The effect of finiteness in the Saffman-Taylor viscous fingering problem

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Abstract

We derive a family of exact time-evolving solutions for the the evolution of a finite blob of fluid confined to a channel in a Hele-Shaw cell. We show rigorously that, for large fluid volume, there are solutions for which one of the interfaces approaches the steady Saffman-Taylor finger solution of arbitrary width $\lambda \in (0, 1)$. On the basis of this, we argue that the far-field effects of a displaced second interface do not provide a selection mechanism for the formation of a width-$\frac{1}{2}$ finger when surface tension, or any other regularization, is ignored.

1 Introduction

Viscous fingering in a Hele-Shaw cell has been the subject of numerous investigations in the literature following the seminal experiment of Saffman & Taylor [1] which showed that a steady finger with width approximately one-half the width of the channel is formed when a less viscous fluid displaces a more viscous one (except in the case of small displacement rate). The reviews by Saffman [2], Bensimon *et al* [3], Homsy [4], Kessler *et al* [5] and Pelce [6] cite much of the existing literature in the mid-eighties. An important aspect of the theoretical development is the notion that it is crucial to include some regularizing effect such as surface tension to explain experimental findings. For a model that ignores three dimensional effects, numerical [7],[8], [9], [10] formal asymptotic calculations [11], [12], [13], [14], [15], [16], [17] as well as recent rigorous mathematical analysis [18], [19] support this contention. Work on more realistic models also suggest the same (see Tanveer [20] for a more comprehensive review of the selection literature).

However, the notion that the zero surface tension solution does not select a finger of a specific width has recently been challenged [21]. Based on some calculations of a special exact solution in the finite-fluid problem for which the relative finger width $\lambda = \frac{1}{2}$, it has been argued that the infinite-fluid limit is singular and that $\lambda = \frac{1}{2}$ can be the only limiting Saffman-Taylor solution as the amount of viscous fluid in the cell tends to infinity. Further, based on a linear stability analysis, it is suggested that if the energy fed into the system is controlled appropriately, this solution would be stable and therefore experimentally relevant. The possibility of finiteness providing a selection mechanism was first suggested by Feigenbaum, Procaccia and Davidovich [22].

In the present paper, we show that there is a class of time-evolving exact zero-surface tension solutions involving a finite region of fluid that behave differently from the solution found by Feigenbaum [21]. This includes solutions that, in the limit of infinite fluid volume, tend to the well-known steady solutions due to Zhuravlev [23] and Saffman & Taylor [1] with any relative finger-width $\lambda \in (0, 1)$. (These steady solutions are popularly known as the Saffman-Taylor finger solutions and will be referred to henceforth in this paper as the ZST solutions). Further, we show that with an appropriate choice of parameters, the initial fluid domains are close to those of Feigenbaum for sufficiently large fluid volume. More specifically, the new class of solutions derived here are given explicitly by a time-evolving conformal mapping of the form (81), where $N \geq 1$ is some positive integer. If the distance between the two interfaces is L, which is assumed finite but large, then the initial shape of the fluid domain described by a properly chosen $N = 1$ solution in (81) is within $O(e^{-\frac{\pi}{2}L})$ of the initial fluid domain of Feigenbaum [21] (see (103) for precise bounds on the difference). We also find that in the special case with $N = 2$, when the fluid volume tends to infinity, the solutions (81) reduce to the family of exact solutions for a single interface found by Howison [30]; these are known to result in the "tip-splitting" of a finger. Thus, since the initial fluid domain with $N = 2$ can be made arbitrarily close to that of an $N = 1$ solution, it follows that a finger-like solution close to the ZST steady shape will be unstable to tip-splitting instabilities.

The mathematical method used here to derive exact solutions is related to methods used in other contexts. Using elliptic function conformal maps from a rectangular pre-image region, Richardson [26] [27] has considered various problems of Hele-Shaw flow in channels involving two free surfaces, but his results do not encompass any finger-like solutions. In this paper, to derive finger-like exact solutions for an evolving finite blob of fluid, we devise a method which is an adaptation of that used previously by Crowdy [25] to study the flow of a fluid annulus in a rotating Hele-Shaw flow. The formulation is based on conformal maps from an annular pre-image region. This is particularly convenient for the specific purposes of this paper because the time-evolving finger-solutions of Saffman [24], relevant in the infinite-fluid case, are retrieved in a simple and natural way as a parameter ρ, arising in the present analysis, tends to zero. This fact will prove crucial in our analysis of the infinite-fluid limit.

2 Mathematical formulation

Let the fluid region $D(t)$ be the finite fluid region in the (x, y)-plane trapped between a channel $-1 \leq y \leq 1$ and having two free interfaces, denoted L (for the left-most interface) and R (for the right-most interface).

The velocity potential ϕ satisfies

$$\nabla^2 \phi = 0, \quad \text{in } D(t), \tag{1}$$

so that the fluid velocity $\mathbf{u} = \nabla\phi$. ϕ is proportional to the fluid pressure which is taken to be constant on L and R. Thus,

$$\phi = \phi_L(t) \quad \text{on } L, \qquad \phi = \phi_R(t) \quad \text{on } R \tag{2}$$

where $\phi_L(t)$ and $\phi_R(t)$ are functions of time (but not space). Without loss of generality, we set $\phi_L = 0$. The kinematic condition on each interface is that the normal velocity of the interface equals the normal fluid velocity. This can be written

$$\mathrm{Im}[z_t \bar{z}_s] = \mathrm{Im}[(u + iv)\bar{z}_s], \quad \text{on} \quad L, R \tag{3}$$

where s is the physical arc-length that increases when the boundary of $D(t)$ is traversed in the positive counter-clockwise sense.
Let the complex potential associated with the flow be $w(z,t)$. Define $W(\zeta,t) \equiv w(z(\zeta,t),t)$, then

$$W(\zeta,t) = -\frac{2V(t)}{\pi}\log\zeta \tag{4}$$

where

$$-\frac{2V(t)}{\pi}\log\rho(t) = \phi_R(t). \tag{5}$$

(5) relates the flow-rate $V(t)$ to the imposed pressure $\phi_R(t)$ on the right-most interface. As pointed out by Feigenbaum [21], two physical problems can be considered: either $\phi_R(t)$ is held fixed in time so that the pressure difference across the fluid region is some specified constant, or the flow-rate $V(t)$ is fixed to be some constant. In either case, $\rho(t)$ is a parameter that must be determined as part of the solution.

Introduce a time-dependent conformal map $z(\zeta,t)$ from the upper-half annulus in the ζ-plane given by $\rho(t) < |\zeta| < 1$, $\mathrm{Im}[\zeta] > 0$ to the finite fluid domain $D(t)$. Let the semi-circle $|\zeta| = 1$ in the upper half-plane map to L and the semi-circle $|\zeta| = \rho(t)$ in the upper half-plane map to R. A schematic is shown in Figure 1.

It can be deduced that $z(\zeta,t)$ has the general form

$$z(\zeta,t) = i - \frac{2}{\pi}\log\zeta + f(\zeta,t) \tag{6}$$

where $f(\zeta,t)$ must be analytic in the annulus $\rho < |\zeta| < 1$ and such that $z(\zeta,t)$ is a univalent conformal map from the upper-half annulus $\rho < |\zeta| <$

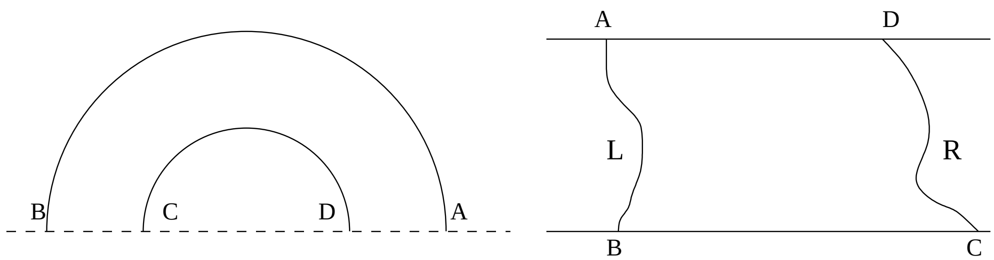

Figure 1: Conformal mapping regions: the map $z(\zeta, t)$ maps the upper-half annulus $\rho < |\zeta| < 1$, $\text{Im}[\zeta] > 0$ in a ζ-plane (left) to the physical fluid domain $D(t)$ (right). The fluid domain has two interfaces labelled L and R. Points labelled with the same letter map to each other under $z(\zeta, t)$.

$1, \text{Im}[\zeta] > 0$ to the physical fluid domain. The real interval $[\rho, 1]$ in the ζ-plane will map to the part of the upper-wall between L and R, while the real interval $[-1, -\rho]$ will map to the part of the lower wall between the two interfaces. Note that this will require that $\text{Im}[f(\zeta, t)] = 0$ for ζ real which implies that

$$\bar{f}(\zeta, t) = f(\zeta, t) \tag{7}$$

and so $\bar{z}_\zeta = z_\zeta$. (We adopt the standard definition of $\bar{g}(\zeta)$ to be that analytic function of ζ which is the complex conjugate of $g(\zeta)$ on the real ζ-axis). By the Schwarz reflection principle, (7) can be used to deduce that $f(\zeta, t)$ is analytic in the entire annulus $\rho < |\zeta| < 1$ and is real on that part of the real axis where $|\zeta| \in (\rho, 1)$. As long as the extended interface, obtained by reflection in the two side-walls, is analytic then it follows that the corresponding conformal map $z(\zeta, t)$ is analytic in $\rho \leq |\zeta| \leq 1$.

On $|\zeta| = 1$,

$$z_s = \frac{i\zeta z_\zeta(\zeta, t)}{|z_\zeta(\zeta, t)|}, \tag{8}$$

while on $|\zeta| = \rho$,

$$z_s = -\frac{i\zeta z_\zeta(\zeta, t)}{\rho |z_\zeta(\zeta, t)|}, \tag{9}$$

where the arclength s increases from A to B on the left interface L and from C to D on the right interface R. It is to be noted that using (8), (9) and the fact that

$$w_z = \frac{W_\zeta}{z_\zeta} = u - iv, \tag{10}$$

it can be deduced that

$$\mathrm{Re}\left[\frac{z_t}{\zeta z_\zeta}\right] = \begin{cases} -\frac{2V(t)}{\pi |z_\zeta|^2}, & \text{on } |\zeta| = 1, \\ -\frac{2V(t)}{\pi \rho^2 |z_\zeta|^2} - \frac{\dot\rho}{\rho}, & \text{on } |\zeta| = \rho. \end{cases} \tag{11}$$

At this point, note that if we replace t in the above equation by a nonlinearly scaled 'time' $\tau = \int_0^t V(t)dt$ and redefine $\dot\rho = \frac{d}{d\tau}\rho$, then the problem becomes mathematically equivalent to choosing $V(t) = 1$. Henceforth, without any loss of generality, we set $V(t) = 1$.

The function in square brackets on the left hand side of (11) is single-valued and analytic everywhere in the upper-half ζ-annulus. It is also real on the real ζ-axis. By the Schwarz reflection principle it must be single-valued and analytic in the entire annulus $\rho < |\zeta| < 1$. This can only be true provided a compatibility condition is satisfied by the data on the right hand side of (11). This condition is that the average of the given data over the circle $|\zeta| = 1$ equals the average of the data over $|\zeta| = \rho$. This takes the form

$$\frac{1}{2\pi i}\oint_{|\zeta|=1} \frac{d\zeta}{\zeta}\left[-\frac{2}{\pi |z_\zeta|^2}\right] = \frac{1}{2\pi i}\oint_{|\zeta|=\rho} \frac{d\zeta}{\zeta}\left[-\frac{2}{\pi\rho^2 |z_\zeta|^2} - \frac{\dot\rho}{\rho}\right]. \tag{12}$$

(12) provides an evolution equation for ρ, i.e.,

$$\dot\rho = \frac{\rho}{\pi^2}\int_0^{2\pi}\left[\frac{1}{|z_\zeta(e^{i\nu}, t)|^2} - \frac{1}{\rho^2 |z_\zeta(\rho e^{i\nu}, t)|^2}\right] d\nu. \tag{13}$$

If (13) is enforced, then

$$z_t(\zeta, t) = \zeta z_\zeta(\zeta, t) I(\zeta, t) \tag{14}$$

where $I(\zeta, t)$ is single-valued and analytic in the annulus $\rho < |\zeta| < 1$. The Villat formula for an annulus can be used to deduce an explicit expression for $I(\zeta, t)$. Using a convenient version of this formula given in Crowdy [25], we obtain

$$I(\zeta, t) = I^+(\zeta, t) - I^-(\zeta, t) + C(t) \tag{15}$$

where

$$I^+(\zeta, t) = \frac{1}{2\pi i}\oint_{|\zeta'|=1} K(\zeta/\zeta', \rho)\left[-\frac{2}{\pi |z_\zeta(\zeta', t)|^2}\right]\frac{d\zeta'}{\zeta'}, \tag{16}$$

$$I^-(\zeta,t) = \frac{1}{2\pi i}\oint_{|\zeta'|=\rho} K(\zeta/\zeta',\rho)\left[-\frac{2}{\pi\rho^2|z_\zeta(\zeta',t)|^2} - \frac{\dot\rho}{\rho}\right]\frac{d\zeta'}{\zeta'}, \quad (17)$$

$$C(t) = -\frac{1}{2\pi i}\oint_{|\zeta'|=\rho}\left[-\frac{2}{\pi\rho^2|z_\zeta(\zeta',t)|^2} - \frac{\dot\rho}{\rho}\right]\frac{d\zeta'}{\zeta'}, \quad (18)$$

and the kernel function $K(\zeta,\rho)$ is given by

$$K(\zeta,\rho) = 1 - 2\zeta\frac{P'(\zeta,\rho)}{P(\zeta,\rho)}. \quad (19)$$

$P(\zeta,\rho)$ is defined by the infinite product expansion

$$P(\zeta,\rho) = (1-\zeta)\prod_{k=1}^{\infty}(1-\rho^{2k}\zeta)(1-\rho^{2k}\zeta^{-1}). \quad (20)$$

$P'(\zeta,\rho)$ denotes the derivative of $P(\zeta,\rho)$ with respect to its first argument. $P(\zeta,\rho)$ will be used again later in the construction of the conformal map solutions.

With use of the infinite product representation, it is clear that

$$\begin{array}{ll} P(\rho^2\zeta,\rho) = -\frac{P(\zeta,\rho)}{\zeta}, & K(\rho^2\zeta,\rho) = K(\zeta,\rho)+2, \\ P(\zeta^{-1},\rho) = -\frac{P(\zeta)}{\zeta}, & K(\zeta^{-1},\rho) = -K(\zeta,\rho). \end{array} \quad (21)$$

These properties will be useful later. In particular from (21), a change of integration variable $\zeta' = e^{i\nu'}$ to $\tilde\zeta = e^{-i\nu'}$ in I^+, a switch from $\zeta' = \rho e^{i\nu'}$ to $\tilde\zeta = \rho e^{-i\nu'}$ in I^- and the symmetry condition (7), we deduce

$$|z_\zeta(e^{-i\nu})|^{-2} = |z_\zeta(e^{i\nu})|^{-2}, |z_\zeta(\rho e^{-i\nu})|^{-2} = |z_\zeta(\rho e^{i\nu})|^{-2}, \quad (22)$$

so that

$$I^+(\zeta^{-1},t) = -I^+(\zeta,t),\;\; I^-(\zeta^{-1},t) = -I^-(\zeta,t) - 2C(t),\;\; I(\zeta^{-1},t) = -I(\zeta,t). \quad (23)$$

In the case where $z(\zeta,t)$ is a conformal mapping function, another property of $I(\zeta,t)$ follows. If $\frac{1}{z_\zeta(\zeta,t)}$ is free of singularities in the ring domain $\mathcal{D}_\delta$ defined as

$$\mathcal{D}_\delta \equiv \{\zeta : (1-\delta)\ \rho \le |\zeta| \le (1+\delta)\}$$

for some $\delta > 0$, then there exists constants C_1 and C_2 independent of any of the parameters so that for $1 \le |\zeta| \le \rho^{-1}$,

$$|\zeta I_\zeta^+| \le \frac{C_1}{\delta^2}\left(\frac{1}{\inf_{|\zeta|=1}|\zeta z_\zeta|^2}\right) + C_2 \sup_{1\le|\zeta|\le 1+\delta}\left|\frac{d}{d\zeta}\left(\frac{1}{z_\zeta(\zeta)z_\zeta(\zeta^{-1})}\right)\right|, \tag{24}$$

$$|\zeta I_\zeta^-| \le \frac{C_1}{\delta^2}\left(\frac{1}{\inf_{|\zeta|=\rho}|\zeta z_\zeta|^2}\right) + \frac{C_2}{\rho^2} \sup_{(1-\delta)\le|\zeta|\le 1}\left|\frac{d}{d\zeta}\left(\frac{1}{z_\zeta(\rho\zeta)z_\zeta(\rho\zeta^{-1})}\right)\right| \tag{25}$$

Consider (24). For $|\zeta| \ge 1 + \frac{\delta}{2}$ the proof follows from differentiating I^+ and using a simple estimate on the integrand. When $1 < |\zeta| \le (1+\frac{\delta}{2})$, we deform the contour to $|\zeta'| = 1+\delta$. In the process, we collect the residue at $\zeta = \zeta'$, noting that the analytic continuation of $|z_\zeta|^2$ off $|\zeta| = 1$ equals $z_\zeta(\zeta)z_\zeta(\zeta^{-1})$. We obtain an estimate for the integral part in the same way as before, the estimate on the residue gives the second term in the inequality above. Similar estimates for I_ζ^- results in (25). In this case, it is convenient to use the alternative representation that follows from the property $K\left(\zeta/(\rho\zeta')\right) = K(\rho\zeta/\zeta') - 2$:

$$I^-(\zeta,t) = -\frac{1}{\pi^2 i\rho^2}\oint_{|\zeta'|=1} K\left(\frac{\rho\zeta}{\zeta'},\rho\right)\frac{1}{z_\zeta(\rho\zeta',t)z_\zeta(\rho/\zeta',t)}\frac{d\zeta'}{\zeta'} + \text{ Constant}$$

which follows from (21) and the fact that the analytic continuation of $|z_\zeta|^2$ off $|\zeta| = \rho$ is $z_\zeta(\zeta)z_\zeta(\rho^2\zeta^{-1})$.

3 Results for the single-interface problem

The Hele-Shaw problem in a channel with just one interface has been well-studied since the work of Saffman and Taylor [1]. In this section, we summarize well-known results for the zero-surface-tension single-interface problem that will be needed in the present paper. Two separate single-interface problems are considered: the case with a single left-most interface L in the limit where the right-most interface R has advanced to $+\infty$, and the case of the single interface R when L has receded to $-\infty$.

3.1 The single left-interface problem

Consider first the case of the evolution of L when R is at $+\infty$. This corresponds to $\rho = 0$ in the equations of §2 which leads to a single equation valid for $|\zeta| < 1$,

$$z_t(\zeta, t) = \zeta z_\zeta(\zeta, t) \mathcal{I}_0[z(.)](\zeta) \tag{26}$$

where the operator $\mathcal{I}_0$ is defined through the expression:

$$\mathcal{I}_0[w(.)](\zeta) \equiv \frac{1}{2\pi i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \frac{\zeta + \zeta'}{\zeta' - \zeta} \left[- \frac{2}{\pi |w_\zeta(\zeta')|^2} \right]. \tag{27}$$

Saffman [24] found a family of time-evolving exact solutions that exist for all times for which the conformal mapping function from the interior of the upper-half semi-circle to the flow region on the right of interface L is given by

$$z(\zeta, t) = i + d - \frac{2}{\pi} \log \zeta + \frac{2}{\pi} (1 - \lambda) \log \left(1 - \zeta^2 \gamma_S^{-2} \right) \equiv z^S(\zeta, t; \gamma_S) \tag{28}$$

where parameters $\gamma_S > 1$ and d are functions of t determined by the transcendental equations:

$$\log \gamma_S + \frac{\pi}{2} t + \lambda (1 - \lambda) \log (1 - \gamma_S^{-2}) = \log \gamma_0 + \lambda (1 - \lambda) \log (1 - \gamma_0^{-2}), \tag{29}$$

$$d(t) = \frac{2}{\lambda \pi} (1 - \lambda) \log \frac{\gamma_S}{\gamma_0} + \frac{t}{\lambda} + d_0. \tag{30}$$

Saffman [24] found that $\gamma_S \to 1^+$ exponentially in t as $t \to \infty$ and that, in this limit, $d \sim \frac{t}{\lambda}$. Therefore, as $t \to +\infty$,

$$z^S(\zeta, t; \gamma_S) \to z_{ST} \equiv i + \frac{t}{\lambda} - \frac{2}{\pi} \log \ \zeta + \frac{2}{\pi} (1 - \lambda) \log \left(1 - \zeta^2 \right) \tag{31}$$

where z_{ST} is the steady solutions found by Zhuravlev [23] and, independently, by Saffman & Taylor [1]. In this way, time-dependent solutions exist that evolve from a near planar interface (for γ_0 chosen sufficiently large) to the Zhuravlev-Saffman-Taylor (ZST) solution with relative finger width λ for any $\lambda \in (0, 1)$. Further, it is known that the zeros of z_ζ^S stay away from $|\zeta| = 1$ for all time so that

$$\inf_{|\zeta|=1} |\zeta z_\zeta^S(\zeta, t; \gamma_S)| > m > 0 \tag{32}$$

for some constant m that is independent of t and γ but which depends on λ.

It is also known (see Tanveer [28]) that the analytic continuation of (26) to the region $|\zeta| > 1$ is of the form:

$$z_t = q_1 z_\zeta + q_2 \tag{33}$$

where

$$q_1(\zeta,t) = \zeta \mathcal{I}_0[z(.,t)](\zeta) \quad , \quad q_2(\zeta,t) = -\frac{4\zeta}{\pi z_\zeta(\zeta^{-1},t)}. \tag{34}$$

If w satisfies the property $|w_\zeta(e^{-i\nu})| = |w_\zeta(e^{i\nu})|$ (as is the case for $w(\zeta) = z(\zeta,t)$) then a simple change of integration variable yields the useful property that

$$\mathcal{I}_0[w(.)](1/\zeta) = -\mathcal{I}_0[w(.)](\zeta). \tag{35}$$

Since Re $\mathcal{I}_0[w(.)](\zeta)$ is a harmonic function of $(\mathrm{Re}(\zeta), \mathrm{Im}(\zeta))$, it follows [28] from the maximum principle that for $|\zeta| > 1$,

$$\inf_{|\zeta|=1} \frac{2}{\pi |w_\zeta|^2} < \mathrm{Re}\,[\mathcal{I}_0[w(.)](\zeta)] < \sup_{|\zeta|=1} \frac{2}{\pi |w_\zeta|^2}. \tag{36}$$

More generally, following similar arguments, it also follows that for two different functions u and v, for $|\zeta| > 1$:

$$\inf_{|\zeta|=1} \left[\frac{2}{\pi |u_\zeta|^2} - \frac{2}{\pi |v_\zeta|^2}\right] < \mathrm{Re}\,[\mathcal{I}_0[u(.)](\zeta) - \mathcal{I}_0[v(.)](\zeta)] < \sup_{|\zeta|=1} \left[\frac{2}{\pi |u_\zeta|^2} - \frac{2}{\pi |v_\zeta|^2}\right] \tag{37}$$

Further, it is known [28] that any singularity $\zeta_s(t)$ of z in the region $|\zeta| > 1$ must move in accordance to the ordinary differential equation

$$\dot{\zeta}_s = -q_1(\zeta_s(t),t) = -\zeta_s \mathcal{I}_0[z(.,t)](\zeta_s(t)). \tag{38}$$

Letting $w(\zeta) = z(\zeta,t)$ in (36), we obtain $\mathrm{Re}[q_1/\zeta] > 0$ and hence all singularities of z continually approach $|\zeta| = 1$ from $|\zeta| > 1$.

Applying (38) to the Saffman solution, the solution $\gamma_S(t)$ of the transcendental equation (29) satisfies

$$\dot{\gamma}_S = -\gamma_S \mathcal{I}_0[z^S(.,t;\gamma_S(t))](\gamma_S(t)). \tag{39}$$

Since the functional form of z^S_ζ is relatively simple, it is possible to use contour integration to calculate

$$\mathcal{I}_0[z^S(.,t;\gamma)](\zeta) = \frac{\pi}{2} \frac{(1 - 2\,\lambda + 3\,\gamma^4 - 2\,\gamma^4\lambda - 3\,\zeta^2\gamma^2 - \zeta^2\gamma^6 + 4\,\zeta^2\gamma^2\lambda)}{(\zeta^2\gamma^2 + 1 - 2\,\lambda)\,(-\gamma^4 + 1 - 4\,\lambda + 4\,\lambda^2)}, \tag{40}$$

which will be useful for later estimates.

Many other exact solutions for the single-interface problem are known, including some that exist for all times. Howison [30] found extensions of the solutions of Saffman, including those of the form

$$z^H(\zeta, t) = i + d_0(t) - \frac{2}{\pi} \log \zeta + \frac{2}{\pi} \sum_{n=1}^{N} a_n \log \left(1 - \zeta^2 \gamma_n^{-2}\right) \tag{41}$$

where $\{|\gamma_n(t)| > 1 | n = 1, ..N\}$ evolve in accordance with (38) and approach the unit circle. In particular, with the choice of initial conditions given by $N = 2$, $a_1 = (1-\lambda)$, $a_2 > 0$, with $\gamma_2(0)$ on the imaginary axis and $\gamma_1(0)$ on the positive real axis such that $|\gamma_2(0)| >> \gamma_1(0) > 1$, the solution can be shown to develop a nearly steady ZST finger of width λ before "tip-splitting" over a longer time-scale. This demonstrates the nonlinear instability of interface L when R is at $+\infty$ and surface tension is ignored. This is not surprising because the initial value problem is ill-posed [30]. Later, it will be shown that the two-interface problem admits solutions which asymptote to the solution (41) of Howison for the left interface in the limit of fluid volume going to ∞. In this way, we will demonstrate that the ZST solution approached over an intermediate time scale for the two-interface problem cannot be stable in any sense.

3.2 The single right-interface problem

Consider now the second single-interface problem for the evolution of R in the limit when L has receded to $-\infty$. This single-interface problem is similar to the single-interface problem just studied, with an important difference; now, the viscous fluid is displacing a less viscous fluid. Except for a 180^o rotation of the geometry, this is equivalent to the previous problem with time reversed.

Consider the conformal map from the upper-half unit $\hat{\zeta}$ semi-circle into the flow domain left of the interface R (with L assumed to be at $-\infty$) with $\hat{\zeta} = \pm 1$ corresponding to the intersection points of the interface R with the lower and upper walls respectively. $\hat{\zeta} = 0$ maps to $z = -\infty$. There is then a time-reversed Saffman solution with a single interface, for which the relevant conformal mapping function is given by the following expression, once we account for the 180^o rotation:

$$z(\hat{\zeta}, t) = -i - \hat{d} + \frac{2}{\pi} \log \hat{\zeta} - \frac{2(1-\lambda)}{\pi} \log \left(1 - \hat{\zeta}^2 \beta_R^{-2}\right) = -z_S(\hat{\zeta}, t; \beta_R) \tag{42}$$

with parameters $\beta_R(t) > 1$ and $\hat{d}$ determined from

$$\log \beta_R - \frac{\pi}{2} t + \lambda(1-\lambda) \log(1-\beta_R^{-2}) = \log \beta_0 + \lambda(1-\lambda) \log(1-\beta_0^{-2}). \quad (43)$$

$$\hat{d}(t) = \frac{2}{\lambda \pi}(1-\lambda) \log \frac{\beta_R}{\beta_0} - \frac{t}{\lambda} + \hat{d}_0. \quad (44)$$

Here $\beta_0 > 1$ and $\hat{d}_0$ are the initial values of β_R and $\hat{d}$ respectively. The singularity $\beta_R(t)$ evolves in accordance to

$$\dot{\beta}_R = \beta_R \mathcal{I}_0 [z_S(., t; \beta_R)](\beta_R) \quad (45)$$

where we have reversed the sign on the right of (38). Hence $\beta_R(t)$, determined from (43), satisfies (45). (This fact will be used later in our discussion of the limit of two interfaces far apart). It is known from an analysis of (43) that β_R continually increases with t and as $t \to +\infty$, while $\beta_R \sim \text{const.} e^{\pi t/2}$. Therefore, as $t \to +\infty$,

$$-z_S(\hat{\zeta}, t, \beta_R) \to -i - t + \frac{2}{\pi} \log \hat{\zeta} \quad (46)$$

which corresponds to a steadily propagating planar front. From the point of view of stability, this makes sense because the planar interface between a more viscous fluid displacing a less viscous one is known to be stable, even with zero surface tension.

For the purposes of the next section, it will be useful to make a transformation of the $\hat{\zeta}$-plane formulation of the single right-interface problem just considered to a ζ-plane formulation where the *interior* of the unit $\hat{\zeta}$-circle maps to the *exterior* of a $|\zeta| = \rho$ circle in the ζ-plane. To do this, introduce the change of variable

$$\hat{\zeta} = e^{i\pi} \frac{\rho}{\zeta}, \quad (47)$$

and the mapping of parameters given by

$$\beta_R = \eta_R^{-1} \rho^{-1}. \quad (48)$$

Then, the single interface R is determined parametrically $(x(\nu), y(\nu))$ through the following equation on $\zeta = \rho e^{i\nu}$:

$$\begin{aligned} x(\nu) + iy(\nu) &= -z^S(-\rho\zeta^{-1}, t; \beta_R) \\ &= -\hat{d} + \frac{2}{\pi} \log \rho + i - \frac{2}{\pi} \log \zeta - \frac{2}{\pi}(1-\lambda) \log\left(1 - \zeta^{-2} \rho^4 \eta_R^2\right). \quad (49) \end{aligned}$$

For later convenience, we also define a time-dependent parameter

$$\rho_{SR} \equiv \rho_0 \left(\frac{\gamma_S \beta_R}{\gamma_0 \beta_0} \right)^{(1-\lambda)/\lambda} \tag{50}$$

where γ_S and β_R are the time-dependent parameters arising in the left and right one-interface problems above, and ρ_0 is some constant. Further, define

$$z^R(\zeta, t; \rho, \eta_R) = -z^S(-\rho\zeta^{-1}, t; \beta_R) - \frac{2}{\pi} \log \left(\frac{\rho}{\rho_{SR}} \right). \tag{51}$$

We now choose the initial condition for the parameter $\hat{d}(t)$ (i.e. $\hat{d}_0$) in (44) so that

$$\hat{d}_0 = \frac{2}{\pi} \log \rho - d_0.$$

Then, it follows from (30), (44) and (50) that

$$z^R(\zeta, t; \rho, \eta_R) = d(t) - \frac{2}{\pi} \log \zeta - \frac{2}{\pi}(1-\lambda) \log(1 - \zeta^{-2}\rho^4\eta_R^2). \tag{52}$$

Using (51) and the fact that β_R continually increases with time, it is easy to show that on $|\zeta| = \rho$,

$$\inf_{|\zeta|=\rho} |\zeta z^R_\zeta(\zeta, t; \eta_R, \rho)| > m > 0 \tag{53}$$

where m is a time-independent constant.

4 Analysis of the two-interface problem

In this section, we give analytical arguments for the existence of exact solutions to the full two-interface problem described in §2 and outline certain properties of these solutions. The arguments of this section provide the rationale for the form of the conformal maps posed in §5.

Using the results from more general obstacle problems [31], [32], [33], [34] to the specific case of Hele-Shaw zero-surface tension evolution, it follows that for a doubly-connected Hele-Shaw domain, as for a simply-connected one, if the initial extended-interface shape is analytic (by "extended" we refer to the shape obtained after reflection about each side-wall), a unique solution of the zero-surface tension Hele-Shaw problem exists for small enough t for which the interface shapes remain analytic. Hence $z(\zeta, t)$ is analytic on $|\zeta| = 1$ as well as on $|\zeta| = \rho$ for small enough time. Thus, at least over the time-interval of existence of an analytic solution, the analytic continuation of equation (14) is possible into the annulus $1 < |\zeta| < \rho^{-1}$. Through standard use of Plemelj formulae (or contour deformation of integrals) this results in

$$z_t(\zeta, t) = \zeta z_\zeta(\zeta, t) I(\zeta, t) - \frac{4\zeta}{\pi z_\zeta(\zeta^{-1}, t)}, \tag{54}$$

while its continuation into the annulus $\rho^2 < |\zeta| < \rho$ is given by

$$z_t(\zeta, t) = \zeta z_\zeta(\zeta, t) I(\zeta, t) - \frac{4\zeta}{\pi \rho^2 z_\zeta(\rho^2 \zeta^{-1}, t)} - \frac{2\dot{\rho}}{\rho} \zeta z_\zeta(\zeta, t). \tag{55}$$

It should be emphasized that while the same formal expression (15)–(18) define I in the annular regions $1 < |\zeta| < \rho^{-1}$, $\rho < |\zeta| < 1$ and $\rho^2 < |\zeta| < \rho$, they are actually different analytic functions (in the sense that they are not the analytic continuations of each other).

Now consider the quantity $H(\zeta, t)$ defined by

$$H(\zeta, t) \equiv z(\rho^2 \zeta, t) - z(\zeta, t). \tag{56}$$

Pick a value of ζ in the annulus $1 < |\zeta| < \rho^{-1}$. (54) will hold. The quantity $\rho^2 \zeta$ will be in the annulus $\rho^2 < |\zeta| < \rho$ so that (55) will also hold with argument $\zeta \mapsto \rho^2 \zeta$. That is,

$$z_t(\rho^2 \zeta, t) + 2\dot{\rho}\rho \zeta z_\zeta(\rho^2 \zeta, t) = \rho^2 \zeta z_\zeta(\rho^2 \zeta, t) I(\rho^2 \zeta, t) - \frac{4\zeta}{\pi z(\zeta^{-1}, t)}. \tag{57}$$

Now, subtracting (54) from (57) produces

$$z_t(\rho^2 \zeta, t) + 2\dot{\rho}\rho \zeta z_\zeta(\rho^2 \zeta, t) - z_t(\zeta, t) = \rho^2 \zeta z_\zeta(\rho^2 \zeta, t) I(\rho^2 \zeta, t) - \zeta z_\zeta(\zeta, t) I(\zeta, t). \tag{58}$$

However, on using the property from (21) that $K(\rho^2 \zeta, t) = K(\zeta, t) + 2$, it can be shown directly from (15) that

$$I(\rho^2 \zeta, t) = I(\zeta, t), \tag{59}$$

so that, using (56), (58) becomes

$$H_t(\zeta, t) = \zeta I(\zeta, t) H_\zeta(\zeta, t). \tag{60}$$

(60) holds for all ζ in $1 < |\zeta| < \rho^{-1}$.

If we choose initial condition so that $H(\zeta, 0) = c$ for some constant c, then it follows that

$$H(\zeta, t) = c. \tag{61}$$

This means that the mapping satisfies the functional equation

$$z(\rho^2\zeta, t) - z(\zeta, t) = c \tag{62}$$

at all times for which a solution exists. This general property in the context of a two interface Hele-Shaw problem was first realized by Richardson [26], [27], though in a differing formulation, and found exact solutions in terms of elliptic functions. Crowdy [25] also exploited this result in a formulation similar to the present in the context of a fluid annulus in a rotating Hele-Shaw cell. Note that although we have only established that (62) holds for points ζ in the annulus $1 < |\zeta| < \rho^{-1}$, by the continuation principle, it also holds everywhere. The arguments above prove the following Lemma:

Lemma 1: *As long as solution exists for which the (extended) interface shapes are analytic, if the associated conformal map $z(\zeta, 0)$ satisfies (62) for some constant c, then $z(\zeta, t)$ satisfies the same condition for $t > 0$.*

Equation (62) furnishes the analytic continuation of $z(\zeta, t)$ into the annulus $\frac{1}{\rho} \leq |\zeta| \leq \frac{1}{\rho^2}$. Indeed, as long as an analytic solution (as far as extended shapes are concerned) exists, the corresponding conformal maping function $z(\zeta, t)$ is free of singularities in the region $\rho \leq |\zeta| \leq 1$. It follows from (62) that it is also free of singularities in $\frac{1}{\rho} \leq |\zeta| \leq \frac{1}{\rho^2}$. From (14) it follows that for $\frac{1}{\rho} < |\zeta| < \frac{1}{\rho^2}$,

$$z_t(\zeta, t) = \left(\frac{2\dot{\rho}}{\rho} + I(\zeta, t)\right) \zeta z_\zeta \tag{63}$$

which is consistent with the absence of singularities of z because $I(\zeta, t)$ is analytic in this ring-domain.

In the region $1 < |\zeta| < \rho^{-1}$, (54) can be written

$$z_t(\zeta, t) = q_1(\zeta, t) z_\zeta(\zeta, t) + q_2(\zeta, t) \tag{64}$$

where we define

$$q_1(\zeta, t) \equiv \zeta I(\zeta, t), \quad q_2(\zeta, t) \equiv -\frac{4\zeta}{\pi z_\zeta(\zeta^{-1}, t)}. \tag{65}$$

$q_1(\zeta,t)$ and $q_2(\zeta,t)$ are analytic everywhere in $1 < |\zeta| < \rho^{-1}$. Taking the derivative of (64) yields

$$\frac{\partial}{\partial t} z_\zeta(\zeta,t) = q_1(\zeta,t)\frac{\partial}{\partial \zeta} z_\zeta(\zeta,t) + q_{1\zeta}(\zeta,t) z_\zeta(\zeta,t) + q_{2\zeta}(\zeta,t). \tag{66}$$

(66) has the form of a first-order linear partial differential equation for $z_\zeta(\zeta,t)$ with coefficients that are known *a priori* to be analytic in region

$$\mathcal{R} = \left\{\zeta : 1 < |\zeta| < \rho^{-1}\right\}.$$

So, all the known properties of such differential equations are valid, including the fact that singularities cannot spontaneously arise and that any singularity ζ_s in this region $\mathcal{R}$ retains its form and must move with characteristic speed, i.e.,

$$\dot{\zeta}_s = -q_1(\zeta_s(t),t) = -\zeta_s(t) I(\zeta_s(t),t) \tag{67}$$

Further, for initial conditions for which $z(\rho^2\zeta,0) - z(\zeta,0) = c$, as long as solutions exist with analytic interfacial shapes, z_ζ is free of singularities in the adjoining regions $\rho \le |\zeta| \le 1$ and $\rho^{-1} \le |\zeta| \le \rho^{-2}$ and hence singularities cannot enter or leave $\mathcal{R}$ from these adjoining regions. We are naturally led to the following Lemma:

Lemma 2: *As long as solutions exist with analytic (extended) interface shapes, if $z(\rho^2\zeta,0) - z(\zeta,0) = c$ for some constant c and $z_\zeta(\zeta,0)$ has only a finite number (N) of poles in $\mathcal{R}$, then $z_\zeta(\zeta,t)$ remains analytic in $\mathcal{R}$, except at N poles, each of which move in accordance to (67).*

By integration, $z(\zeta,t)$ is deduced to have a finite number of (time-evolving) logarithmic singularities in $\mathcal{R}$.

Suppose a typical logarithmic singularity is at the point $\gamma(t)$ in $\mathcal{R}$ so that

$$z(\zeta,t) = a(t)\log(\zeta - \gamma(t)) + \text{analytic}, \quad \text{as } \zeta \to \gamma(t). \tag{68}$$

On direct substitution into (54) and comparison of singularities, it can be deduced that $a(t)$ and $\gamma(t)$ satisfy the ordinary differential equations

$$\dot{a}(t) = 0, \quad \dot{\gamma}(t) = -\gamma(t) I(\gamma(t),t). \tag{69}$$

Note that $a(t) = a(0)$ is a conserved quantity.

5 A class of exact solutions

Consider the initial condition

$$z(\zeta,0) = i - \frac{2}{\pi}\left(\log\zeta + (1-\lambda)\log\left[\frac{P(\zeta/\eta_0,\rho_0)P(-\zeta/\eta_0,\rho_0)}{P(\zeta/\gamma_0,\rho_0)P(-\zeta/\gamma_0,\rho_0)}\right]\right) \qquad (70)$$

where $\rho_0 > 0$ and $\rho_0^{-1} > \gamma_0, \eta_0 > 1$ are chosen appropriately so that $z(\zeta,0)$ is univalent in the domain $\rho_0 < |\zeta| < 1$ and $z_\zeta(\zeta,0) \neq 0$ for $\rho_0 \leq |\zeta| \leq 1$. That such a choice is possible will be obvious later in §6 for sufficiently small ρ_0. In that case $z(\zeta,0)$ is a conformal mapping function from the half-ring region $\{\zeta : \rho_0 < |\zeta| < 1, \mathrm{Im}[\zeta] > 0\}$ to a domain in the z-plane shown in Figure 1. Further, from examination of the conformal map (70), it is clear that the corresponding interface shapes are symmetric about the channel centerline. Further, from the properties (21) of $P(\zeta)$, it is clear that in the domain $1 < |\zeta| < \rho_0^{-1}$, $z_\zeta(\zeta,0)$ has just four poles at $\zeta = \pm\gamma_0, \pm\eta_0$.

$$z(\rho_0^2\zeta,0) - z(\zeta,0) = c \equiv -\frac{2}{\pi}\left(\log\ [\rho_0^2] + \log\left(\frac{\eta_0}{\gamma_0}\right)^{2(1-\lambda)}\right) \qquad (71)$$

Uniqueness of solutions and the symmetry of the equations and initial conditions guarantee that the interfaces remain symmetric, corresponding to the the positive imaginary-ζ-axis, for $t > 0$. In particular, this implies that

$$|z_\zeta(\rho e^{i\pi/2+i\nu})|^{-2} = |z_\zeta(\rho e^{i\pi/2-i\nu})|^{-2}, |z_\zeta(e^{i\pi/2+i\nu})|^{-2} = |z_\zeta(e^{i\pi/2-i\nu})|^{-2}. \quad (72)$$

Using Lemmas 1 and 2, it follows that as long as a solution with analytic interfaces exists, the corresponding conformal map $z(\zeta,t)$ will satisfy

$$z(\rho^2\zeta,t) - z(\zeta,t) = c \qquad (73)$$

where $\rho(t)$ evolves according to (13), and $z_\zeta(\zeta,t)$ will have four poles at $\gamma(t)$, $\tilde{\gamma}(t)$, $\eta(t)$ and $\tilde{\eta}(t)$ in the annular region $1 < |\zeta| < \rho^{-1}$ where each pole (generically represented by ζ_s) evolves according to (67) with respective initial conditions $\gamma(0) = \gamma_0$, $\tilde{\gamma}(0) = -\gamma_0$, $\eta(0) = \eta_0$ and $\tilde{\eta}(0) = -\eta_0$.

Note that the symmetry condition (72), together with symmetry about the real ζ axis as expressed in (7), imply that

$$|z_\zeta(-\zeta',t)|^{-2} = |z_\zeta(\zeta',t)|^{-2}$$

on both $|\zeta'| = 1$ and $|\zeta'| = \rho$. From the integral expressions in (15)–(17), it follows that $I(-\zeta, t) = I(\zeta, t)$. Hence if $\zeta_s = \gamma(t)$ satisfies (69), so does $\zeta_s = -\gamma(t)$, the latter satisfying the same initial condition as $\tilde{\gamma}$. Therefore, by the uniqueness of the solutions of the ordinary differential equation (which follows from the fact that I has bounded derivatives in $\mathcal{R}$, at least for the interval of time when a zero of z_ζ stays away from $|\zeta| = 1$ and $|\zeta| = \rho$ – see (24)) it follows that $\tilde{\gamma}(t) = -\gamma(t)$. Similar arguments show that $\tilde{\eta}(t) = -\eta(t)$. Thus, the only singularities of the analytically-continued conformal mapping function $z(\zeta, t)$ in the annular region $\mathcal{R}$ (corresponding to the unique analytic interfacial solution) are poles at $\zeta = \pm\gamma(t)$ and $\zeta = \pm\eta(t)$, where

$$\frac{d\gamma}{dt} = -\gamma(t) I(\gamma(t), t), \; \frac{d\eta}{dt} = -\eta(t) I(\eta(t), t), \tag{74}$$

with initial conditions $\gamma(0) = \gamma_0$, $\eta(0) = \eta_0$.

Now we define $\mathcal{Z}(\zeta, t)$ through the decomposition

$$z(\zeta, t) = i - \frac{2}{\pi}\left(\log\zeta + (1-\lambda)\log\left[\frac{P(\zeta/\eta, \rho)P(-\zeta/\eta, \rho)}{P(\zeta/\gamma, \rho)P(-\zeta/\gamma, \rho)}\right]\right) + \mathcal{Z}(\zeta, t) \tag{75}$$

where $\lambda \in (0, 1)$ is constant, $\gamma(t)$ and $\eta(t)$ evolve according to (74) and ρ evolves according to (13). We will take $|\eta_0| > |\gamma_0| > 1$. ¿From the initial condition (70), it follows that $\mathcal{Z}(\zeta, 0) = 0$.

Since $z_\zeta(\zeta, 0)$ has simple poles at $\zeta = \pm\gamma_0, \pm\eta_0$, with residues equalling $-\frac{2}{\pi}(1-\lambda)$ at $\pm\gamma_0$ and $\frac{2}{\pi}(1-\lambda)$ at $\pm\eta_0$, it follows from Lemma 2 that $z_\zeta(\zeta, t)$ will have simple poles at $\pm\gamma(t)$ and $\pm\eta(t)$ with time-invariant residues. Since the ζ-derivative of the first term on the right of (75) has simple poles at these points (and *only* these points in $\mathcal{R}$) with the same residues, it follows that $\mathcal{Z}(\zeta, t)$ is free of singularities in $\mathcal{R}$. It is also single-valued in $\mathcal{R}$ since $z + \frac{2}{\pi}\log\zeta$ and terms other than $\log\zeta$ and $\mathcal{Z}$ on the right of (75) are known to be single-valued in $\mathcal{R}$. Further, the condition

$$z(\rho^2\zeta, t) - z(\zeta, t) = z(\rho^2\zeta, 0) - z(\zeta, 0) = -\frac{2}{\pi}\left(\log\ [\rho_0^2] + \log\left(\frac{\eta_0}{\gamma_0}\right)^{2(1-\lambda)}\right) = c \tag{76}$$

implies

$$\mathcal{Z}(\rho^2\zeta, t) - \mathcal{Z}(\zeta, t) = g(t) \equiv \frac{2}{\pi}\left(\log\ \left(\frac{\rho(t)}{\rho_0}\right)^2 + \log\left(\frac{\eta(t)\gamma_0}{\eta_0\gamma(t)}\right)^{2(1-\lambda)}\right) \tag{77}$$

Thus, $\zeta \mathcal{Z}_\zeta$ is a doubly-periodic function of $\log \zeta$ with no singularities in the fundamental cell $\rho \leq |\zeta| < \frac{1}{\rho}$, $0 \leq \arg[\zeta] < 2\pi$. It therefore equals some constant $k(t)$ (by Liouville's theorem for elliptic functions). This implies that $\mathcal{Z}(\zeta, t) = d(t) + k(t) \log \zeta$ where, from symmetry about the real ζ-axis, $d(t)$ and $k(t)$ must be real functions of time. However, since we know that $z(\zeta, t)$ necessarily has a logarithmic singularity at $\zeta = 0$ of the known form $-\frac{2}{\pi} \log \zeta$ (which has already been explicitly included in the decomposition (75)) it follows that $k(t) = 0$. Thus, $\mathcal{Z}(\zeta, t) = d(t)$. In summary, the conformal mapping function corresponding to the unique analytic Hele-Shaw interfacial solution must remain of the form

$$z(\zeta, t) = d(t) + i - \frac{2}{\pi} \left(\log \zeta + (1 - \lambda) \log \left[\frac{P(\zeta/\eta, \rho) P(-\zeta/\eta, \rho)}{P(\zeta/\gamma, \rho) P(-\zeta/\gamma, \rho)} \right] \right) \tag{78}$$

where $\rho(t)$, $\eta(t)$ and $\gamma(t)$ are determined from (13) and (74). An ordinary differential equation for $d(t)$ can be derived by evaluating the equation (14) at an arbitrary point in the annulus $\rho < |\zeta| < 1$.

We have proved the following theorem:

Theorem 1: *If the initial Hele-Shaw shape corresponds to the conformal map $z(\zeta, 0)$ given by (77) with $\rho_0 > 0$ and $1 < \gamma_0 < \eta_0 < \rho_0^{-1}$ chosen so as to ensure that the $z(\zeta, 0)$ is appropriately univalent and the corresponding (extended) interface shapes are analytic the, over the time interval $0 \leq t \leq T$ for which a unique interfacial solution exists, the corresponding conformal mapping function $z(\zeta, t)$ is of the form (75) where $\gamma(t)$, $\eta(t)$ and $\rho(t)$ are determined from (13) and (74). Also, such a solution must satisfy (76) with $1 < \gamma, \eta < \rho^{-1}$.*

Since $\mathcal{Z}(\zeta, t) = d(t)$, it follows from (77) that $g(t) = 0$ and therefore for any $t > 0$,

$$c = -\frac{2}{\pi} \left(\log \rho^2 + \log \left(\frac{\eta}{\gamma} \right)^{2(1-\lambda)} \right). \tag{79}$$

Instead of determining $\gamma(t)$, $\eta(t)$ and $\rho(t)$ from (13) and (74), it is clearly possible to replace any one of the three equations by (79). For instance, given c from the initial configuration, (79) provides an algebraic equation for η, i.e.,

$$\eta = \gamma \left(\frac{e^{-\pi c/2}}{\rho^2} \right)^{\frac{1}{2(1-\lambda)}}. \tag{80}$$

From the arguments above, use of (80) is equivalent to solving the second equation in (74) for η provided we use (74) to determine ρ. (80) is a nonlinear algebraic relation between the conformal mapping parameters. Of course, parameters c and λ must be chosen carefully to ensure that η_0, like γ_0, is in the annulus $1 < |\zeta| < \rho^{-1}$. Note that, although the map depends on the four parameters λ, ρ, γ and η, there are two conserved quantities associated with the motion, namely, λ and c.

Using the same methods as above, it is not difficult to prove that one can construct more general classes of exact solution with z_ζ possessing $4N$ poles in $1 < |\zeta| < \rho^{-1}$. The generalized conformal maps take the form

$$z(\zeta, t) = i + d(t) - \frac{2}{\pi}\left(\log\zeta + (1-\lambda)\log\left[\frac{\prod_{j=1}^{N} P(\zeta/\eta_j)P(-\zeta/\eta_j)}{\prod_{j=1}^{N} P(\zeta/\gamma_j)P(-\zeta/\gamma_j)}\right]\right) \quad (81)$$

where each γ_j and η_j evolve in accordance with (67) as before. Generalization for solutions that correspond to shapes that are not necessarily symmetric about the channel centerline is of the form:

$$z(\zeta, t) = i + d(t) - \frac{2}{\pi}\left(\log\zeta + (1-\lambda)\log\left[\frac{\prod_{j=1}^{N} P(\zeta/\eta_j^{(1)})P(\zeta/\eta_j^{(2)})}{\prod_{j=1}^{N} P(\zeta/\gamma_j^{(1)})P(\zeta/\gamma_j^{(2)})}\right]\right) \quad (82)$$

where each $\gamma_j^{(1)}$, $\gamma_j^{(2)}$ and $\eta_j^{(1)}$ and $\eta_j^{(2)}$ evolve in accordance with (67) as before.

6 The infinite-fluid limit: $\rho_0 \to 0$

We will now consider the solution (75) in the asymptotic limit of parameters $\rho_0 \to 0^+$ when $\gamma_0 = O(1)$ and $\eta_0\rho_0 = O(1)$, with parameters constrained by the inequality

$$1 < \gamma_0 \ll \eta_0 < \rho_0^{-1}. \quad (83)$$

In this limit, the amount of viscous fluid in the Hele-Shaw cell becomes infinite and this is clear from equation (5) since $\phi_R \to \infty$ (for $V = 1$).

It is convenient to define

$$\beta = \frac{1}{\rho\eta} \quad (84)$$

On use of (79), ρ can be expressed in terms of γ and β as:

$$\rho = \rho_0 \left(\frac{\beta\gamma}{\beta_0 \gamma_0} \right)^{(1-\lambda)/\lambda} \tag{85}$$

Using (74) and the fact that $I(\eta^{-1}, t) = -I(\eta, t)$ (see (23)), it follows that β evolves in accordance to

$$\dot{\beta} = -\beta \left[I(\eta^{-1}, t) + \frac{\dot{\rho}}{\rho} \right]. \tag{86}$$

Given the restriction on η, it is clear that $\beta > 1$.

We now wish to consider the limit of $\rho_0 \to 0^+$, with $\gamma_0, \beta_0 = O(1)$. This will be done in the next subsection using formal arguments. A mathematically rigorous treatment of the same arguments is given in section 6.2.

6.1 Formal results

We look at the form of some approximations to the exact solutions when ρ is small. Clearly, if ρ_0 is small, from the continuity of the solutions in time, there will be an interval of time for which ρ remains small and each of $\gamma = O(1)$ and $\beta = O(1)$. On $|\zeta| = 1$, we note that

$$\begin{aligned} \left|\log\left[P(\zeta\eta^{-1})P(-\zeta\eta^{-1})\right]\right| &\leq C_0 \rho^2 \beta^2, \\ \left|\log\left[P(\zeta\gamma^{-1})P(-\zeta\gamma^{-1})\right] - \log\left(1 - \frac{\zeta^2}{\gamma^2}\right)\right| &\leq C_0 \rho^2 \end{aligned} \tag{87}$$

while for $|\zeta| = \rho$,

$$\begin{aligned} \left|\log\left[P(\zeta\gamma^{-1})P(-\zeta\gamma^{-1})\right]\right| &\leq C_0 \rho^2 \gamma^2, \\ \left|\log\left[P(\zeta\eta^{-1})P(-\zeta\eta^{-1})\right] - \log\left(1 - \frac{\rho^4\eta^2}{\zeta^2}\right)\right| &\leq C_0 \rho^2 \end{aligned} \tag{88}$$

for some constant C_0 independent of any of the parameters. These results mean that on the left interface corresponding to $|\zeta| = O(1)$,

$$\begin{aligned} |z(\zeta, t) - z^S(\zeta, t; \gamma)\,| &\leq K_1 \rho^2 \beta^2, \\ |\zeta z_\zeta(\zeta, t) - \zeta z_\zeta^S(\zeta, t; \gamma)\,| &\leq K_2 \rho^2 \beta^2 \end{aligned} \tag{89}$$

while on the right interface, $|\zeta| = \rho$,

$$\begin{aligned} |z(\zeta,t) - z^R(\zeta,t;\rho,\eta)\ | &\le K_3\rho^2\gamma^2, \\ |\zeta z_\zeta(\zeta,t) - \zeta z_\zeta^R(\zeta,t;\rho,\eta)\ | &\le K_4\rho^2\gamma^2. \end{aligned} \tag{90}$$

In (89) and (90), K_1 through K_4 are constants independent of ρ, η, γ and t.

The evolution of parameters $\gamma(t)$ and $\eta(t)$ can be simplified. We note that

$$K(\zeta,\rho) = \frac{1+\zeta}{1-\zeta} + 2\sum_{j=1}^{\infty}\left(\frac{\rho^{2j}\zeta}{(1-\rho^{2j}\zeta)} - \frac{\rho^{2j}}{\zeta(1-\rho^{2j}\zeta^{-1})}\right). \tag{91}$$

For all ζ in the annulus $1 < |\zeta| \le |\gamma|$ and for $|\zeta'| = 1$,

$$\left|K(\zeta/\zeta',\rho) - \frac{\zeta'+\zeta}{\zeta'-\zeta}\right| \le C_1\rho^2, \tag{92}$$

while for $|\zeta'| = \rho$,

$$|K(\zeta/\zeta',\rho) + 1| \le C_2\rho. \tag{93}$$

It follows that for $1 < |\zeta| \le |\gamma|$,

$$|I(\zeta,t) - \mathcal{I}_0[z(.,t)](\zeta)| \le C_1\rho\left(\frac{1}{\inf_{|\zeta|=1}|\zeta z_\zeta|^2} + \frac{1}{\inf_{|\zeta|=\rho}|\zeta z_\zeta|^2}\right) \tag{94}$$

where $\mathcal{I}_0$ is defined in (27). Using (32) and (89), (94) implies that for small enough ρ with $\gamma = O(1)$, $\beta = O(1)$

$$|I(\zeta,t) - \mathcal{I}_0[z(.,t)](\zeta)|| \le C_1\rho \tag{95}$$

for some constant C_1, independent of any of the parameters.

From (89) and (95), we obtain

$$|I(\gamma(t),t) - \mathcal{I}_0[z^S(.,t;\gamma(t))](\gamma(t))| \le C_1\left(\rho + \rho^2\beta^2\right) \tag{96}$$

and hence for small ρ,

$$\dot{\gamma} = -\gamma I(\gamma(t),t) \sim -\gamma\mathcal{I}_0[z^S(.,t;\gamma(t))](\gamma(t)). \tag{97}$$

We note from the expression (40) that $\mathcal{I}_0[z^S(.,t;\gamma)](\gamma)$ is a continuous function of γ. Therefore, as long as ρ remains small and β and γ remain order 1,

$\gamma \sim \gamma_S$, where γ_S satisfies (39) and hence is determined by (29). Thus the left interface L shape will approach the Saffman-solution shape as $\rho \to 0$.

On the other hand, for $\rho < |\zeta| \leq \eta^{-1}$, using similar approximations to $K(\zeta/\zeta', \rho)$ and lower bounds on $|\zeta z_\zeta|$ for $\zeta = \rho$ that derive from (53) and (90), it follows that

$$\left| I(\zeta, t) + \frac{\dot{\rho}}{\rho} - \mathcal{I}_R[z(., t)](\zeta) \right| \leq C_1 \rho \beta \tag{98}$$

where

$$\begin{aligned} \mathcal{I}_R[z](\zeta) &\equiv \frac{1}{\pi^2 i} \oint_{|\zeta'|=\rho} \frac{d\zeta'}{\zeta'} \frac{\zeta + \zeta'}{\zeta' - \zeta} \left[\frac{1}{|\zeta' z_\zeta(\zeta', t)|^2} \right] \\ &= -\frac{1}{\pi^2 i} \oint_{|\zeta'|=1} \frac{d\zeta'}{\zeta'} \frac{\zeta' + \rho\zeta^{-1}}{\zeta' - \rho\zeta^{-1}} \left[\frac{1}{\rho^2 |z_\zeta\left(\rho\zeta'^{-1}, t\right)|^2} \right]. \end{aligned} \tag{99}$$

On use of (51), and using $|z^S_\zeta(\zeta, t; \beta)|^{-2} = |z^S_\zeta(-\zeta, t; \beta)|^{-2}$ which follows from the symmetries about the real and imaginary ζ-axes (the latter corresponding to channel-centerline),

$$\begin{aligned} \mathcal{I}_R[z^R(., t; \rho, \eta)](\eta^{-1}) &= \mathcal{I}_0[z^S\left(., t; (\rho\eta)^{-1}\right)](\rho\eta) \\ &= -\mathcal{I}_0[z^S\left(., t; (\rho\eta)^{-1}\right)]((\rho\eta)^{-1}) \end{aligned} \tag{100}$$

where the latter equality follows from property (35) with $w = z^S$. On use of (86), it follows that over the interval of time $[0, T]$ when ρ remains small and $\beta, \gamma = O(1)$,

$$\dot{\beta} = -\beta \left[I(\eta^{-1}, t) + \frac{\dot{\rho}}{\rho} \right] \sim \beta \mathcal{I}_0[z^S(., t; \beta)](\beta). \tag{101}$$

Using the continuity of $\mathcal{I}_0[z^S(., t; \beta)](\beta)$ with respect to β, as is evident from (40), it follows that $\beta \sim \beta_R$, where β_R satisfies (45) and is therefore determined by the transcendental equation (43). Therefore, over this interval of time, the right interface stays close to a time-reversed Saffman solution $z^R(\zeta, t; \rho_{SR}, \eta_R)$, which is a solution to the single interface problem when L has receded to $-\infty$.

To estimate T, the length of the time interval for which the solution is close to the Saffman single-interface solution, we plug approximations $\beta = \beta_R$, $\gamma = \gamma_S$ into (85) to obtain

$$\rho \sim \rho_{SR}(t) \tag{102}$$

where $\rho_{SR}(t)$ is given by (50). γ_S is known to decrease monotonically to 1 and β_R increases monotonically and approaches an $e^{\pi t/2}$ growth. Thus ρ remains small over the time interval $[0, T]$ when $T \ll -\log \rho_0$. Another way to interpret this result is that once T is chosen, there exists ρ_0 small enough so that the assumptions $\rho \ll 1$ and $\gamma, \beta = O(1)$ are self-consistent and the left interface shape L stays close to the Saffman solution. When $1 \ll t \ll -\log \rho_0$ the left interface approaches the Zhuravlev-Saffman-Taylor (ZST) steady solution of width λ, without any restriction on λ, aside from being in the interval $(0, 1)$. This steady shape propagates with speed $\frac{1}{\lambda}$. The right interface, corresponding to z^R, will then approach a planar front propagating with speed 1. However, the asymptotic approach of the ZST steady solution is only over an intermediate time-scale. Regardless of the amount of fluid, *i.e.* no matter how small ρ_0 is, eventually, at times of $O(-\log \rho_0)$ or larger, the left interface L, which is moving faster, will catch up with the right interface R. No steady solution is possible when the two interfaces interact strongly.

Also, it is to be noted that the mathematics does not prevent the choice $\gamma_0 = 1$, in which case $\gamma(t) = 1$ for later time. In this case, the left interface extends to $-\infty$ and the corresponding left-interface shape as $\rho_0 \to 0$, with $\eta_0 = O(1/\rho_0)$ is that of the steady ZST solution of arbitrary width λ even when t is not large. In this case, the left interface shape deviates from the ZST steady shape only when $t = O(-\log \rho_0)$. In the special case where we chose $\lambda = \frac{1}{2}$, $\gamma_0 = 1$, $\eta_0 = \rho_0^{-1/2}$ in the initial condition (70) for the solution (78), we can be more precise about the difference between the initial shapes and that described in [21]. The conformal map for the solution of Feigenbaum can be written, in the present formulation, as

$$z^F(\zeta) = i - \frac{2}{\pi} \ln \ \zeta + \frac{1}{\pi} \ \ln \ (1 - \zeta^2)$$

Using the property that for $\rho_0 < \frac{1}{2}$

$$\left| \ln \left(\frac{P(\zeta, \rho_0) P(-\zeta, \rho_0)}{1 - \zeta^2} \right) \right| \leq \frac{5}{3} \rho_0^4 \left(|\zeta|^2 + |\zeta|^{-2} \right),$$

it is not difficult to deduce rigorously that

$$\sup_{|\zeta|=1, |\zeta|=\rho_0} \left| z(\zeta, 0) - z^F(\zeta, 0) \right| \leq \frac{5}{3} \rho_0 \leq 2 e^{-\frac{\pi}{2} L} \tag{103}$$

where

$$L = |z^F(i) - z^F(i\rho_0)| = -\frac{2}{\pi}\ln\ \rho_0 + \frac{1}{\pi}\ln\left(\frac{1+\rho_0^2}{2}\right)$$

is the non-dimensional distance from the tip of the finger to the right interface.

Now, if we carry out similar arguments for the solution (81) for $N = 2$, with $\eta_1(0)$ and $\gamma_1(0) \geq 1$ real and with $\eta_2(0)$, $\gamma_2(0)$ each imaginary with the ordering

$$1 \leq \gamma_1(0) \ll |\gamma_2(0)| \ll |\eta_2(0)| \ll \eta_1(0)$$

then it is possible to show that, for small enough ρ_0, the left interface asymptotes to a Howison solution z^H given by (41) for $N = 2$, which for $1 \ll t \ll -\log\rho_0$ will result in tip-splitting of the left interface. Since at the initial time, with $\gamma_1(0) = 1$, the left interface shape is an arbitrarily small deviation from a ZST shape when ρ_0 tends to zero, it follows that the left interface, corresponding to a ZST solution, is unstable in this nonlinear sense for any λ (including $\lambda = \frac{1}{2}$). In every case, at least within the family of solutions shown, the limit where the amount of viscous fluid tends to infinity results in the recovery of the single-interface dynamics over an intermediate time-scale – and the latter problem is known to be ill-posed [30].

6.2 Rigorous results for $\rho_0 \to 0$

The arguments in the last subsection, as far as the Saffman-type solution is concerned, can be put on a rigorous foundation. We will prove the following theorem:

Theorem 2: *For any choice of $\gamma_0, \beta_0 > 1$ and time interval $[0, T]$, there exists ρ_0 small enough, with $-\log\rho_0$ sufficiently large compared to T, such that $\gamma(t)$, $\beta(t)$ satisfy the following inequalities:*

$$\left|\frac{\gamma^2}{\gamma_S^2} - 1\right| \leq C_1\rho_{SR}(T)\beta_R(T),$$
$$\left|\frac{\beta^2}{\beta_R^2} - 1\right| \leq C_1\rho_{SR}(T)\beta_R(T), \qquad (104)$$
$$\left|\frac{\rho}{\rho_{SR}} - 1\right| \leq C_1\rho_{SR}(T)\beta_R(T)$$

for constant C_1 independent of any parameters. Also,

$$\sup_{|\zeta|=1} |z(\zeta,t) - z^S(\zeta,t;\gamma_S)| \le C_1 \rho_{SR}(T)\beta_R(T)$$
$$\sup_{|\zeta|=\rho} |z(\zeta,t) - z^R(\zeta,t;\rho,\eta_R)| \le C_1 \rho_{SR}^2. \tag{105}$$

The proof of this theorem will follow after some preliminary Lemmas.

Definitions: We introduce the following definitions:

$$\begin{aligned}
T_0(t) &\equiv \mathcal{I}_0[z^S(.,t;\gamma(t))](\gamma(t)) - \mathcal{I}_0[z^S(.,t;\gamma_S(t))](\gamma_S(t)),\\
T_1(t) &\equiv I(\gamma(t),t) - \mathcal{I}_0[z^S(.,t;\gamma(t))](\gamma(t)),\\
S_0(t) &\equiv \mathcal{I}_0[z^S(.,t;\beta(t)](\beta(t)) - \mathcal{I}_0[z^S(.,t;\beta_R(t)](\beta_R(t)),\\
S_1(t) &\equiv I(\eta^{-1}(t),t) + \frac{\dot\rho}{\rho} - \mathcal{I}_R[z^R(.,t;\rho(t),\eta(t))](\eta^{-1}(t)),
\end{aligned} \tag{106}$$

and

$$a \equiv \frac{\gamma^2}{\gamma_S^2} - 1, \quad b \equiv \frac{\beta^2}{\beta_R^2} - 1. \tag{107}$$

Note that a, for example, gives a measure of the difference between the parameter γ in the full two-interface exact solution and the parameter γ_S in the single left-interface solution. Similarly, b measures the difference between β and β_R.

On use of (40), and the definition of T_0 and a, it follows that

$$T_0(t) = \frac{2\pi\lambda(1-\lambda)a(2+a)}{\gamma_S^4\left[(1+a)^2 - (2\lambda-1)^2\gamma_S^{-4})\right]\left[1-(2\lambda-1)^2\gamma_S^{-4}\right]}. \tag{108}$$

It is convenient to break up $T_0(t)$ as follows:

$$T_0(t) = T_{00}(t) + T_{01}(t)$$

where

$$T_{00}(t) = \frac{4\pi\lambda(1-\lambda)a}{\gamma_S^4\left[1-(2\lambda-1)^2\gamma_S^{-4}\right]^2}.$$

Note that T_{00}/a is a known positive function of t, bounded from above and below, that asymptotes to a constant for large t. Also, it is to be noted that $T_{01} \equiv T_0 - T_{00} = O(a^2)$ for small a, i.e., there exists constant C_1, independent of t so that $|T_{01}| < C_1 a^2$ for small enough a. Also, from (40) and the definition of S_0 and b,

$$S_0(t) = \frac{2\pi\lambda(1-\lambda)b(2+b)}{\beta_R^4\left[(1+b)^2-(2\lambda-1)^2\beta_R^{-4})\right]\left[1-(2\lambda-1)^2\beta_R^{-4}\right]}. \tag{109}$$

As before, it is convenient to break up S_0 as

$$S_0 = S_{00} + S_{01}$$

where

$$S_{00}(t) = \frac{4\pi\lambda(1-\lambda)b}{\beta_R^4\left[1-(2\lambda-1)^2\beta_R^{-4}\right]^2}.$$

Then, from the asymptotic exponential growth of β_R, it is clear that $|S_{00}/b| < C_1\ e^{-2\pi t}$ and $|S_{01}| < C_1\ e^{-2\pi t}|b|^2$.

Lemma 3: *a and b, as defined above, satisfy the following set of integral equations*

$$a = -2\int_0^t dt' e^{-\omega(t)+\omega(t')}\left[T_1(1+a)+T_{01}(1+a)+T_{00}a\right](t')dt' \equiv \mathcal{N}_1[a,b](t)$$

$$b = 2\int_0^t dt' e^{\alpha(t)-\alpha(t')}\left[-S_1(1+b)+S_{01}(1+b)+S_{00}b\right](t')dt' \equiv \mathcal{N}_2[a,b](t)$$

where ρ occurring in T_1, S_1 is determined from

$$\rho = \rho_{SR}\left([1+a][1+b]\right)^{(1-\lambda)/(2\lambda)}$$

and

$$\omega(t) = 2\int_0^t [T_{00}/a](t')dt',$$

$$\alpha(t) = 2\int_0^t [S_{00}/b](t')dt'.$$

Proof: We know from (39) and (74) that

$$\frac{\dot\gamma}{\gamma} - \frac{\dot\gamma_S}{\gamma_S} = -(T_0+T_1).$$

This implies

$$\dot a = -2(1+a)(T_0+T_1)$$

with initial condition $a(0) = 0$. Inverting the operator $\mathcal{L}_1$, defined by $\mathcal{L}_1[a] = \dot{a} + 2T_{00}$, we obtain the integral equation for a, as in the Lemma.

Similarly, using (45) and (86), we obtain

$$\frac{\dot{\beta}}{\beta} - \frac{\dot{\beta}_R}{\beta_R} = (S_0 - S_1)$$

which implies

$$\dot{b} = 2(1+b)(S_0 - S_1).$$

Inverting the linear operator $\mathcal{L}_2$ defined by $\mathcal{L}_2[b] = \dot{b} - 2S_{00}$, with initial condition $b(0) = 0$, we obtain the integral equation for b stated in the Lemma. On use of (50), we note

$$\rho = \rho_{SR}\left(\frac{\gamma^2\beta^2}{\gamma_S^2\beta_R^2}\right)^{(1-\lambda)/(2\lambda)},$$

implying

$$\frac{\rho}{\rho_{SR}} = [(1+a)(1+b)]^{(1-\lambda)/\lambda}$$

This determines ρ in terms of a and b in the integral equations of the Lemma.

Lemma 4: *If we define the operator* $\mathcal{L}$ *as*

$$\mathcal{L}[q](t) = 2\int_0^t e^{-\omega(t)+\omega(t')}q(t')dt'$$

for $t \in [0, T]$*, then in the* sup *norm over this interval,*

$$\|\mathcal{L}[q]\| \leq \left\|\frac{q}{T_{00}a^{-1}}\right\|$$

Similarly, if we define the operator $\hat{\mathcal{L}}$ *as*

$$\hat{\mathcal{L}}[q](t) = 2\int_0^t e^{\alpha(t)-\alpha(t')}q(t')dt'$$

and $\|e^{-\Lambda t}q(t)\|$ *is bounded independent of* T *for some* Λ*, then for* $\Lambda > 0$*,*

$$\|\hat{\mathcal{L}}[q]\| \leq \frac{C_0}{\Lambda}\|e^{-\Lambda t}q\|e^{\Lambda T}$$

while for $\Lambda < 0$*,*

$$\|\hat{\mathcal{L}}[q]\| \leq \frac{C_0}{|\Lambda|} \left\| e^{-\Lambda t} q \right\|$$

for some constant C_0 *independent of* T

Proof: For the first part, note that

$$\mathcal{L}[q](t) = \int_{\exp[-\omega(t)]}^{1} d\left[e^{-\omega(t)+\omega(t')}\right] \frac{q(t')}{\omega'(t')}.$$

Since $\omega' = 2\frac{T_{00}}{a}$ and $\omega(t)$ is a positive function, the first part of the Lemma follows from a simple estimate on the term on the right hand side.

For the second part, we note the same type of estimate is not helpful, since unlike $\omega'(t)$, which is bounded below by a non-zero constant, α' tends to zero, as $t \to +\infty$. In this case, we note that because of the exponential growth in t of $\beta_R(t)$, $\alpha(t)$ asymptotes to a constant and so $e^{\alpha(t)-\alpha(t')}$ is bounded above by a constant independent of T. Then a simple estimate shows

$$|\hat{\mathcal{L}}[q](t)| \leq C_0 \|q e^{-\Lambda t}\| \int_0^t e^{\Lambda t'} dt',$$

which gives the desired result.

Lemma 5: *Consider the space* $\mathcal{S}$ *of continous functions of* t*,* $(a(t), b(t))$*, in the interval* $[0, T]$*, with* $\|(a,b)\| = \sup_{t \in [0,T]} |a(t)| + |b(t)|$*.* $\mathcal{S}$ *is clearly a Banach space. Consider the ball* $\mathcal{B}_{\rho_0,T}$ *where*

$$\|(a,b)\| \leq \tilde{K} \rho_{SR}(T) \beta_R(T) \equiv A(T)\rho_0$$

for a suitable constant $\tilde{K}$*, independent of* T*. Then, the mapping* $\mathbf{N} : \mathcal{S} \to \mathcal{S}$*, defined by*

$$\mathbf{N}[(a,b)] = (\mathcal{N}_1[a,b], \mathcal{N}_2[a,b])$$

is a contraction map from $\mathcal{B}_{\rho_0,T}$ *to itself when* ρ_0 *is small enough so that* $-\log \rho_0$ *is sufficiently large compared to* T*. Therefore, there exists a unique solution* (a, b) *to the integral equations in Lemma 3 in the ball* $\mathcal{B}_{\rho_0,T}$ *for sufficiently small* ρ_0*.*

Proof: Note that for $(a,b) \in \mathcal{B}_{\rho_0,T}$, ρ_0 can be chosen small enough to ensure $\rho_{ST}(T)\beta_R(T)$, $\|a\|$ and $\|b\|$ sufficiently small. This will ensure γ, β and ρ will remain close to γ_S, β_R and ρ_{ST}. Using (96),

$$|T_1| \leq C_1 \left\{ \rho_{SR} \left[(1+a)(1+b) \right]^{(1-\lambda)/\lambda} + \rho_{SR}^2 \beta_R^2 \sqrt{1+b} \left[(1+a)(1+b) \right]^{2(1-\lambda)/\lambda} \right\}. \tag{110}$$

From (98) and an equality similar to (37) for $\mathcal{I}_R$ applied to (90), it follows that

$$|S_1| \leq C_1 \Big\{ \beta_R \rho_{SR} (1+b)^{1/2} \left[(1+a)(1+b) \right]^{(1-\lambda)/\lambda} + \rho_{SR}^2 \gamma_S^2 (1+a)^{1/2} \left[(1+a)(1+b) \right]^{2(1-\lambda)/\lambda} \Big\}. \tag{111}$$

Also, note that

$$\begin{array}{ll} |T_{01}(1+a)| \leq C_1 a^2, & |T_{00} a| < C_1 a^2 \\ |S_{01}(1+b)| \leq C_1 \beta_R^{-4} b^2, & |S_{00} b| < C_1 \beta_R^{-4} b^2 \end{array} \tag{112}$$

It follows that

$$\left\| \frac{a}{T_{00}} \left(T_1(1+a) + T_{01}(1+a) + T_{00} a \right) \right\| \leq 2C_2 \left[\|T_1\| + \|a\|^2 \right]$$

From Lemma 4, it is clear that

$$\|\mathcal{N}_1[a,b]\| < 2C_2 \left\{ \rho_{SR}(T) \beta_R(T) + \|a\|^2 \right\}$$

Again,

$$|-S_1(1+b) + S_{01}(1+b) + S_{00} b| \leq 2C_1 \beta_R \rho_{SR} + 2C_1 \rho_{SR}^2 + 2C_2 b^2 e^{-2\pi t}$$

Using Lemma 4, with $\Lambda = \frac{\pi}{2\lambda}$ in the first term, $\Lambda = \frac{\pi(1-\lambda)}{\lambda}$in the second $\Lambda = -2\pi$ for the last term on the right hand side of the above equation, we obtain

$$\|\mathcal{N}_2[a,b]\| < \hat{C}_2 \left[\rho_{SR}(T) \beta_R(T) + [\rho_{SR}(T)]^2 + \|b\|^2 \right] < C_2 \left[\rho_{SR}(T) \beta_R(T) + \|b\|^2 \right] \tag{113}$$

If we choose ρ_0 small enough so that

$$C_2 \rho_{SR}(T) \beta_R(T) < \frac{1}{4}$$

and we choose $\tilde{K} = 4C_2$, then it is clear from the inequalities above that $\mathbf{N}$ maps $\mathcal{B}_{\rho_0,T}$ to itself. Using similar calculations, the boundedness of I_ζ and $\mathcal{I}_{0_\zeta}$, $\mathcal{I}_{R_\zeta}$ which follows from the fact that the zeros of z_ζ, z^S_ζ and ζz^R_ζ stay away from the boundary over the time interval $[0, T]$, and the facts that

$$\frac{\partial}{\partial\gamma}(z_\zeta - z^S_\zeta)$$

on $|\zeta| = 1$ and

$$\frac{\partial}{\partial\beta}(z_\zeta - z^R_\zeta)$$

on $|\zeta| = \rho$ are small, it is not difficult to show that $\mathbf{N}$ is a contraction map. Hence, from the well-known contraction mapping theorem, there is a unique fixed point of $\mathbf{N}$ in this ball. The proof is complete.

Finally, the proof of Theorem 2 follows easily from Lemma 5.

7 Calculations of the two-interface solutions

This section presents some numerical calculations of the exact solutions to the two-interface problem in an effort to illustrate the results of the preceding rigorous analysis. They are performed as follows. Three (arbitrary) choices of $\lambda = 0.3, 0.5$ and 0.7 are considered. In all cases, the initial parameter values $\rho_0 = 10^{-4}$ and $\gamma_0 = 1.1$ are chosen. Then, a value of c is picked so that the corresponding value of η_0 is consistent with the theoretical condition

$$\eta_0 \gg \gamma_0. \tag{114}$$

The values of c used are, respectively, $c = 4.25, 6.5$ and 8.5 leading to values of η_0 of the order of 5000 (which, it is noted, is much greater than γ_0 but less than ρ_0^{-1}). The very small choice of ρ_0 can be expected to lead to initially well-separated interfaces.

The evolution of the conformal mapping parameters is computed as follows. $\dot{\rho}$ is computed using (13) while $\dot{\gamma}$ is calculated using (74). A trapezoidal rule is used to compute all integrals arising in these two equations because this method gives super-algebraic convergence for periodic integrands on periodic domains, which is the case here. The corresponding value of η is found from the algebraic condition (80). A backward Euler method is used to advance the parameters in time.

Figures 2 – 4 show the evolution for the three distinct values of λ. A clearly-defined finger in the left-most interface is seen to develop having width λ specified *a priori*. By the time the left-most interface has deformed into a well-developed finger with well-defined asymptotic width, the right-most interface is still well-separated from the tip of the finger and is still ostensibly flat.

These calculations of the exact solutions to the two-interface problem are consistent with the rigorous analysis and conclusions presented in previous sections.

8 Discussion

The preceding results demonstrate rigorously the existence of a large family of time-dependent solutions involving a finite amount of fluid in a Hele-Shaw channel; these evolve rather differently from the translationally invariant solution of Feigenbaum [21]. As the fluid volume becomes progressively larger, we prove that that there is an intermediate time scale, far smaller than the time it takes for left interface to catch up with the right interface, where the above $N = 1$ family of solutions approaches a ZST (or Saffman-Taylor) steady solution of relative finger width λ, without any restriction on λ. Further, we have shown that there exist more general solutions (the $N = 2$ solution, for instance) that can be made close to the $N = 1$ solution initially, but which subsequently evolve into multiple fingers by a tip-splitting instability over an intermediate time scale. We therefore conclude that the $N = 1$ solutions are unstable.

Since experimental observation of steady fingers is always under conditions in which the two interfaces are well-separated, our overall conclusion is that a zero-surface tension theory, even with a finite volume of fluid, cannot explain the experimentally-observed finger selection.

If a zero-surface tension model were to be physically relevant (a contention which we do not support), it is difficult to argue that the particular translationally invariant solution due to Feigenbaum [21] is any more relevant than those found above in explaining experimentally observed steady fingers when the two interfaces are still far apart. Further, we have found precise upper bounds (103) on the differences between the the initial shape described by a particular solution in our family of solutions and that of Feigenbaum. This difference, which is less than $2e^{-\pi L/2}$ (=0.00016 for $L = 6$) is beyond experimental precision even at this moderate value of L, the initial distance between the left finger tip and the right interface.

In addition, within the above class of exact solutions, one recovers, in the asymptotic limit where the fluid volume tends to infinity, the class of solutions found earlier by Howison [30]. This latter class includes one where a ZST solution exhibits "tip-splitting". Thus the $\lambda = \frac{1}{2}$ ZST solution is nonlinearly unstable – a result that is not at all surprising in view of the ill-posed dynamics, which we now briefly discuss.

Even when there is a finite, but large, amount of viscous fluid in the channel the initial value problem can be expected to be ill-posed. If $\dot{\rho}/\rho > 0$ – as is the case for the special Saffman-like solution (since $\dot{\rho}/\rho$ closely tracks $\dot{\rho}_{SR}/\rho_{SR}$) and is surely the case for many other solutions – it is clear that $\mathrm{Re}[I(\zeta,t)]$ on the boundary of the ring $\mathcal{R}$: $1 < |\zeta| < \rho^{-1}$ is positive since

$$\mathrm{Re}[I(\zeta,t)] = -\mathrm{Re}[I(\zeta^{-1},t)] = \frac{2}{\pi|z_\zeta(\zeta^{-1},t)|^2} > 0 \quad \text{on } |\zeta| = 1,$$

and on the outer boundary of $\mathcal{R}$,

$$\mathrm{Re}[I(\zeta,t)] = -\mathrm{Re}[I(\zeta^{-1},t)] = \frac{2}{\pi\rho^2|z_\zeta(\zeta^{-1},t)|^2} + \frac{\dot{\rho}}{\rho} > 0 \quad \text{on } |\zeta| = \rho^{-1},$$

while $\mathrm{Re}[I(\zeta,t)]$ is a harmonic function of $(\mathrm{Re}[\zeta], \mathrm{Im}[\zeta])$ in $\mathcal{R}$. Applying the maximum principle for harmonic functions, any singularity ζ_s in $\mathcal{R}$ will move according to the property:

$$\mathrm{Re}\left[\frac{\dot{\zeta}_s}{\zeta_s}\right] = -\mathrm{Re}\left[I(\zeta_s,t)\right] < 0.$$

Stated differently, singularities will continuously move inwards towards $|\zeta| = 1$, corresponding to the left interface L. For a single left-interface problem, it has been argued [28] how this fundamental property is characteristic of the ill-posedness of the initial value problem when the interface shape is specified to some finite precision. We expect similar dynamics to occur in the case of a large finite amount of fluid.

Given the ill-posed nature of the dynamics and the expected structural instability of the zero-surface tension problem (in the sense that the solution set is not continuous in the surface tension parameter at exactly zero surface-tension), we believe one should not attach physical significance to any one particular solution of the zero-surface tension problem. Explicit simple examples have been given [20] to show the limitations of using physical intuition to make assertions about a structurally unstable, or even a nearly structurally unstable, system. Even within the context of Hele-Shaw flows, there are explicit examples where the spectrum of the linear stability operator is discontinuous at exactly zero surface tension [35].

Thus, it is our belief that physical arguments for selection based on the energetics of any solution of the zero-surface tension Hele-Shaw model are precarious since the behavior of such solutions is not exhibited by the physical system in the limit as surface tension tends to zero. A real physical system always has some surface tension (or some other appropriate regularizing effect) that is ignored in the zero-surface tension model. Such effects are known to change the expected behavior of the system (based on a zero-surface tension model) in surprising ways. This explains, for instance, why regularization in the form of arbitrarily small anisotropic surface tension can lead to selection of narrow fingers (much narrower than the celebrated $\lambda = \frac{1}{2}$). Different types of small regularization lead to different types of selection mechanism; an unregularized model (e.g. the zero-surface tension Hele-Shaw model) cannot possibly explain the dependence of selection on the type of small regularization introduced.

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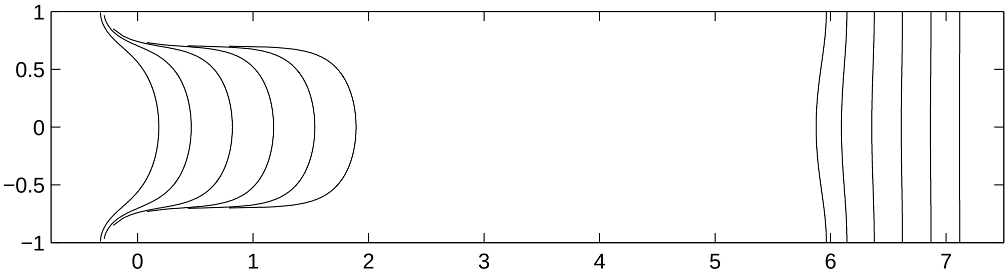

Figure 2: Evolution of a finite blob of fluid in a Hele-Shaw cell. Times shown are $t = 0, 0.25, 0.5, 0.75, 1, 1.25$. Initial conditions are $\lambda = 0.7, \gamma_0 = 1.1, \rho_0 = 10^{-4}, c = 8.5$.

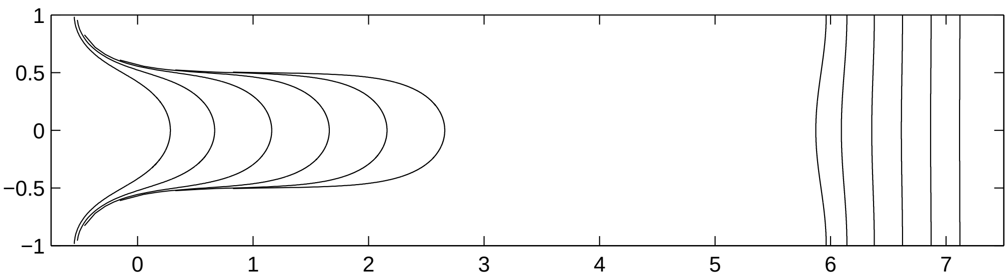

Figure 3: Evolution of a finite blob of fluid in a Hele-Shaw cell. Times shown are $t = 0, 0.25, 0.5, 0.75, 1, 1.25$. Initial conditions are $\lambda = 0.5, \gamma_0 = 1.1, \rho_0 = 10^{-4}, c = 6.5$.

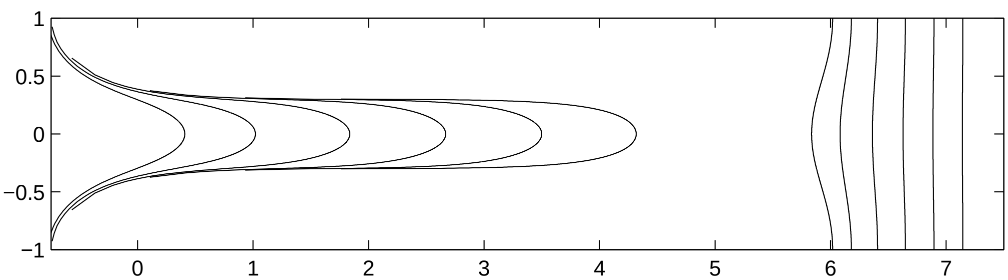

Figure 4: Evolution of a finite blob of fluid in a Hele-Shaw cell. Times shown are $t = 0, 0.25, 0.5, 0.75, 1, 1.25$. Initial conditions are $\lambda = 0.3, \gamma_0 = 1.1, \rho_0 = 10^{-4}, c = 4.25$.