Exponential Asymptotics for nonlinear ODEs

Example: A Simple Nonlinear Equation

\[ y' + y = \frac{1}{x^2} + y^2, \quad y \to 0 \text{ as } |x| \to \infty, \quad \text{arg } x \in \left[-\pi, \frac{\pi}{2}\right) \]

Apply dominant balance procedure

\[ y \sim \frac{1}{x^2} \text{ to leading order; full asymptotics: } y \sim \sum_{n=2}^{\infty} \frac{a_{n,0}}{x^n} \equiv \tilde{y}_0 \]

Seek correction \( y_E \ll x^{-n} \). Plugging \( y = y_0 + y_E \)

\[ y'_E + y_E = 2y_0y_E + y_E^2 \]

To the leading order for large \( x \), exponential correction:

\[ y_E \sim Ce^{-x} \]
Exponential Asymptotics for a nonlinear ODE-2

With exponential corrections for \( \arg x \in (0, \frac{\pi}{2}) \)

\[
y \sim \sum_{k=0}^{\infty} (Ce^{-x})^k \tilde{y}_k , \text{ where } \tilde{y}_k = \sum_{n=0}^{\infty} \frac{a_{n,k}}{x^n}
\]

Formally, plugging

\[
y(x) = \sum_{k=0}^{\infty} (Ce^{-x})^k y_k(x)
\]

into ODE and equating powers of \( e^{-x} \), for \( k \geq 1 \):

\[
y'_k - (k - 1)y_k = 2y_0y_k + \sum_{j=1}^{k-1} y_j y_{k-j}
\]

Formally, \( y_k \sim a_{0,k} + \frac{a_{1,k}}{x} + \ldots = \sum_{n=0}^{\infty} \frac{a_{n,k}}{x^n} \equiv \tilde{y}_k \)
Borel Transform of differential Equation

\[ y' + y = \frac{1}{x^2} \quad ; \quad \text{On transformation} \quad -pY(p) + Y(p) = p \]

So, \( Y(p) = \frac{p}{1 - p}, \quad y(x) = \int_0^\infty e^{i\phi} Y(p)e^{-px}dp \)

\[ y' + y = \frac{1}{x^2} + y^2 \]

On transformation:

\[(1-p)Y(p) = p + Y*Y(p) \quad , \quad \text{where} \quad Y*Y(p) = \int_0^p Y(s)Y(p-s)ds \]

or \( Y(p) = \frac{p}{1 - p} + \frac{Y*Y(p)}{1 - p} \equiv \mathcal{N}[Y](p) \)

Theorem: \( Y(p) \) analytic for \( |p| < \delta \). Along any ray \( p = re^{i\phi}, \infty > r > 0 \),
\[ |Y(p)| < A e^{b|p|}, \text{for constant } A \text{ and } b. \text{ Further, pole singularity at } p = 1 \]
Exponential Asymptotics for Nonlinear ODEs

General Setting

\[ y' = f(x^{-1}, y) \quad y \in \mathbb{C}^n, \quad x \in \mathbb{C} \]

\( f \) analytic at \((0, 0)\), satisfying generic conditions:

(ii) Eigenvalues \( \lambda_j \) of \( \hat{\Lambda} \equiv -\left\{ \frac{\partial f_i}{\partial y_j}(0, 0) \right\}_{i,j=1,2,...n} \) linearly independent over \( \mathbb{Z} \) and \( \arg \lambda_j \) are all different. W.L.O.G

\[ y' = -\hat{\Lambda}y + \frac{1}{x}\hat{\Lambda}y + g(x^{-1}, y) \]

\( \hat{\Lambda} = \text{diag}\{\lambda_j\} \), \( \hat{\Lambda} = \text{diag}\{\alpha_j\} \) constant matrices.

\( g(x^{-1}, y) = O(x^{-2}) + O(|y|^2) \) as \( x \to \infty \) and \( y \to 0 \)
Exponential Asymptotics for Nonlinear ODEs

Costin, 1998 results

\[ \tilde{y}(x) = \sum_{k \in (\mathbb{N} \cup \{0\})^n} C^k e^{-\lambda_k \alpha_k x^\alpha_k} \tilde{y}_k(x) \]

\( \tilde{y}_k \) factorially divergent formal power series:

\[ \tilde{y}_k(x) = \sum_{n=0}^{\infty} \frac{\tilde{a}_{k;n}}{x^n} \]
Solve

$$\nabla^2 \phi = 0$$  \hfill (1)

in the upper-half plane domain $y > 0$ satisfying the following condition: $\phi$ and all its derivatives are continuous as $y \to 0^+$ for any $x$. On the boundary of the domain,

$$\epsilon \phi_{xxx}(x, 0) + (1 - x^2 + \alpha)\phi_x(x, 0) - 2x\phi_y(x, 0) = 1 \hfill (2)$$

where $\epsilon > 0$ and $\alpha$ is real in the interval $(-1, \infty)$. As $x^2 + y^2 \to \infty$, $(x^2 + y^2) |\nabla \phi|$ is bounded for $y > 0$ and higher derivatives of $\phi$ all vanish.
Conversion into an ODE problem:

It is known that \( W(x + iy) \equiv \phi_x(x, y) - i\phi_y(x, y) \) is an analytic function in the upper half \( z \)-plane. The boundary condition implies

\[
Re \{\epsilon W'' + [(1 - iz)^2 + a] W - 1\} = 0
\]
on \( \text{Im} \, z = 0 \).

Since \{..\} is analytic in the upper-half plane, applying Schwarz reflection principle it is entire. From given conditions, bounded as well. From Liouville’s theorem, w.l.o.g

\[
\epsilon W'' + [(1 - iz)^2 + a] W = 1
\]
Formulation of problem

Question of existence of solution to PDE equivalent to that of the following problem:

Find analytic $W(z)$ satisfying ODE in $\text{Im} z > 0$ with

$$W(z) \to 0 \text{ as } z \to \infty \text{ for } \text{Arg} z \in (0, \pi)$$

For $\epsilon = 0$, $W = W_0(z)$, where

$$W_0 = \left( (1 - iz)^2 + \alpha \right)^{-1}$$

Singular at $z = \pm \sqrt{\alpha} - i$ (in the lower-half plane). For $\epsilon << 1$:

$$W \sim W_0 + \epsilon W_1 +$$

with $W_n$ for $n \geq 1$ determined recursively from

$$W_n = - \left( (1 - iz)^2 + \alpha \right)^{-1} W_{n-1}(z)$$
Quantization condition

Formal expansion consistent for $Imz \geq 0$ for any $a \in (-1, \infty)$.

Problem only at $z = \pm \sqrt{a} - i$ in the lower-half plane. Naive conclusion: solution exist for any $a \in (-1, \infty)$.

Not true. Solution exists if and only if

$$a = 2 \left( 2n + \frac{3}{2} \right) \varepsilon^{1/2} \text{ for integer } n \geq 0$$

Introduce scaled variables:

$$1 - iz = 2^{-1/2} \varepsilon^{1/4} Z \ ; \ W = 2^{-1} \varepsilon^{-1/2} G(Z) \ ; \ a = 2\varepsilon^{1/2} \alpha$$

$$G'' - \left( \frac{1}{4} Z^2 + \alpha \right) G = -1$$

$$G(Z) \to 0 \text{ as } Z \to \infty \text{ for } Arg \ Z \in (-\pi/2, \pi/2)$$
Associated Homogeneous Equation

Solution to associated homogeneous equation $G_1(\alpha, Z), G_2(\alpha, Z), G_3(\alpha, Z)$ related to parabolic cylinder function: $U(\alpha, Z)$:

$$G_1(\alpha, Z) = U(\alpha, Z) ; \quad G_2(\alpha, Z) = e^{-i\pi \left(\frac{\alpha}{2} - \frac{1}{4}\right)} U(-\alpha, iZ) ;$$

$$G_3(\alpha, Z) = e^{i\pi \left(\frac{\alpha}{2} - \frac{1}{4}\right)} U(-\alpha, -iZ)$$

$G_1(\alpha, Z) \sim Z^{-\alpha - 1/2} e^{-Z^2/4}$; as $Z \to \infty$, for $arg Z \in (-3\pi/4, 3\pi/4)$

$G_2(\alpha, Z) \sim Z^{\alpha - 1/2} e^{Z^2/4}$; as $Z \to \infty$, for $arg Z \in (-5\pi/4, \pi/4)$

$G_3(\alpha, Z) \sim Z^{\alpha - 1/2} e^{Z^2/4}$; as $Z \to \infty$, for $arg Z \in (-\pi/4, 5\pi/4)$
Solution to scaled ODE

From integral representation of $U(\alpha, Z)$,

$$G_2(Z) - G_3(Z) = 2 \, i \, k \, G_1(Z), \quad \text{where} \quad k = \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{2\pi}} \cos(\pi\alpha)$$

There exists unique solution to scaled ODE, satisfying

$$G(Z) \to 0 \text{ as } Z \to \infty \quad \text{for} \quad \text{Arg } Z \in (-\pi/2, 0]$$

and is given by

$$G(Z) = G_p(Z) \equiv G_1(Z) \int_{-i\infty}^{Z} G_2(\xi) d\xi - G_2(Z) \int_{\infty}^{Z} G_1(\xi) d\xi$$
Another solution to scaled ODE

Similar arguments lead to the following lemma:

There exists a unique solution to scaled ODE, satisfying

\[ G(Z) \to 0 \text{ as } Z \to \infty \text{ for } \arg Z \in [0, \pi/2) \]

and is given by

\[ G(Z) = \hat{G}_p(Z) \equiv G_1(Z) \int_0^Z G_3(\xi)d\xi - G_3(Z) \int_0^Z G_1(\xi)d\xi \]

Thus solution to original problem iff

\[ \hat{G}_p(Z) = G_p(Z) \]

But,

\[ G_p(Z) - \hat{G}_p(Z) = G_1(Z)A(\alpha) \]

\[ A(\alpha) = 0 \text{ iff } \alpha = \alpha_n \equiv 2n + \frac{3}{2} \text{ for } n \geq 0 \]
Conclusion for PDE-I

Problem structurally unstable at $\epsilon = 0$; solution set discontinuous at $\epsilon = 0$.

Consider

$$\epsilon W'' + \left[ (1 - iz)^2 + a + \frac{\epsilon_1}{(1 - iz)^2} \right] W = 1$$

As before, we seek $W(z) \to 0$ as $z \to \infty$ for $\text{Im } z > 0$.

Scaling gives

$$G'' - \left( \frac{1}{4} Z^2 + \alpha + \frac{\gamma}{Z^2} \right) G(Z) = 1$$

$$\gamma = \frac{\epsilon_1}{\epsilon}$$

Seek solution so that $G(Z) \to 0$, when $Z \to \infty$ for $\text{arg } Z \in (-\pi/2, \pi/2)$. 
Conclusion for perturbed PDE

Solution exists only for
\[ \alpha = \hat{\alpha}_n(\gamma) \]

When \( \gamma \to 0 \), \( \hat{\alpha}_n \to \alpha_n = (2n + 3/2) \).

Otherwise, \( \alpha \) depends on \( \gamma \) when \( \gamma \) is sizable.

\( \epsilon_1/(1 - iz)^2 \) is smaller than other terms on \( \text{Im } z = 0 \) for 
\( \epsilon << \epsilon_1 << \epsilon^{1/2} \), yet important!

Apparently small terms cause large effects near structural unstable point.