]^{3/2} \\
&\qquad \times \left[G\chi^{3/2} + (\epsilon^{4/7}\chi)^{3/2}\tilde{A}_4(\epsilon^{4/7}\chi)\right] \\
&\quad + (\epsilon^{4/7}\chi)^{3/2}\tilde{A}_4(\epsilon^{4/7}\chi)\left[1 - \frac{G'}{\chi} + \epsilon^{4/7}\chi\tilde{A}_2(\epsilon^{4/7}\chi)\right]^{3/2} \\
&\quad + \epsilon^{4/7}\chi\tilde{A}_5(\epsilon^{4/7}\chi)\,,
\end{aligned}
\tag{4.4}
$$

where $\tilde{A}_j(\epsilon^{4/7}\chi)$ are analytic functions in $\epsilon^{4/7}\chi$.

The additional change of variable

$$
\chi = \left(\frac{7}{4}\eta\right)^{4/7}, \quad \chi^{3/2}G(\chi) = -\eta\phi(\eta)\,,
\tag{4.5}
$$

leads to

$$
\begin{aligned}
\mathcal{L}\phi &\equiv \frac{d^2\phi}{d\eta^2} + \frac{5}{7\eta}\frac{d\phi}{d\eta} - \left(1 + \frac{45}{196\eta^2}\right)\phi \\
&= -\frac{1}{\eta} - \frac{33}{196\eta^2}\phi + \phi\left\{\left[1 + \frac{4}{49\eta}\phi + \frac{4}{7}\phi'\right]^{3/2} - 1\right\} \\
&\quad + \frac{(\epsilon\eta)^{4/7}}{\eta} E\big((\epsilon\eta)^{2/7}, \phi, \phi', \eta^{-1}\big)\,.
\end{aligned}
\tag{4.6}
$$

Remark 4.1. It is to be noted that E has a convergent series in ϕ and ϕ',

$$
E = \sum_{j_1, j_2 \geq 0}^{\infty} E_{j_1, j_2}\left((\epsilon\eta)^{2/7}, \frac{1}{\eta}\right)\phi^{j_1}(\phi')^{j_2}\,,
\tag{4.7}
$$

where we can choose ρ and C independently of ϵ and η so that

$$
|E_{j_1, j_2}| < C\rho^{-j_1 - j_2}
$$

in the domain $q_1/\epsilon \geq |\eta| \geq R$ for R sufficiently large and ϵ small for some q_1 independent of ϵ.

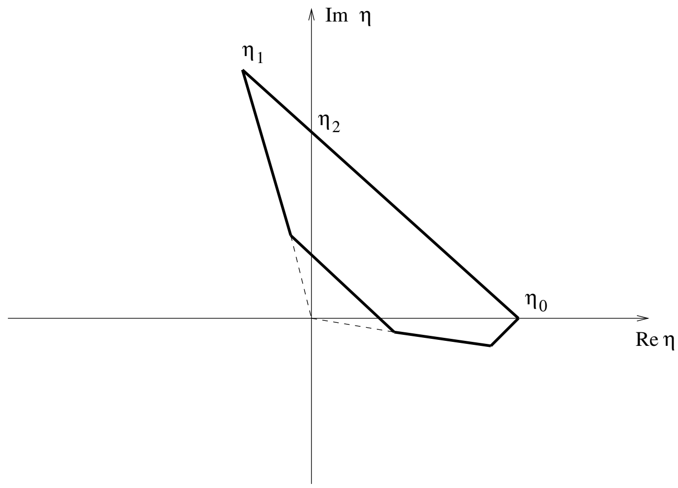

FIGURE 4.1. Region $\mathcal{R}_{2,R}$.

THEOREM 4.2 *Let $F(\xi)$ be the solution in Theorem* 1.3. *After the changes of variables* (4.2), (4.3)*, and* (4.5)*, $\phi(\eta, \epsilon, a)$ satisfies* (4.6) *for $q_0\epsilon^{-1} \le |\eta| \le q_1\epsilon^{-1}$, at least for $0 \le \arg\eta \le 5\pi/8$ (where $q_0, q_1 = O(\gamma - b)^{7/4}$ but are independent of ϵ). In that domain, $\phi(\eta,\epsilon), \phi'(\eta,\epsilon) = O(\epsilon)$ as $\epsilon \to 0^+$. Furthermore, in this domain $|\eta\phi| = O((\gamma-b)^{3/2})$, $|\eta\phi'| = O((\gamma-b)^{3/4}, \frac{\epsilon}{(\gamma-b)^{\frac{1}{4}}})$. Also, on the positive real η-axis in the interval $q_0\epsilon^{-1} \le \eta \le q_1\epsilon^{-1}$, $\operatorname{Im}\phi = 0$.*

PROOF: From (2.11) and the definition of $\mathcal{G}$, the solution $F(\xi)$ in Theorem 1.3 clearly satisfies (4.1). Notice from transformation (4.2) and (4.5), if $\eta = O(\epsilon^{-1})$, $\xi + i\gamma = O(1)$ and if $0 \le \arg\eta \le 5\pi/8$, then $0 \le \arg(\gamma + i\xi) \le 5\pi/14$, and for suitable $q_0, q_1 = O(\gamma - b)^{7/4}$, this corresponds to $\xi \in \mathcal{R}^-$ close to $\xi = -ib$, where F is known to satisfy (4.1) with $\|F\|_0 = O(\epsilon^2)$ and $\|F'\|_1 = O(\epsilon)$. Hence $\phi(\eta,\epsilon)$ must satisfy transformed equation (4.6). Also, from (4.2), (4.3), and (4.5), it is clear that $\phi(\eta,\epsilon), \phi'(\eta,\epsilon) = O(\epsilon)$ as $\epsilon \to 0^+$, and that $\eta\phi = O(\gamma - b)^{3/2}$ and $\eta\phi' = O((\gamma-b)^{3/4}, \epsilon(\gamma-b)^{-\frac{1}{4}})$. Since $F(\xi)$ is real at least on the imaginary ξ-axis segment $[-ib, 0]$, it follows from (4.2) that for suitable q_0 and q_1, $\operatorname{Im}\phi = 0$ for η real and positive, at least when $q_0/\epsilon \le \eta \le q_1/\epsilon$. □

DEFINITION 4.3 $\mathcal{R}_{2,R} = \{\eta : R < \operatorname{Im}\eta + \operatorname{Re}\eta < \tilde{k}_0\epsilon^{-1}, \arg\eta \in [0, \frac{5\pi}{8}); -\operatorname{Im}\eta + R < \operatorname{Re}\eta < \operatorname{Im}\eta + \tilde{k}_0\epsilon^{-1}, \arg\eta \in (-\frac{\pi}{8}, 0]\}$, where $q_0 < \tilde{k}_0 < q_1$ (see Figure 4.1).

DEFINITION 4.4 We define $\tilde{\phi}(\eta)$ (suppressing the ϵ-dependence) as the solution $\phi(\eta,\epsilon)$ in Theorem 4.1.

DEFINITION 4.5

$$\phi_1(\eta) = \eta^{-5/14}e^{-\eta}\,, \qquad \phi_2(\eta) = \eta^{-5/14}e^{\eta}\,. \tag{4.8}$$

$\phi_1(\eta)$ and $\phi_2(\eta)$ satisfy the following equation exactly:

$$\mathcal{L}\phi \equiv \frac{d^2\phi}{d\eta^2} + \frac{5}{7\eta}\frac{d\phi}{d\eta} - \left(1 + \frac{45}{196\eta^2}\right)\phi = 0\,.$$

The Wronskian of ϕ_1 and ϕ_2 is

$$\mathcal{W}(\phi_1, \phi_2)(\eta) = 2\eta^{-5/7} \tag{4.9}$$

Equation (4.6) can be rewritten as

$$\mathcal{L}\phi = \mathcal{N}_1(\phi, \phi', \epsilon) \tag{4.10}$$

where the operator $\mathcal{N}_1$ is defined by

$$\begin{aligned}\mathcal{N}_1(\phi, \phi', \epsilon)[\eta] = -\frac{1}{\eta} - \frac{33}{196\eta^2}\phi + \phi\left\{\left[1 + \frac{4}{49\eta}\phi + \frac{4}{7}\phi'\right]^{3/2} - 1\right\} \\ + \frac{(\epsilon\eta)^{4/7}}{\eta}E\big((\epsilon\eta)^{2/7}, \phi, \phi', \eta^{-1}\big)\,.\end{aligned} \tag{4.11}$$

DEFINITION 4.6

$$\eta_0 = \tilde{k}_0\epsilon^{-1}\,, \quad \eta_1 = \tilde{k}_0\epsilon^{-1}\frac{\sin\frac{\pi}{4}}{\sin\frac{\pi}{8}}e^{\frac{5i\pi}{8}}\,, \quad \eta_2 = i\tilde{k}_0\epsilon^{-1}\,. \tag{4.12}$$

LEMMA 4.7 *The solution $\tilde{\phi}(\eta)$ as defined earlier satisfies the following integral equation*:

$$\begin{aligned}\tilde{\phi} = &-\phi_1(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[t]dt \\ &+ \phi_2(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[t]dt \\ &- \phi_1(\eta)\frac{(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1))}{2\eta_1^{-5/7}} \\ &+ \phi_2(\eta)\frac{(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0))}{2\eta_0^{-5/7}}\,.\end{aligned} \tag{4.13}$$

PROOF: By using a variation of parameters on (4.6), with $\phi(\eta, \epsilon)$ replaced by $\tilde{\phi}(\eta)$, we get

$$\begin{aligned}\tilde{\phi}(t) = &-\phi_1(t)\int_{\eta_2}^{t}\frac{\phi_2(s)}{2s^{-5/7}}\mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[s]ds \\ &+ \phi_2(t)\int_{\eta_2}^{t}\frac{\phi_1(s)}{2s^{-5/7}}\mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)[s]ds + A_1\phi_1(t) + A_2\phi_2(t)\,.\end{aligned} \tag{4.14}$$

After evaluating (4.14) and its derivative at $t = \eta_2$ and solving for A_1 and A_2, we have

$$A_1 = \frac{\tilde{\phi}(\eta_2)\phi_2'(\eta_2) - \tilde{\phi}'(\eta_2)\phi_2(\eta_2)}{2\eta_2^{-5/7}},$$
$$A_2 = \frac{\tilde{\phi}(\eta_2)\phi_1'(\eta_2) - \tilde{\phi}'(\eta_2)\phi_1(\eta_2)}{2\eta_2^{-5/7}}. \tag{4.15}$$

However, on using integration by parts twice, we get

$$\begin{aligned} &-\phi_1(t)\int_{\eta_2}^{\eta_1}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{L}\tilde{\phi}(t)dt + \phi_2(t)\int_{\eta_2}^{\eta_0}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{L}\tilde{\phi}(t)dt \\ &= \Bigg\{-\phi_1(t)\frac{(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1))}{2\eta_1^{-5/7}} \\ &\quad + \phi_2(t)\frac{(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0))}{2\eta_0^{-5/7}} - A_1\phi_1(t) - A_2\phi_2(t)\Bigg\}. \end{aligned} \tag{4.16}$$

Using $\mathcal{L}\tilde{\phi} = \mathcal{N}_1(\tilde{\phi}, \tilde{\phi}', \epsilon)$ in (4.16) and using this expression in (4.14), we get (4.13) and hence the lemma follows. □

DEFINITION 4.8

$$\mathbf{W} = \big\{\phi : \phi(\eta) \text{ is analytic in } \mathcal{R}_{2,R} \text{ and continuous in its closure, with } \|\phi\| := \sup_{\mathcal{R}_{2,R}} |\eta\phi(\eta)| < \infty\big\}. \tag{4.17}$$

LEMMA 4.9 *Let $N \in \mathbf{W}$ and define*

$$\psi_1(\eta) := \phi_1(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}N(t)dt, \quad \psi_2(\eta) := \phi_2(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}N(t)dt,$$
$$\psi_3(\eta) := \phi_1'(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}N(t)dt, \quad \psi_4(\eta) := \phi_2'(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}N(t)dt.$$

Then $\|\psi_1\| \le K\|N(t)\|$, $\|\psi_2\| \le K\|N(t)\|$, $\|\psi_3\| \le K\|N(t)\|$, and $\|\psi_4\| \le K\|N(t)\|$, where K is some constant independent of R and ϵ.

PROOF: From the nature of the domain $\mathcal{R}_{2,R}$, any point $\eta \in \mathcal{R}_{2,R}$ can be connected to η_0 by a straight line within $\mathcal{R}_{2,R}$. So on line $t(s)$, parametrized by arc length s, $\mathrm{Re}(t(s) - \eta)$ increases from η to η_0 so that $\frac{d}{ds}\mathrm{Re}(t(s) - \eta) > C_1 > 0$, where C_1 is a constant independent of ϵ. Further, on this straight line, $0 < C_2 < |t/\eta|$, where C_2 is independent of ϵ. Then

$$\begin{aligned} |\psi_2(\eta)| &= \left|\int_{\eta_0}^{\eta} t^{-9/14}\eta^{-5/14}e^{-(t-\eta)}(tN(t))dt\right| \\ &\le C_2^{-9/14}|\eta|^{-1}\|N\|\int_0^1 \frac{d\{e^{-\mathrm{Re}(t(s)-\eta)}\}}{\frac{d}{ds}\mathrm{Re}\,t(s)} \le C_1^{-1}C_2^{-9/14}|\eta|^{-1}\|N\|. \end{aligned}$$

So $\|\psi_2\| \le K\|N\|$, and similarly for ψ_4 since $|\phi_2'|/|\phi_2| \le C$.

Also, it is clear that any point η can be connected to η_1 by a straight line entirely within $\mathcal{R}_{2,R}$, and on such a path $t(s)$, $\frac{d}{ds}\,\mathrm{Re}(\eta - t(s)) > C_1 > 0$, where C_1 is independent of ϵ. Furthermore, on this straight line, $|t/\eta| > C_2 > 0$, where C_2 is independent of ϵ. So

$$\begin{aligned}|\psi_1(\eta)| &= \left|\int_{\eta_1}^{\eta} t^{-9/14}\eta^{-5/14}e^{-(\eta-t)}tN(t)dt\right|\\ &\le C_2^{-9/14}C_1^{-1}|\eta|^{-1}\|N\| \le K|\eta|^{-1}\|N\|\,,\end{aligned}$$

and similarly for $\|\psi_3\|$ since $|\phi_1'/\phi_1| \le C$. □

DEFINITION 4.10 We define ϕ_3 and ϕ_4 so that

$$\phi_3(\eta) = -\phi_1(\eta)\frac{(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1))}{2\eta_1^{-5/7}}\,, \tag{4.18}$$

$$\phi_4(\eta) = \phi_2(\eta)\frac{(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0))}{2\eta_0^{-5/7}}\,. \tag{4.19}$$

LEMMA 4.11

$$\begin{aligned}\|\phi_3'\|, \|\phi_3\| &\le C_1\big[|\eta_1\tilde{\phi}(\eta_1)| + |\tilde{\phi}'(\eta_1)\eta_1|\big]\,,\\ \|\phi_4'\|, \|\phi_4\| &\le C_1\big[|\eta_0\tilde{\phi}(\eta_0)| + |\tilde{\phi}'(\eta_0)\eta_0|\big]\,,\end{aligned}$$

where C_1 is independent of ϵ.

PROOF: Since $|\eta_1| > |\eta|$, it follows that

$$\left|\frac{\phi_1(\eta)\phi_2(\eta_1)}{\eta_1(\eta_1^{-5/7})}\right| = \exp\big[\mathrm{Re}\,\eta_1 - \mathrm{Re}\,\eta\big]|\eta_1|^{-9/14}|\eta|^{-5/14} \le C_1|\eta|^{-1}\,.$$

Also, since $|\eta| < C|\eta_0|$, for constant C independent of ϵ,

$$\left|\frac{\phi_2(\eta)\phi_1(\eta_0)}{\eta_0(\eta_0^{-5/7})}\right| = \exp\big[\mathrm{Re}\,\eta - \mathrm{Re}\,\eta_0\big]|\eta_0|^{-9/14}|\eta|^{-5/14} \le C_2|\eta|^{-1}\,.$$

Since $\phi_2'(\eta) = \phi_2(\eta)[1 - 5/(14\eta)]$ and $\phi_1'(\eta) = \phi_1(\eta)[-1 - 5/(14\eta)]$, the same arguments as above show that

$$\left|\frac{\phi_1(\eta)\phi_2'(\eta_1)}{\eta_1(\eta_1^{-5/7})}\right| \le C_2|\eta|^{-1} \quad \text{and} \quad \left|\frac{\phi_2(\eta)\phi_1'(\eta_0)}{\eta_0(\eta_0^{-5/7})}\right| \le C_1|\eta|^{-1}\,.$$

Hence, the lemma follows from the definition of ϕ_3 and ϕ_4 in Definition 4.10. □

DEFINITION 4.12

$$\mathbf{W}_\sigma \equiv \{\phi \in \mathbf{W} : \|\phi\| < \sigma\}\,.$$

DEFINITION 4.13

$$D\left(\phi, \phi', \frac{1}{\eta}\right) \equiv \phi\left\{\left(1 + \frac{4}{49\eta}\frac{\phi}{\eta} + \frac{4}{7}\phi'\right)^{3/2} - 1\right\} = \sum_{j_1+j_2\geq 2} \tilde{A}_{j_1,j_2}(\eta)\phi^{j_1}(\phi')^{j_2}\,.$$

We define a constant $\tilde{\rho}$ that is independent of η and ϵ in the domain $\mathcal{R}_{2,R}$ so that

$$|\tilde{A}_{j_1,j_2}(\eta)| < C\tilde{\rho}^{-j_1-j_2}\,.$$

LEMMA 4.14 *If $\phi, \phi' \in \mathbf{W}_\sigma$, then $\mathcal{N}_1(\phi, \phi', \epsilon) \in \mathbf{W}_\sigma$ and*

$$\|\mathcal{N}_1(\phi, \phi', \epsilon)\| < 1 + C\left(\frac{1}{R^3}\sigma + \tilde{k}_0^{4/7} + \frac{\sigma^2}{R\rho^2}\right)$$

for R large enough so that $\sigma/\tilde{\rho}R, \sigma/\rho R \leq \frac{1}{2}$, where ρ and $\tilde{\rho}$ are as in Remark 4.1 *and Definition* 4.13*.*

PROOF: In (4.11),

$$\left\|\frac{-33}{196\eta^2}\phi\right\| < \frac{c}{R^3}\|\phi\|\,, \qquad \left\|\frac{1}{\eta}\right\| \leq 1\,.$$

The norm of the first nonlinear term in (4.11) can be estimated by noting

$$\begin{aligned}\left|\eta D\left(\phi, \phi', \frac{1}{\eta}\right)\right| &= \left|\sum_{j_1+j_2\geq 2} \tilde{A}_{j_1,j_2}(\eta)\phi^{j_1}\phi'^{j_2}\eta\right| \\ &\leq C\sum_{j_1+j_2\geq 2}\frac{1}{|\eta|^{j_1+j_2-1}}\frac{1}{\tilde{\rho}^{j_1+j_2}}|\eta\phi|^{j_1}|\eta\phi'|^{j_2} \leq C\frac{\sigma^2}{R\tilde{\rho}^2}\,.\end{aligned}$$

The norm of the second nonlinear term in (4.11) can be estimated by using (4.7) and noting

$$\begin{aligned}\left|(\epsilon\eta)^{4/7}E\right| &\leq (\epsilon\eta_0)^{4/7}\left|\sum_{j_1,j_2\geq 0}^{\infty} E_{j_1,j_2}\left((\epsilon\eta)^{2/7}, \frac{1}{\eta}\right)\phi^{j_1}(\phi')^{j_2}\right| \\ &\leq C(\epsilon\eta_0)^{4/7}\left(\sum_{j_1,j_2\geq 0}^{\infty}\frac{1}{R^{j_1+j_2}}\frac{1}{\rho^{j_1+j_2}}\sigma^{j_1+j_2}\right) \leq C\tilde{k}_0^{4/7} = O(\gamma - b)\,,\end{aligned}$$

where ρ is as defined in Remark 4.1. The lemma follows from combining the above results. □

LEMMA 4.15 *If $\phi \in \mathbf{W}_\sigma$, $\psi \in \mathbf{W}_\sigma$, $\phi' \in \mathbf{W}_\sigma$, and $\psi' \in \mathbf{W}_\sigma$, then for $R > \{\frac{2\sigma}{\tilde{\rho}}, \frac{2\sigma}{\rho}\}$,*

$$\begin{aligned}&\left\|\mathcal{N}_1(\phi, \phi', \epsilon) - \mathcal{N}_1(\psi, \psi', \epsilon)\right\| \leq \\ &\qquad C\left[\frac{1}{R^3} + \frac{\sigma}{\tilde{\rho}^2 R} + \tilde{k}_0^{4/7}\right](\|\phi - \psi\| + \|\phi' - \psi'\|)\,,\end{aligned}$$

where ρ and $\tilde{\rho}$ are as in Remark 4.1 *and Definition* 4.13 *and C is independent of ϕ, ψ, and ϵ.*

PROOF:

$$\left\| \frac{-33}{196\eta^2}(\phi - \psi) \right\| \le \frac{C}{R^3} \|(\phi - \psi)\| .$$

Note that

$$\begin{aligned}
\left|\eta(\phi^{j_1}\psi^{j_2} - \psi^{j_1}\phi^{j_2})\right| &= \left|\eta\phi^{j_1}(\phi'^{j_2} - \psi'^{j_2}) + \eta\psi'^{j_2}(\phi^{j_1} - \psi^{j_1})\right| \\
&\le \|\phi\|^{j_1} \frac{1}{R^{j_1+j_2-1}} j_2(\|\phi'\| + \|\psi'\|)^{j_2-1} \|\psi' - \phi'\| \\
&\quad + \frac{\|\psi\|^{j_2}}{R^{j_1+j_2-1}} j_1(\|\psi\| + \|\phi\|)^{j_1-1} \|\psi - \phi\| \\
&\le \left(\frac{\sigma}{R}\right)^{j_1+j_2-1} \left[j_2\|\psi' - \phi'\| + j_1\|\phi - \psi\| \right].
\end{aligned} \tag{4.20}$$

So in (4.11),

$$\begin{aligned}
&\left| \eta D\left(\phi, \phi', \frac{1}{\eta}\right) - \eta D\left(\psi, \psi', \frac{1}{\eta}\right) \right| \\
&\quad \le \sum_{j_1+j_2\ge 2} |\tilde{A}_{j_1,j_2}(\eta)| \left|\eta(\phi^{j_1}\phi'^{j_2} - \psi^{j_1}\psi'^{j_2})\right| \\
&\quad \le C \sum_{j_1+j_2\ge 2} \frac{1}{\tilde{\rho}^{j_1+j_2}} \left(\frac{\sigma}{R}\right)^{j_1+j_2-1} \left\{ j_2\|\psi' - \phi'\| + j_1\|\psi - \phi\| \right\} \\
&\quad \le \frac{C}{\tilde{\rho}^2} \frac{\sigma}{R} \left[\|\psi' - \phi'\| + \|\psi - \phi\| \right] \quad \text{for } R > \frac{2\sigma}{\tilde{\rho}} .
\end{aligned}$$

From (4.7) and (4.20)

$$\begin{aligned}
&\left| (\epsilon\eta)^{4/7} \left[E\left((\epsilon\eta)^{4/7}, \frac{1}{\eta}, \phi, \phi'\right) - E\left((\epsilon\eta)^{2/7}, \frac{1}{\eta}, \psi, \psi'\right) \right] \right| \\
&\quad \le C|\epsilon\eta_0|^{4/7} \sum_{j_1,j_2\ge 0} \frac{1}{\rho^{j_1+j_2}} \left(\frac{\sigma}{R}\right)^{j_1+j_2-1} \left\{ j_2\|\psi' - \phi'\| + j_1\|\psi - \phi\| \right\} \\
&\quad \le C\tilde{k}_0^{4/7} \left[\|\psi' - \phi'\| + \|\psi - \phi\| \right] \quad \text{for } R > \frac{2\sigma}{\tilde{\rho}} .
\end{aligned}$$

□

Consider the integral equation in the domain $\mathcal{R}_{2,R}$:

$$\begin{aligned}
\phi(\eta) &= \mathcal{L}_1\phi(\eta) - \phi_1(\eta) \frac{(\phi_2(\eta_1)\tilde{\phi}'(\eta_1) - \phi_2'(\eta_1)\tilde{\phi}(\eta_1))}{2\eta_1^{-5/7}} \\
&\quad + \phi_2(\eta) \frac{(\phi_1(\eta_0)\tilde{\phi}'(\eta_0) - \phi_1'(\eta_0)\tilde{\phi}(\eta_0))}{2\eta_0^{-5/7}} ,
\end{aligned} \tag{4.21}$$

where

$$\mathcal{L}_1\phi \equiv -\phi_1(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)[t]dt + \phi_2(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)[t]dt\,. \tag{4.22}$$

DEFINITION 4.16

$$\mathbf{E}^1 := \mathbf{W}\oplus\mathbf{W}\,,\quad \|(\phi,\phi')\|_{\mathbf{E}^1} = \|\phi\| + \|\phi'\|\,. \tag{4.23}$$

This is clearly a Banach space. Similarly, $\mathbf{E}^1_\sigma = \{(\phi,\phi')\in\mathbf{E}^1 \text{ with } \|(\phi,\phi')\|\le\sigma\}$.

Define

$$\mathcal{M}:\mathbf{E}^1\to\mathbf{E}^1\,,\quad \mathcal{M}(\phi,\phi') = \big(\mathcal{M}_1(\phi,\phi'),\mathcal{M}_2(\phi,\phi')\big)\,,$$

where

$$\mathcal{M}_1(\phi,\phi') = \mathcal{L}_1\phi(\eta) + \phi_3(\eta) + \phi_4(\eta)\,, \tag{4.24}$$

$$\mathcal{M}_2(\phi,\phi') = \mathcal{L}_2\phi(\eta) + \phi_3'(\eta) + \phi_4'(\eta)\,, \tag{4.25}$$

where

$$\mathcal{L}_2\phi \equiv -\phi_1'(\eta)\int_{\eta_1}^{\eta}\frac{\phi_2(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)dt + \phi_2'(\eta)\int_{\eta_0}^{\eta}\frac{\phi_1(t)}{2t^{-5/7}}\mathcal{N}_1(\phi,\phi',\epsilon)(t)dt\,. \tag{4.26}$$

THEOREM 4.17 *For fixed $\sigma \ge 4K$, where K is as defined in Lemma 4.9, there exists $\tilde{k}_0$ small enough but independent of ϵ (i.e., b is chosen so that $\gamma - b$ is small but independent of ϵ), and R large enough so that for any ϵ small enough, $\mathcal{M}$ is a contraction mapping from $\mathbf{E}^1_\sigma$ to $\mathbf{E}^1_\sigma$.*

PROOF: Using Lemmas 4.9, 4.11, and 4.14 in (4.24) and (4.25), it follows that

$$\begin{aligned}\|\mathcal{M}(\phi,\phi')\| &= \|\mathcal{M}_1(\phi,\phi')\| + \|\mathcal{M}_2(\phi,\phi')\|\\ &\le 2K\left[1 + C\left(\frac{\sigma}{R^3} + \tilde{k}_0^{4/7} + \frac{\sigma^2}{R\tilde{\rho}^2}\right)\right]\\ &\quad + C_1\big[|\eta_0\tilde{\phi}(\eta_0)| + |\tilde{\phi}'(\eta_0)\eta_0| + |\eta_1\tilde{\phi}(\eta_1)| + |\eta_1\tilde{\phi}'(\eta_1)|\big]\,.\end{aligned} \tag{4.27}$$

From Theorem 4.2,

$$\eta\tilde{\phi}(\eta),\eta\tilde{\phi}'(\eta) = O\left((\gamma-b)^{3/4},\frac{\epsilon}{(\gamma-b)^{\frac{1}{4}}}\right)$$

for $\eta = \eta_0$ or η_1. But since $(\gamma-b)^{7/4} = O(\tilde{k}_0)$, it follows that $\tilde{k}_0$ can be chosen small enough (but independent of ϵ) and R can be chosen large enough so that the right-hand side of (4.27) is less than $4K$ for small enough ϵ.

Furthermore, from Lemma 4.9 and Lemma 4.15,

$$
\begin{aligned}
&\left\|\mathcal{M}_{1,2}(\phi_1,\phi_2)-\mathcal{M}_{1,2}(\phi_2,\phi_2')\right\| \\
&\quad\le K\|\mathcal{N}(\phi_1,\phi_1',\epsilon)-\mathcal{N}(\phi_2,\phi_2',\epsilon)\| \\
&\quad\le KC\big[\|\phi_1-\phi_2\|+\|\phi_1'-\phi_1'\|\big]\left[\frac{1}{R^3}+\frac{\sigma}{\tilde{\rho}^2R}+\tilde{k}_0^{4/7}\right],
\end{aligned}
$$

so

$$
\begin{aligned}
&\left\|\mathcal{M}(\phi_1,\phi_1')-\mathcal{M}(\phi_2,\phi_2')\right\| \le \\
&\qquad 2KC\left[\frac{1}{R^3}+\frac{\sigma}{\tilde{\rho}^2R}+\tilde{k}_0^{4/7}\right]\left\|\left(\phi_1-\phi_2,\phi_1'-\phi_2'\right)\right\|,
\end{aligned}
$$

which is a contraction for $\tilde{k}_0$ small and R large. □

Remark 4.18. Note that R can be chosen large enough and $\tilde{k}_0$ small enough and the theorem holds for all small ϵ. In other words, the choice of R and $\tilde{k}_0$ can be made independent of ϵ, although the theorem also holds if $R = O(1/\epsilon)$ for sufficiently small ϵ and $\tilde{k}_0$.

COROLLARY 4.19 *Integral equation* (4.21) *has the unique analytic solution* $\phi(\eta)$ *and* $\phi(\eta) = \tilde{\phi}(\eta)$ *in the domain* $\mathcal{R}_{2,R}$.

PROOF: The unique solution ϕ follows from Theorem 4.17 using the contraction mapping theorem. If we choose $R = O(1/\epsilon)$ suitably, then Theorem 4.2 applies to domain $\mathcal{R}_{2,R}$, and from Lemma 4.7, $\phi = \tilde{\phi}$. From analytic continuation, $\phi - \tilde{\phi} = 0$ everywhere on $\mathcal{R}_{2,R}$ even when R is independent of ϵ but large. □

LEMMA 4.20 *The solution* $\tilde{\phi}(\eta)$ *satisfies* $\operatorname{Im}\tilde{\phi}(\eta) = 0$ *for* $R < \eta < q_1/\epsilon$ *for sufficiently large* R *and small enough* ϵ *for* R *independent of* ϵ.

PROOF: From Corollary 4.19, it follows that $\tilde{\phi}(\eta)$ is analytic, in particular, on the real axis for $R < \eta < \tilde{k}_0/\epsilon$. However, from Theorem 4.2, $\operatorname{Im}\tilde{\phi} = 0$ for $q_0/\epsilon \le \eta \le q_1/\epsilon$. Since $\tilde{k}_0 > q_0$, the lemma follows. □

LEMMA 4.21 *For any fixed* η *in the domain* $\{\eta : \operatorname{Re}\eta + \operatorname{Im}\eta > R,\ 5\pi/8 > \arg\eta > -\pi/8\}$ *and* $\lim_{\epsilon\to 0}\tilde{\phi}(\eta,\epsilon) = \phi_0(\eta)$, *where* $\phi_0(\eta)$ *satisfies*

$$
\begin{aligned}
(4.28)\quad &\phi_0(\eta) = \\
&\quad -\phi_2(\eta)\int_{\infty e^{5i\pi/8}}^{\eta}\frac{\mathcal{N}_1(\phi_0,\phi_0',0)[t]}{2t^{-5/7}}\,dt+\phi_1(\eta)\int_{\infty}^{\eta}\frac{\mathcal{N}_1(\phi_0,\phi_0',0)[t]}{2t^{-5/7}}\,dt\,.
\end{aligned}
$$

PROOF: The lemma follows from (4.13) by taking the limit as $\epsilon \to 0$ and using Theorem 4.2; $\tilde{\phi}(\eta_1)$, $\tilde{\phi}'(\eta_1)$, $\tilde{\phi}(\eta_0)$, and $\phi'(\eta_0)$ all tend to 0 while

$$
\frac{\phi_1(\eta)\phi_2(\eta_1)}{2\eta_1^{-5/7}} \to 0 \quad\text{and}\quad \frac{\phi_2(\eta)\phi_1(\eta_0)}{2\eta_0^{-5/7}} \to 0 \quad \text{as } \epsilon \to 0
$$

since $\eta_1, \eta_0 \to \infty$. □

COROLLARY 4.22 *$\phi_0(\eta)$ satisfies the differential equation*

$$\mathcal{L}_1 \phi_0 = \mathcal{N}_1\big(\phi_0, \phi_0', 0\big) = \frac{-33}{196\eta^2} - \frac{1}{\eta} + \phi_0 \left\{ \left(1 + \frac{4}{49}\frac{\phi_0}{\eta} + \frac{4}{7}\phi_0' \right)^{3/2} - 1 \right\} \tag{4.29}$$

with $\eta\phi_0(\eta)$ finite as $\eta \to \infty$, at least for $\arg \eta \in (-\pi/8, 5\pi/8)$.

PROOF: $\mathcal{L}_1\phi_0 = \mathcal{N}_1(\phi_0, \phi_0', 0)$ follows simply from applying $\mathcal{L}_1$ to (4.28). Since $|\eta\phi(\eta)|$ was bounded independently of ϵ in the domain $\mathcal{R}_{2,R}$, it follows that as $\epsilon \to 0$, $|\eta\phi_0|$ is also bounded at least for $\arg \eta \in (-\pi/8, 5\pi/8)$. □

Remark 4.23. It is known from the general theory worked out by Costin [7] that (4.29) has a unique solution with asymptotic expansion

$$\phi_0 \sim \sum_{j=1}^{\infty} \frac{a_j}{\eta^j} \quad \text{valid for } -\tfrac{\pi}{2} < \arg \eta < \pi \,,$$

and that on the positive real axis

$$\operatorname{Im} \phi_0 \sim S \eta^{-5/14} e^{-\eta} \tag{4.30}$$

for some Stokes constant S (which is a numerical constant independent of any parameter) that can be computed.

However, applying transformation (4.2), (4.3), and (4.5) and going back to variable χ and G, it is clear that $\lim_{\epsilon \to 0} G(\chi(\eta)) = G_0(\chi(\eta))$ and that $G_0(\chi)$ satisfies

$$G_0'' = 1 + (\chi - G_0')^{3/2} G_0 \,. \tag{4.31}$$

If we use transformation

$$V_0(\chi) = (\chi - G_0')^{-1/2} \,, \tag{4.32}$$

then it follows from (4.31) that $V_0(\chi)$ satisfies

$$2V_0''(\chi) = \chi - V_0^{-2} \tag{4.33}$$

with $V_0(\chi) \to \chi^{-1/2}$ as $\chi \to \infty$, at least for $5\pi/14 \geq \arg \chi \geq 0$.

Combescot et al. [5] considered (4.33), and by computing many terms in the asymptotic expansion for large χ, were able to use a Borel summation procedure to compute the constant $\tilde{S}$ in the asymptotic expression

$$\operatorname{Im} V_0(\chi) \sim \tilde{S} \chi^{-\frac{3}{8}} e^{-\frac{4}{7}\chi^{7/4}} \tag{4.34}$$

for large positive χ. The number $\tilde{S}$ was found to be nonzero. By using the transformation from χ to η, it follows that S in (4.31) must also be nonzero.

LEMMA 4.24 *For all sufficiently small ϵ,*

$$\operatorname{Im} \tilde{\phi}(\eta, \epsilon) \neq 0 \quad \textit{for any } \eta \in \left(R, \frac{q_1}{\epsilon} \right). \tag{4.35}$$

PROOF: Since $\lim_{\epsilon\to 0}\tilde{\phi}(\eta) = \phi_0(\eta)$, $\lim_{\epsilon\to 0} \operatorname{Im}\tilde{\phi}(\eta,\epsilon) = \operatorname{Im}\phi_0(\eta) \neq 0$ from (4.30), since S is nonzero. $\square$

COROLLARY 4.25 $\operatorname{Im} F \neq 0$ *on some imaginary* ξ*-axis segment* $[-ib, -ib']$ *for some* $b' < b$*.*

PROOF: On using transformation (4.2), (4.3), and (4.5), the interval $(R, q_1/\epsilon)$ in η corresponds to an $\operatorname{Im}\xi$-axis interval that includes $[-ib, -ib']$ for some suitably chosen $b' < b$. So, at least on this segment, $\operatorname{Im} F(\xi) = \operatorname{Im}\phi(\eta(\xi)) \neq 0$. $\square$

PROOF OF THEOREM 1.8: We have shown that any classical solution $F(\xi)$, if it exists, is analytic in $\mathcal{R}\cup\bar{\mathbb{C}}^{+}$ and belongs to $\mathbf{A}_0$. It is also analytic in the $\operatorname{Im}\xi$-axis segment $[-ib, i\infty)$. From successive Taylor expansions on the imaginary ξ-axis, starting at $\xi = 0$, it follows that the symmetry Condition 3 implies $\operatorname{Im} F = 0$ for $\xi \in [-ib, i\infty)$. But this contradicts the previous corollary for all sufficiently small ϵ. Hence the proof of Theorem 1.8 follows. $\square$

Appendix A: Proof of Some Lemmas

LEMMA A.1 *Let* $g \in \mathcal{C}^1(-\infty,\infty)$ *such that* $\|(\xi - 2i)^\tau g\|_\infty < \infty$ *for some* $0 < \tau < 1$*, and let* $\||\xi - 2i|^{\tau+1} g'\|_\infty < \infty$ *as well. Then, for any* $k \in (0, \frac{1}{2}]$*,*

$$\text{(A.1)} \qquad \|(\xi - 2i)^\tau \mathcal{H}(g)\|_\infty \le C_1 \ln\frac{1}{k}\|(\xi - 2i)^\tau g\|_\infty + C_2 k \|(\xi - 2i)^{\tau+1} g'\|_\infty\,,$$

where C_1 *and* C_2 *are independent of* k *and* $\mathcal{H}$ *is the Hilbert transform operator defined as*

$$\text{(A.2)} \qquad \mathcal{H}(g)[\xi] \equiv \frac{1}{\pi}(P)\int_{-\infty}^{\infty} \frac{g(\xi+\xi')}{\xi'}\,d\xi'\,.$$

PROOF: We first take $\xi \ge 1$. Denote $k' = 2 - k$; clearly $\frac{3}{2} \le k' < 2$. We break up the integral in (A.2) into four parts:

$$\text{(A.3)} \qquad \int_{-\infty}^{\infty} = \int_{-k\xi}^{k\xi} + \int_{-k'\xi}^{-k\xi} + \int_{-\infty}^{-k'\xi} + \int_{k\xi}^{\infty} \frac{1}{\pi}\frac{g(\xi+\xi')}{\xi'}\,d\xi'\,.$$

Consider the first term

$$\begin{aligned} &\left|\frac{1}{\pi}(P)\int_{-k\xi}^{k\xi}\frac{g(\xi+\xi') - g(\xi)}{\xi'}\,d\xi'\right| \\ &\quad\le \left|\frac{1}{\pi}\int_{-k\xi}^{k\xi} g'(\bar{\xi}+\xi)d\xi'\right| \\ &\quad\le \frac{1}{\pi}\|(\xi - 2i)^{1+\tau} g'\|_\infty \int_{-k\xi}^{k\xi} |\bar{\xi}+\xi-2i|^{-\tau-1}\,d\xi'\,, \end{aligned}$$

where $\bar{\xi} \in (-k\xi, k\xi)$. But

$$\int_{-k\xi}^{k\xi} |\bar{\xi}+\xi-2i|^{-\tau-1}\,d\xi' \le |\xi(1-k) - 2i|^{-1-\tau} 2k\xi \le C_2 k|\xi - 2i|^{-\tau}\,,$$

where C_2 can be made independent of $k \in (0, \frac{1}{2}]$. Hence

$$\left| \frac{1}{\pi} \int_{-k\xi}^{k\xi} \frac{g(\xi+\xi') - g(\xi)}{\xi'} \, d\xi' \right| \le k C_2 |\xi - 2i|^{-\tau} \| (\xi - 2i)^{1+\tau} g' \|_\infty .$$

Now consider the second term. Make the change of variable $\xi' + \xi = \xi''$ and let $L = (1-k)\xi$; we get

$$\frac{1}{\pi} \int_{-L}^{L} \frac{g(\xi'')}{(\xi'' - \xi)} \, d\xi'' .$$

We write this integral as

$$\frac{1}{\pi} \int_{-L}^{L} \frac{g(\xi'')}{(\xi'' - \xi)} \, d\xi'' = \frac{1}{\pi} \int_{-L}^{L} \left[\frac{g(\xi'')}{(\xi'' - \xi)} + \frac{g(\xi'')}{\xi} \right] d\xi'' - \frac{1}{\pi \xi} \int_{-L}^{L} g(\xi'') d\xi'' \tag{A.4}$$

and estimate each term separately. The second term on the right-hand side above can be estimated as

$$\left| \frac{1}{\pi \xi} \int_{L}^{L} |g(\xi')(\xi' - 2i)^\tau| |\xi' - 2i|^{-\tau} \, d\xi' \right| \le C \frac{L^{1-\tau}}{\xi} \| (\xi - 2i)^\tau g \|_\infty \le C \xi^{-\tau} \| (\xi - 2i) g \|_\infty , \tag{A.5}$$

where C can be made independent of k. Now consider the first term in (A.4):

$$\begin{aligned} &\left| \frac{1}{\pi} \int_{-L}^{L} \left[\frac{g(\xi'')}{\xi'' - \xi} + \frac{g(\xi'')}{\xi} \right] d\xi'' \right| \\ &\quad = \left| \frac{1}{\pi} \int_{-L}^{L} \frac{g(\xi'')\xi''}{(\xi - \xi'')\xi} \, d\xi'' \right| \\ &\quad \le C \| (\xi - 2i)^\tau g \|_\infty \int_{-L}^{L} \frac{|\xi''|^{1-\tau}}{(\xi - \xi'')\xi} \, d\xi'' \\ &\quad \le C \| (\xi - 2i)^\tau g \|_\infty \xi^{-\tau} \left[\int_{(1-k)}^{1-k} \frac{|\hat{\xi}|^{1-\tau}}{(1 - \hat{\xi})} \, d\hat{\xi} \right] \\ &\quad \le C_1 \ln \frac{1}{k} |\xi - 2i|^{-\tau} \| (\xi - 2i)^\tau g \|_\infty , \end{aligned} \tag{A.6}$$

where C_1 is independent of k, and the $\ln \frac{1}{k}$ term accounts for the behavior of the estimate on the right-hand side as $k \to 0^+$.

We now estimate the third term in (A.3):

$$\left|\frac{1}{\pi}\int_{-\infty}^{-k'\xi}\frac{g(\xi'+\xi)}{\xi'}\,d\xi'\right| = \left|\frac{1}{\pi}\int_{k'\xi}^{\infty}\frac{g(-\xi'+\xi)}{\xi'}\,d\xi'\right|$$
$$\le \|(\xi-2i)^{\tau}g\|_{\infty}\frac{1}{\pi}\int_{k'\xi}^{\infty}\frac{(\xi'-\xi)^{-\tau}}{\xi'}\,d\xi'$$
$$\le C\|(\xi-2i)^{\tau}g\|_{\infty}\xi^{-\tau}\,,$$

where C above can be chosen independently of k.

Now consider the fourth term in (A.3).

$$\left|\frac{1}{\pi}\int_{k\xi}^{\infty}\frac{g(\xi'+\xi)}{\xi'}\,d\xi'\right| \le \|(\xi-2i)^{\tau}g\|_{\infty}\frac{1}{\pi}\int_{k\xi}^{\infty}\frac{(\xi'+\xi)^{-\tau}}{\xi'}\,d\xi'$$
$$\le \|(\xi-2i)^{\tau}g\|_{\infty}\xi^{-\tau}\int_{k}^{\infty}\frac{(1+\hat{\xi})^{-\tau}}{\hat{\xi}}\,d\hat{\xi}$$
$$\le C_1\ln\frac{1}{k}\|\xi^{\tau}g\|_{\infty}\xi^{-\tau}\,,$$

where C_1 is chosen independently of k and $\ln\frac{1}{k}$ accounts for the asymptotic behavior of the integral on the right-hand side as $k\to 0^+$. Combining all the terms above, we obtain the proof of the lemma for $\xi > 1$. Now for $0\le\xi\le 1$, we split the integral in (A.2) into

$$\frac{1}{\pi}\int_{-k}^{k}\frac{g(\xi'+\xi)-g(\xi)}{\xi'}\,d\xi' + \frac{1}{\pi}\int_{k}^{\infty}\frac{g(\xi'+\xi)}{\xi'}\,d\xi' + \frac{1}{\pi}\int_{-\infty}^{-k}\frac{g(\xi'+\xi)}{\xi'}\,d\xi'\,.$$

The first term yields

$$\left|\frac{1}{\pi}\int_{-k}^{k}\frac{g(\xi'+\xi)-g(\xi)}{\xi'}\,d\xi'\right| \le \frac{1}{\pi}\int_{-k}^{k}|g'(\hat{\xi})d\xi'| \le C_2 k\|(\xi-2i)^{1+\tau}g'\|_{\infty}\,,$$

where C_2 is independent of ϵ.

For the second term, we have

$$\frac{1}{\pi}\int_{k}^{\infty}\frac{g(\xi'+\xi)}{\xi'}\,d\xi' \le C\|(\xi-2i)^{\tau}g\|_{\infty}\int_{k}^{\infty}\frac{|\xi'+\xi-2i|^{-\tau}}{\xi'}\,d\xi'$$
$$\le C\|(\xi-2i)^{\tau}g\|_{\infty}\int_{k}^{\infty}\frac{(\xi'^2+4)^{-\tau/2}}{\xi'}\,d\xi'$$
$$\le C_1\ln\frac{1}{k}\|(\xi-2i)^{\tau}g\|_{\infty}\,,$$

where C_1 is independent of k and $\ln\frac{1}{k}$ accounts for the asymptotic behavior of the right-hand side estimate as $k\to 0^+$.

For the third term,

$$\begin{aligned}\left|\frac{1}{\pi}\int_{-\infty}^{-k}\frac{g(\xi'+\xi)}{\xi'}\,d\xi'\right| &\le \left|\frac{1}{\pi}\int_{k}^{\infty}\frac{g(-\xi'+\xi)}{\xi'}\,d\xi'\right| \\ &\le \|(\xi-2i)^{\tau}g\|_{\infty}\frac{1}{\pi}\int_{k}^{\infty}\frac{|\xi'-\xi-2i|^{-\tau}}{\xi'}\,d\xi' \\ &\le C_1\ln\frac{1}{k}\|(\xi-2i)^{\tau}g\|_{\infty}\,,\end{aligned}$$

where C_1 is made independent of k by using the asymptotic behavior of the right-hand side estimate as $k \to 0^+$.

For $\xi < 0$, we note that

$$\mathcal{H}(g)[\xi] = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\xi')\frac{d\xi'}{\xi'-\xi} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(-\xi')d\xi'}{\xi'-(-\xi)}\,,$$

which is the negative of the Hilbert transform of the function $g(-\xi)$ evaluated at the point $-\xi > 0$. Since $g(-\xi)$ satisfies the same conditions as those given for $g(\xi)$ in this lemma, it follows that all bounds also hold for $\xi < 0$. $\square$

LEMMA A.2 *Let* $g \in \mathcal{C}^2(-\infty,\infty)$ *such that* $\|(\xi-2i)^{\tau}g\|_{\infty}$ *and* $\|(\xi-2i)^{\tau+2}g''\|_{\infty}$ *exist for some* $\tau \in (0,1)$. *Then*

$$\left\|(\xi-2i)^{\tau+1}\mathcal{H}(g')\right\|_{\infty} \le C_2\left\|(\xi-2i)^{\tau+2}g''\right\|_{\infty} + C_0\left\|(\xi-2i)^{\tau}g\right\|_{\infty}. \tag{A.7}$$

PROOF: First, we consider the case $\xi > 1$. Then we decompose

$$\begin{aligned}&\mathcal{H}(g')[\xi] \\ &\quad = \frac{1}{\pi}\int_{-\xi/2}^{\xi/2}\left[\frac{g'(\xi+\xi')-g'(\xi)}{\xi'}\right]d\xi' - \frac{2}{\pi\xi}\left(g\left(\frac{3}{2}\xi\right)+g\left(\frac{\xi}{2}\right)\right) \\ &\qquad + \frac{1}{\pi}\left(\int_{\xi/2}^{\infty}+\int_{-\infty}^{-\frac{3}{2}\xi}\right)\frac{g(\xi+\xi')}{\xi'^2}\,d\xi' + \frac{1}{\pi}\int_{-\frac{3}{2}\xi}^{-\frac{\xi}{2}}\frac{g(\xi+\xi')}{\xi'^2}\,d\xi'\,.\end{aligned} \tag{A.8}$$

Using arguments similar to Lemma A.1 for $k = \frac{1}{2}$, it is clear that the first term on the right of (A.8) is bounded by

$$\left|\frac{1}{\pi}\int_{-\xi/2}^{\xi/2}\left[\frac{g'(\xi+\xi')-g'(\xi)}{\xi'}\right]d\xi'\right| \le C_1|\xi-2i|^{-\tau-1}\|(\xi-2i)^{\tau+2}g''\|_{\infty}\,.$$

The second term of (A.8) is easily seen to be bounded by

$$\left|\frac{2}{\pi\xi}\left(g\left(\frac{3}{2}\xi\right)+g\left(\frac{\xi}{2}\right)\right)\right| \le C|\xi-2i|^{-\tau-1}\|(\xi-2i)^{\tau}g\|_{\infty}\,.$$

Using arguments similar to Lemma A.1, with $k = \frac{1}{2}$, the third term in (A.8) is also bounded:

$$\left| \frac{1}{\pi} \left(\int_{\xi/2}^{\infty} + \int_{-\infty}^{-\frac{3}{2}\xi} \right) \frac{g(\xi + \xi')}{\xi'^2}\, d\xi' \right| \leq$$
$$C\xi^{-1-\tau} \|(\xi - 2i)^\tau g\|_\infty \left[\int_{1/2}^{\infty} \frac{(1+\hat{\xi})^{-\tau}}{\hat{\xi}^2}\, d\hat{\xi} + \int_{3/2}^{\infty} \frac{(\hat{\xi} - 1)^{-\tau}}{\hat{\xi}^2}\, d\hat{\xi} \right].$$

Now, with the change of variable $\xi' + \xi = \xi''$, the last term on the right of (A.8) can be bounded by

$$\left| \int_{-\xi/2}^{\xi/2} \frac{g(\xi'')}{(\xi'' - \xi)^2}\, d\xi'' \right| = C\xi^{-1-\tau} \|(\xi - 2i)^\tau g\|_\infty \int_{-1/2}^{1/2} \frac{|\hat{\xi}|^{-\tau}}{(1-\hat{\xi})^2}\, d\hat{\xi}\,. \tag{A.9}$$

Therefore, combining the bounds on each term, we get

$$\left| \mathcal{H}(g')[\xi] \right| \leq |\xi - 2i|^{-1-\tau} \left(C_2 \|(\xi - 2i)^{\tau+2} g''\|_\infty + C_0 \|(\xi - 2i)^\tau g\|_\infty \right). \tag{A.10}$$

We now consider $0 \leq \xi \leq 1$. In this case, it is convenient to write

$$\begin{aligned} \pi\mathcal{H}(g')[\xi] = &\int_{-1}^{1} \frac{g'(\xi' + \xi) - g'(\xi)}{\xi'}\, d\xi' - [g(\xi+1) + g(\xi-1)] \\ &+ \left(\int_1^\infty + \int_{-\infty}^{-1} \right) \frac{g(\xi + \xi')}{\xi'^2}\, d\xi'\,. \end{aligned} \tag{A.11}$$

Consider the first term in (A.11):

$$\left| \int_{-1}^{1} \frac{g'(\xi' + \xi) - g'(\xi)}{\xi'}\, d\xi' \right| \leq C_1 \left\| (\xi - 2i)^{2+\tau} g'' \right\|_\infty. \tag{A.12}$$

For the second term,

$$\left| g(\xi + 1) + g(\xi - 1) \right| \leq C \|(\xi - 2i)^\tau g\|_0\,. \tag{A.13}$$

For the third term in (A.11),

$$\begin{aligned} &\left| \left(\int_1^\infty + \int_{-\infty}^{-1} \right) \frac{g(\xi + \xi')}{\xi'^2} \right| \leq \\ &\qquad C\|(\xi - 2i)^\tau g\|_\infty \left[\int_1^\infty \xi'^{-2-\tau}\, d\xi' + \int_1^\infty \frac{(\xi' - 1)^{-\tau}}{\xi'^2}\, d\xi' \right]. \end{aligned} \tag{A.14}$$

By combining the above inequalities, it follows that (A.10) holds for $0 \leq \xi \leq 1$ as well. Also, it is to be noted that as in Lemma A.1, for $\xi < 0$, $\mathcal{H}(g)[\xi]$ can be related to the Hilbert transform of $g(-\xi)$ evaluated at $-\xi$. Thus, the same inequalities as above hold for $\xi < 0$. Therefore, (A.10) holds for all $\xi \in (-\infty, \infty)$ and the lemma follows. $\square$

LEMMA A.3 *Let $g \in \mathcal{C}^3(-\infty,\infty)$ such that $\|(\xi-2i)^\tau g\|_\infty$, $\|(\xi-2i)^{\tau+1}g'\|_\infty$, and $\|(\xi-2i)^{\tau+3}g'''\|_\infty$ are each bounded for some $\tau \in (0,1)$; then*

$$\left\|(\xi-2i)^{\tau+2}\mathcal{H}(g'')\right\|_\infty \le C_3\left\|(\xi-2i)^{\tau+3}g'''\right\|_\infty + C_1\left\|(\xi-2i)^{\tau+1}g'\right\|_\infty + C_0\|(\xi-2i)^\tau g\|_\infty\,. \tag{A.15}$$

PROOF: For $\xi > 1$, we decompose

$$\begin{aligned}\pi\mathcal{H}(g'')[\xi] &= \\ &\int_{-\xi/2}^{\xi/2}\frac{g''(\xi'+\xi)-g''(\xi)}{\xi'}\,d\xi' + \frac{4}{\xi^2}\left[-g\left(\frac{3}{2}\xi\right)+g\left(\frac{\xi}{2}\right)\right] \\ &\qquad -\frac{2}{\xi}\left[g'\left(\frac{3}{2}\xi\right)+g'\left(\frac{\xi}{2}\right)\right] \\ &+\left(\int_{\xi/2}^{\infty}+\int_{-\infty}^{-\frac{3}{2}\xi}\right)\frac{2g(\xi'+\xi)}{\xi'^3}\,d\xi' + \int_{-\xi/2}^{\xi/2}\frac{2g(\xi'')}{(\xi''-\xi)^3}\,d\xi''\,.\end{aligned} \tag{A.16}$$

For $\xi > 1$, we get from the estimates for each term in the above, using the same procedure as in Lemma A.2,

$$\begin{aligned}\big||\mathcal{H}(g'')[\xi]\big| &\le \\ &|\xi-2i|^{-2-\tau}\big\{C_3\|(\xi-2i)^{3+\tau}g'''\|_\infty + C_1\|(\xi-2i)^{1+\tau}g'\|_\infty \\ &\qquad + C_0\|(\xi-2i)^\tau g\|_\infty\big\}\,.\end{aligned} \tag{A.17}$$

For $0 \le \xi \le 1$, we decompose

$$\begin{aligned}\pi\mathcal{H}(g'')[\xi] &= \\ &\int_{-1}^{1}\frac{g''(\xi'+\xi)-g''(\xi)}{\xi'}\,d\xi' + \left(\int_1^\infty+\int_{-\infty}^{-1}\right)\frac{2g(\xi+\xi')}{\xi'^3}\,d\xi' \\ &+\big[g(\xi-1)-g(\xi+1)-g'(\xi+1)-g'(\xi-1)\big]\,.\end{aligned} \tag{A.18}$$

As before in Lemma A.2, each term can be estimated, and one obtains (A.17) for $0 \le \xi \le 1$ as well. Again, for $\xi < 0$, $\mathcal{H}(g)[\xi]$ can be related to the Hilbert transform of $g(-\xi')$ evaluated at $-\xi$; hence the inequality (A.17) is valid in that case as well. Therefore, the lemma follows. □

LEMMA A.4 *If F satisfies Conditions 1 through 3 and Assumption 1, then*

$$\sup_{\xi\in(-\infty,\infty)}|\xi+2i|^{1+\tau}|F'| < \infty\,.$$

PROOF: Define $g(\xi) = \epsilon^2\,\mathrm{Im}\ln[1+F'/H]$ on the real ξ-axis. From Condition 1,

$$g'(\xi) = -\epsilon^2\,\mathrm{Im}\frac{H'}{H}(\xi) + |F'+H|\,\mathrm{Re}\,F = O(\xi^{-\tau-1}) \quad \text{as } \xi \to \pm\infty\,.$$

Hence, on integration, $g(\xi) = O(\xi^{-\tau})$ as $\xi \to \pm\infty$. We note that $\ln(1 + F'/H)$ is analytic in $\mathbb{C}^+$, and so on the real ξ-axis, $\epsilon^2 \operatorname{Re} \ln(1 + F'/H) = \mathcal{H}(g)[\xi]$. Since the conditions for Lemma A.1 are met by $g(\xi)$, it follows that

$$\mathcal{H}(g)[\xi] = O(\xi^{-\tau}) \quad \text{as } \xi \to \pm\infty\,,$$

and therefore $\epsilon^2 \ln(1 + F'/H) = O(\xi^{-\tau})$ as $\xi \to \pm\infty$, which implies $F' = O(\xi^{-1-\tau})$. The lemma follows since F is continuously differentiable in $(-\infty, \infty)$. □

LEMMA A.5 *If f is analytic in the upper half-plane $\mathbb{C}^+$ and continuous on $\bar{\mathbb{C}}^+$, the closure of $\mathbb{C}^+$, and $\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau_1} |f(\xi)| = \delta < \infty$ for some $\tau_1 > 0$, then*

$$\sup_{\xi\in\mathbb{C}^+} |\xi + 2i|^{\tau_1} |f(\xi)| = \delta\,. \tag{A.19}$$

On the other hand, if f is analytic in the lower half-plane $\mathbb{C}^-$ and continuous on $\bar{\mathbb{C}}^-$ with $\sup_{\xi\in(-\infty,\infty)} |\xi - 2i|^{\tau_1} |f(\xi)| = \delta < \infty$, then

$$\sup_{\xi\in\mathbb{C}^-} |\xi - 2i|^{\tau_1} |f(\xi)| \le \delta\,. \tag{A.20}$$

PROOF: Since f is analytic in the upper half-plane, $\sup_{\xi\in\bar{\mathbb{C}}^+} |f(\xi)| \le M_0$. Let us define integer $n = \operatorname{Int}[\tau_1/2] + 2$. Choosing $h_{\epsilon_1}(\xi) = 1/(1 - i\xi\epsilon_1)^{2n}$, we note that

$$|h_{\epsilon_1}(\xi)| = \frac{1}{(1 + \epsilon_1 \operatorname{Im}\xi)^2 + \epsilon_1^2 (\operatorname{Re}\xi)^2)^n} \le 1\,.$$

Consider $g(\xi) = f(\xi)(\xi + 2i)^{\tau_1} h_{\epsilon_1}(\xi)$ and domain $\mathbf{D} := \{\operatorname{Im}\xi \ge 0, |\xi| \le 2^{\tau_1/2} M_0/\epsilon_1^2 \delta\}$. We will assume that ϵ_1 is small enough so that $2^{\tau_1/2} M_0/\epsilon_1^2 \delta > 1$. On the circular part of $\partial\mathbf{D}$,

$$|g(\xi)| \le \frac{M_0[(\operatorname{Re}\xi)^2 + (\operatorname{Im}\xi + 2)^2]^{\tau_1/2}}{(1 + \epsilon_1 \operatorname{Im}\xi)^2 + (\operatorname{Re}\xi)^2 \epsilon_1^2} \le \frac{2^{\tau_1/2} M_0[(\operatorname{Re}\xi)^2 + (\operatorname{Im}\xi)^2]^{\tau_1/2}}{\epsilon_1^2((\operatorname{Re}\xi)^2 + (\operatorname{Im}\xi)^2)^n} \le \delta\,.$$

On the straight part of $\partial\mathbf{D}$, $|g| \le \delta$. So $|g| \le \delta$ inside $\mathbf{D}$, from the maximum principle. Also, outside $\mathbf{D}$, but for $\operatorname{Im}\xi \ge 0$, it is clear that $|g| \le \delta$. So for $\operatorname{Im}\xi \ge 0$, we have $|g| \le \delta$. So for any fixed ξ, as $\epsilon_1 \to 0$, $g(\xi) \to f(\xi)(\xi + 2i)^{\tau_1}$. So $|f(\xi)||\xi + 2i|^{\tau_1} \le \delta$ for all $\xi \in \bar{\mathbb{C}}^+$. The proof of the second part is very similar. □

PROOF OF LEMMA 1.5: (1.7) and (1.8) follow from Lemma A.5 on using Lemma A.4. Since $g(\xi) = \epsilon^2 \operatorname{Im} \ln(1 + F'/H)$ satisfies

$$g' = -\epsilon^2 \operatorname{Im} \frac{H'}{H} + |F' + H| \operatorname{Re} F\,,$$

it is clear that $g' = O(\xi^{-1-\tau})$ as $\xi \to \pm\infty$, and

$$g'' = -\epsilon^2 \operatorname{Im} \left(\frac{H'}{H}\right)' + |F' + H| \operatorname{Re} F' + |F' + H| \operatorname{Re}\left[\frac{F'' + H'}{F' + H}\right] \operatorname{Re} F\,.$$

Since $\mathcal{H}(g')[\xi]$ is a priori $O(\xi^{-1})$ as $\xi \to \pm\infty$, it follows that

$$\operatorname{Re}\left[\epsilon^2 \frac{d}{d\xi} \ln(F' + H) - \epsilon^2 \frac{H'}{H}\right] = \mathcal{H}(g')[\xi] = O(\xi^{-1}) \quad \text{at best.}$$

Therefore,

$$\operatorname{Re} \frac{F'' + H'}{F' + H} = O(\xi^{-1}) \quad \text{for large } |\xi|.$$

Also, using large-$|\xi|$ behavior, $\operatorname{Im}(H'/H)' = O(\xi^{-3})$, $|H + F'| = O(\xi^{-1})$, and using $\operatorname{Re} F = O(\xi^{-\tau})$ and (1.8) to obtain $\operatorname{Re} F' = O(\xi^{-1-\tau})$, it follows that $g'' = O(\xi^{-2-\tau})$. From using Lemma A.2, it follows that $\mathcal{H}(g')[\xi] = O(\xi^{-1-\tau})$. So,

$$\frac{F'' + H'}{F' + H} - \frac{H'}{H} = g' + i\mathcal{H}(g)[\xi] = O(\xi^{-1-\tau})\,.$$

Therefore, $F'' = O(\xi^{-2-\tau})$ as $\xi \to \pm\infty$ and hence

$$\sup_{\xi \in (-\infty,\infty)} |\xi - 2i|^{\tau+2} |F''(\xi)| \equiv \delta_2 < \infty\,.$$

Using Lemma A.5, with f replaced by F'', the proof of Lemma 1.5 is complete. □

Appendix B: Proofs of the Properties of Function $P(\xi)$

In this section we discuss properties of the following function:

$$P(\xi) = \int_{-i\gamma}^{\xi} iL^{1/2}(t)dt = i \int_{-i\gamma}^{\xi} \frac{(\gamma - it)^{3/4}(\gamma + it)^{\frac{1}{4}}}{(1 + t^2)}\, dt\,. \tag{B.1}$$

We choose the branch cut $\{\xi : \xi = \rho i, \rho > \gamma\}$, $-\pi \le \arg(\gamma + i\xi) \le \pi$, for the function $(\gamma + i\xi)^{\frac{1}{4}}$, and the branch cut $\{\xi : \xi = -\rho i, \rho > \gamma\}$, $-\pi \le \arg(\gamma - i\xi) \le \pi$, for the function $(\gamma - i\xi)^{3/4}$.

Proof of Property 1: First consider $\xi \in (-\infty, 0)$,

$$\operatorname{Re} P(\xi) = \int_{\xi}^{0} \frac{(\gamma^2 + t^2)^{1/2}}{(1 + t^2)} \sin\left\{\frac{1}{2}\arg(\gamma - it)\right\} dt + \operatorname{Re} P(0)\,.$$

Clearly, $\operatorname{Re} P(-\infty) = \infty$ since $\arg(\gamma - it) \to \frac{\pi}{2}$ as $t \to -\infty$, and $\operatorname{Re} P(\xi)$ decreases as ξ increases since $\arg(\gamma - it) \in (0, \frac{\pi}{2})$.

For $-b < \rho < 0$,

$$P(\rho i) = -\int_0^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{\frac{1}{4}}}{(1 - t^2)}\, dt + P(0)\,,$$

so

$$\operatorname{Re} P(\rho i) = -\int_0^{\rho} \frac{(\gamma + t)^{3/4}(\gamma - t)^{\frac{1}{4}}}{(1 - t^2)}\, dt + \operatorname{Re} P(0)\,.$$

On inspection, as ρ increases in the interval $(-b, 0)$, $\operatorname{Re} P(i\rho)$ decreases. □

PROOF OF PROPERTY 2:

$$P'(t) = \frac{e^{\frac{\pi}{4}i}(t+i\gamma)^{3/4}(t-i\gamma)^{\frac{1}{4}}}{(t+i)(t-i)}\,.$$

It is to be noted that $|t-2i||P'(t)|$ has nonzero upper and lower bounds in the domain $\mathcal{R}$. Furthermore, on a ray $t(s) = \xi - se^{i\varphi}$, $0 \le s < \infty$, where $0 \le \varphi < \frac{\pi}{2}$, as $s \to \infty$, it is clear from the behavior of $P'(t)$ for large t that since $\arg P'(t(s)) \sim -5\pi/4 - \varphi$,

$$\begin{aligned}\frac{d}{ds}\operatorname{Re} P(t(s)) &= \operatorname{Re}[P'(t(s))e^{i\varphi+i\pi}]\\ &= |P'(t(s))|\cos[\arg P'(t(s)+\pi+\varphi)] > \frac{C}{|t(s)-2i|}\end{aligned}$$

satisfies Property 2. □

PROOF OF PROPERTY 3: $P' \sim i\gamma$ as $\xi \to 0$, so $P(\xi) = P(0) + i\gamma\xi + O(\nu^2)$. Therefore, on $\xi = -\nu + se^{-i\pi/4}$, $0 \le s \le \sqrt{2}\nu$,

$$P(\xi) \sim P(0) + i\gamma(-\nu + se^{-i\pi/4}) + O(\nu^2) \sim P(0) - i\gamma\nu + \gamma se^{i\pi/4} + O(\nu^2)\,,$$

$$\frac{d}{ds}\operatorname{Re} P(\xi(s)) \sim \gamma\cos\pi/4 + O(\nu) > C > 0\,,$$

with C independent of ν and ϵ for sufficiently small ν. □

PROOF OF PROPERTY 4: *Step* 1. For $0 < \gamma < 1$

$$P'(\xi) = i(\gamma + i\xi)^{\frac{1}{4}}(\gamma - i\xi)^{3/4}\frac{1}{(\xi^2+1)}\,.$$

On $l^- = \{\xi : \xi = -ib - e^{i\pi/4}s\}$, it suffices to consider $\arg(-e^{i\pi/4}P')$ and ensure it is in $(-\frac{\pi}{2}, \frac{\pi}{2})$, modulo an additive multiple of 2π. This will ensure Property 4, since $|P'||\xi - 2i|$ has a lower bound in the region $\mathcal{D}$.

Consider

$$\xi^2 + 1 = 1 - b^2 + is^2 + 2e^{3i\pi/4}bs = \left(1 - b^2 - \sqrt{2}\,bs\right) + i\left(\sqrt{2}\,bs + s^2\right),$$

$$\arg(\xi^2+1) = \pi - \arctan\left(\frac{\sqrt{2}\,bs + s^2}{\sqrt{2}\,bs - (1-b^2)}\right).$$

Put $s = \sqrt{2}\,b\rho$ to get

$$\arg(\xi^2+1) = \pi - \arctan\left[\frac{2b^2(\rho^2+\rho)}{2b^2\left(\rho - \frac{1-b^2}{2b^2}\right)}\right] = \pi - \arctan\left[\frac{\rho^2+\rho}{\rho - q}\right]$$

where $q = (1-b^2)/2b^2$. In the range $\rho > q$, the minimum of the function $(\rho^2+\rho)/(\rho-q)$ is $(\sqrt{1+q} + \sqrt{q})^2$. Since

$$q = \frac{1-b^2}{2b^2} \ge \frac{1-\gamma^2}{2\gamma^2} := q_{\min}\,,$$

define

$$\theta_{\min} = \tan^{-1}\left(\sqrt{1+q_{\min}} + \sqrt{q_{\min}}\right)^2 > \frac{\pi}{4}\,;$$

then

$$-\pi + \theta_{\min} \le -\arg(\xi^2+1) \le 0\,. \tag{B.2}$$

Since $\arg(\xi - i\gamma) \in (-\frac{3}{4}\pi, -\frac{\pi}{2}]$,

$$\arg i(\gamma + i\xi)^{\frac{1}{4}} = \arg\left[e^{i\pi/2}e^{i\pi/8}(\xi - i\gamma)^{\frac{1}{4}}\right] \in \left(\frac{7\pi}{16}, \frac{\pi}{2}\right]. \tag{B.3}$$

Let $\epsilon_2 = \frac{1}{2}(\theta_{\min} - \frac{\pi}{4})$. Near $\xi = -i\gamma$,

$$P'(\xi) = \frac{i(2\gamma)^{\frac{1}{4}}}{(1-\gamma^2)}(\gamma - i\xi)^{3/4}\{1 + O(\gamma - i\xi)\}\,. \tag{B.4}$$

Clearly, there exists R_0 large enough (depending on b) so that for $\xi \in l^-$,

$$\frac{3}{4}\arg(\gamma - i\xi) \in \left[\frac{9}{16}\pi - \epsilon_2, \frac{9}{16}\pi\right) \quad \text{for } |\xi + i\gamma| \ge R_0 \tag{B.5}$$

and

$$\frac{3}{4}\arg(\gamma - i\xi) \in \left[0, \frac{9}{16}\pi - \epsilon_2\right) \qquad \text{for } |\xi + i\gamma| < R_0\,.$$

From a geometric consideration, it is clear that $R_0(b) \to 0$ as $b \to \gamma^-$. We choose b close enough to γ so that the approximation in (B.4) is good enough to ensure that on l^-,

$$\arg(P'(\xi)e^{-3\pi i/4}) \in \left(-\frac{5\pi}{16}, \frac{7\pi}{16}\right) \quad \text{for } |\xi + i\gamma| < R_0(b)\,.$$

On the other hand, on l^- for $|\xi + i\gamma| \ge R_0(b)$, by using (B.5), along with (B.2) and (B.3), it follows that

$$\arg\left(e^{i\frac{-3\pi}{4}}P'(\xi)\right) \in \left(-\frac{3\pi}{4} + \frac{\pi}{4} + \left(\theta_{\min} - \frac{\pi}{4}\right) - \epsilon_2, \frac{5\pi}{16}\right) \subset \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \tag{B.6}$$

Step 2. Now consider $\gamma > 1$ (i.e., $\lambda > \frac{1}{2}$) but make $\gamma - 1$ small enough so that we can choose b so that $10(\gamma - 1) \le |b-1|$. (See [34] for an alternate proof without this restriction). We want to show that on ray $l^- \equiv \{\xi = -bi - se^{i\pi/3}, 0 \le s < \infty\}$,

$$\frac{d}{ds}\operatorname{Re} P(\xi(s)) = \operatorname{Re}\left\{P'(\xi)e^{-i2\pi/3}\right\} > \frac{C}{|\xi(s) - 2i|} > 0\,. \tag{B.7}$$

We note that since $|\xi - 2i||P'|$ is bounded above and below by nonzero constants, it suffices to show that

$$\arg\left(P'(\xi(s))e^{-i2\pi/3}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \mod 2\pi\,.$$

Note that

$$P'(\xi) = i\frac{(\xi - i\gamma)}{(\xi + i)}\left(\frac{\gamma + i\xi}{\gamma - i\xi}\right)^{\frac{1}{4}}\frac{1}{(1+i\xi)}. \tag{B.8}$$

Let $B(s)$ be the positive angle between $\xi(s) + \gamma i$ and $\xi(s) + i$; then by geometry

$$\arg\left(e^{-i2\pi/3}P'(\xi)\right) \in \left(-B - \frac{5\pi}{12}, 0\right),$$

and we can see that $B < \frac{\pi}{12}$ implies (B.7).

Let $d_1 = |b-1| + |\gamma - 1|$; by geometry

$$\cos B = \frac{\left(s^2 - 2s|b-1|\sin\frac{\pi}{3} + |b-1|^2\right) + \left(s^2 - 2s\,d_1\sin\frac{\pi}{3} + d_1^2\right) - |\gamma - 1|^2}{2\sqrt{\left(s^2 - 2s|b-1|\sin\frac{\pi}{3} + |b-1|^2\right)}\sqrt{\left(s^2 - 2sd_1\sin\frac{\pi}{3} + d_1^2\right)}}.$$

Let

$$t = \frac{|\gamma - 1|^2 s}{|b-1|\sin\frac{\pi}{3}}, \qquad d = \frac{|\gamma - 1|}{|b-1|},$$

$$\cos B = \frac{\left(t - 1 - \frac{d}{2}\right)^2 + (1+d)\cot^2\frac{\pi}{3} - \frac{1}{4}d^2}{\sqrt{(t-1)^2 + \cot^2\frac{\pi}{3}}\sqrt{(t-1-d)^2 + (1+d)^2\cot^2\frac{\pi}{3}}}.$$

The minimum of the above function over $0 < t < \infty$, $d \le 0.1$, is 0.9688749307, but $\cos\frac{\pi}{12} = 0.9330127$, so $B < \frac{\pi}{12}$.

$\square$

PROOF OF PROPERTIES 5 AND 6: Recall that in our proof of Property 4, we showed that there exist ϕ_0 and b with $\frac{\pi}{4} \le \phi_0 < \frac{\pi}{2}$, $0 < b < \min\{1, \gamma\}$, so that on $\xi = -ib - e^{i\phi_0}s$

$$\arg[P'e^{-i(\pi - \phi_0)}] \in (-\theta_1, \theta_2) \subset \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right) \quad \text{where } 0 < \theta_1, \theta_2 < \tfrac{\pi}{2};$$

without loss of generality, we assume that $\pi/4 \le \theta_1, \theta_2 < \pi/2$. Then it is clear that on $\xi = -ib - e^{i\phi_0}s$,

$$\arg P' \in (\pi - \phi_0 - \theta_1, (\pi - \phi_0) + \theta_2).$$

Note that $[\frac{\pi}{2}, \frac{3\pi}{4}) \subset (\pi - \phi_0 - \theta_1, (\pi - \phi_0) + \theta_2)$. On the real axis, $\arg P' = \arg i + \frac{1}{2}\arg(\gamma - i\xi) \in \frac{\pi}{2} + (0, \frac{\pi}{4}) = (\frac{\pi}{2}, \frac{3\pi}{4})$. On the imaginary axis between O and ib, $\arg P' = \frac{\pi}{2}$. In all cases, on the boundary of the domain $\mathcal{R}^-$, bounded by the negative real axis, the imaginary axis between 0 and ib, and the line $\xi = -bi - e^{i\phi_0}s$, we have $\arg P' \in (\pi - \phi_0 - \theta_1, \pi - \phi_0 + \theta_2)$. On $\xi = -bi - se^{i\phi}$ for $\phi < \phi_0$, as $s \to \infty$, we have

$$\arg(\xi + i\gamma) \to (\pi + \phi), \quad \arg(\xi - i\gamma) \to (-\pi + \phi),$$
$$\arg(\xi + i) \to (\pi + \phi), \quad \arg(\xi - i) \to (-\pi + \phi).$$

So, as $s \to \infty$, $\arg P' \to \frac{3\pi}{4} - \phi \in (\frac{3\pi}{4} - \phi_0, \frac{3\pi}{4})$. So as $\xi \to \infty$ and $\xi \in \mathcal{R}^-$, $\arg P' \in (\frac{3\pi}{4} - \phi_0, \frac{3\pi}{4}) \subset (\frac{\pi}{2}, \frac{3\pi}{4})$. Using the maximum principle, $\arg P' \in (\pi - \phi_0 - \theta_1, \pi - \phi_0 + \theta_2)$ everywhere inside the domain $\mathcal{R}^-$.

Now if we choose $\mathcal{P}(\xi, -\infty) = \{t : t = \xi - e^{i\phi_0}s, 0 < s < \infty\}$, it is clear on $\mathcal{P}$ we have

$$\frac{d}{ds}(\mathrm{Re}\, P) = |P'| \cos\left[\arg(P' e^{-i(\pi-\phi_0)})\right] > \frac{C}{|\xi - 2i|} > 0\,,$$

where C can be made independent of γ for γ in a compact subset of $(0, \infty)$. Hence Property 6 follows.

Now to find $\mathcal{P}(\xi, -\nu)$ so that $\mathrm{Re}\, P$ decreases monotonically from ξ to $-\nu$, we use line $\mathcal{P}_0 = \{t = \xi + e^{i\phi_0}s, s > 0\}$ where

$$\frac{d}{ds}\,\mathrm{Re}\, P = |P'| \cos[\arg P' + \phi_0] \le -\frac{C}{|\xi - 2i|} < 0\,.$$

This line intersects $\partial\mathcal{D}$ at some point $\xi_1 \in \partial\mathcal{D}$. Now clearly ξ_1 can be connected to $\xi = -\nu$ by $\mathcal{P}_1(\xi_1, -\nu)$ on a path coinciding with $\partial\mathcal{D}$ so that $\mathrm{Re}\, P$ decreases monotonically from ξ_1 to $-\nu$ such that

$$-\frac{d}{ds}(\mathrm{Re}\, P) > \frac{C}{|\xi - 2i|} > 0\,.$$

Then $\mathcal{P}(\xi, -\nu) = \mathcal{P}_0(\xi, \xi_1) + \mathcal{P}_1(\xi_1, -\nu)$. Reversing this path leads to the desired path $\mathcal{P}(-\nu, \xi)$ having Property 5. $\square$

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