1 Conic section

Shapes such as ellipses, parabola, hyperbola are referred to as conic section since they result from intersection of a plane with a right circular cone.

A right circular cone surface results from rotation of the straight line $z = kx$ about the z axis. Since on the surface of a cone, the vertical height z only depends on $r = \sqrt{x^2 + y^2}$, the distance from the cone-axis, *i.e.* $z = f(r)$, by taking $y = 0$ and $x > 0$, we conclude $z = f(x) = kx$. Therefore,

$$z = kr = k\sqrt{x^2 + y^2} \tag{1}$$

We can extend this cone to $z < 0$, to the *lower nappe*, for which $z = -kr$. Thererefore, with the understanding that $\sqrt{\cdot}$ gives a non-negative number, the entire cone (both upper and lower *nappe*) is described by

$$z = \pm k\sqrt{x^2 + y^2} \tag{2}$$

The formula of a plane, except when it is vertical, is given by

$$z = ax + by + c \tag{3}$$

On the cone surface, both (2)-(3) are satisfied we have

$$ax + by + c = \pm k \left(x^2 + y^2\right)^{1/2} \tag{4}$$

Completing squares, we get

$$(a^2 - k^2)x^2 + (b^2 - k^2)y^2 + c^2 + 2abxy + 2acx + 2bcy = 0 \tag{5}$$

In the next two subsection, we will show that in all cases, (5) is the equation of a conic.

1.1 Case of $ab = 0$

Consider the case $ab = 0$ in (5), *i.e.* a and/or $b = 0$. Clearly, (5) is of the form

$$Ax^2 + By^2 + Dx + Ey + F = 0 \tag{6}$$

In this case, the simplest sub-case is $A = 0 = B$ when (6) degenerates into a straight line. The next simplest case is when A or $B = 0$. Suppose $B = 0$ and $A \neq 0$, then then (6) implies

$$\left(x + \frac{D}{2A}\right)^2 = -Ey + F \tag{7}$$

which will be a equation of a parabola when $E \neq 0$.

If in this subcase, if in addition $E = 0$, (7) reduces simply to the degenerate case of a pair of straight lines $x = -\frac{D}{2A} \pm \sqrt{F}$ when $F > 0$ and one straight line for $F = 0$ and no curve at all of $F < 0$. euation

Now, consider the case when both $A, B \neq 0$. Then, we can complete the square in (6) and write

$$A\left(x + \frac{D}{2A}\right)^2 + B\left(y + \frac{E}{2B}\right)^2 = \frac{D^2}{4A} + \frac{E^2}{4B} - F \tag{8}$$

Now, first consider A, B are of the same sign, which we may take to be positive, without loss of generality (otherwise, we multipy (6) by -1 and then the new equation will have $A, B > 0$) Then if

$$\rho := \frac{D^2}{4A} + \frac{E^2}{4B} - F > 0 \tag{9}$$

on division by the right hand side leads to the equation of the ellipse in the form

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \ , \quad \text{where} \tag{10}$$

$(x_0, y_0) = \left(-\frac{D}{2A}, -\frac{E}{2B}\right)$, $a^2 = \frac{\rho}{A}$, $b^2 = \frac{\rho}{B}$. If in (11), $\rho = 0$, the ellipse shrinks to a single point, and if $\rho < 0$, there are no curve satisfying the equation (8) since the left hand side is non-negative while the righthand side $\rho < 0$.

In the case, when A, B in (8) have opposite signs, then division of (8) by $\rho \neq 0$ (even if it is negative) results in either

$$\frac{(x-x_0^2}{a^2} - \frac{(y-y_0^2}{b^2} = 1 \ , \quad \text{or} \ \frac{(y-y_0^2}{a^2} - \frac{(x-x_0^2}{b^2} = 1 \tag{11}$$

depending on whether $\frac{A}{\rho} > 0$ or $\frac{A}{\rho} < 0$. In either case, (11) describes parabola, in the first case, with focii on the x-axis, and in the second case, focii is on the y-axis. The degenerate case $\rho = 0$ with A, B having opposite signs results in (8) corresponding to a pair of straight lines

$$y + \frac{E}{2B} = \pm\sqrt{-\frac{A}{B}}\left(x + \frac{D}{2A}\right) \tag{12}$$

1.2 Rotation of coordinate system

Consider change of variables $(x, y) \to (u, v)$ where

$$x = du + ev \ , y = -eu + dv \ , \quad \text{where } d = \cos\theta \ , e = \sin\theta \ . \tag{13}$$

As explained pictorially in class, and as you will work out in the homework, where (u, v) are the coordinates in a coordinate system that is rotated clockwise by an angle θ with respect to the original $x - y$ coordinate system. Then (5) transforms into

$$\begin{aligned} &\left\{\left(a^2-k^2\right)d^2 + \left(b^2-k^2\right)e^2 - 2abde\right\}u^2 + \left\{\left(a^2-k^2\right)e^2 + \left(b^2-k^2\right)d^2 + 2abed\right\}v^2 \\ &\quad + 2\left\{de(a^2-b^2) + ab(d^2-e^2)\right\}uv + (2acd - 2bce)\,u + (2ace + 2bcd)\,v + c^2 \ = \ 0 \end{aligned} \tag{14}$$

With choice of rotation angle θ that ensures

$$\tan(2\theta) = -\frac{2}{ab}\left(a^2 - b^2\right) \ , \tag{15}$$

which also implies

$$\tan(2\theta) = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2de}{d^2-e^2} \ , \tag{16}$$

it follows on inspection that the coefficient of uv term disappears in (14) and and we obtain an equation of the form

$$a'u^2 + b'v^2 + c'u + d'v = e' \tag{17}$$

This is precisely where the discussion in the previous subsection holds, and we therefore have a conic.

2 A more geometric characterization

Parabola is characterized by the set of points equidistant from a straight line, called a *directrix* and a fixed point called *focus.*

For instance, suppose the focus is chosen on the y axis at a location F with coordinates $(0, k)$ and the directrix is given by the line $y = -k$. Then, from the geometric characterization, distance PF must be the same as the distance of P from the straight line $y = -k$, which is given by $|y + k|$:

$$\sqrt{(x-0)^2 + (y-k)^2} = |y+k| \tag{18}$$

Squaring both sides we have

$$x^2 + y^2 - 2yk + k^2 = y^2 + 2yk + k^2 \tag{19}$$

which on cancellation gives

$$4ky = x^2 \ , \quad y = \frac{1}{4k}x^2 \tag{20}$$

which is the equation of a parabola. In this representation, the point $(0, 0)$ is the vertex, $y = -k$ is the directrix and $(0, k)$ is the focus. If we perform a translation such that the vertex location to (x_0, y_0) without rotating the figure, then the parabola equation will look like

$$4k(y - y_0) = (x - x_0)^2 \tag{21}$$

More generally, if we are given an equation like

$$2x^2 + x + y - 2 = 0 \tag{22}$$

and ask you to determine vertices, directrix and focal location, you will complete the square and write it as

$$2\left(x + \frac{1}{4}\right)^2 = -\left(y - \frac{17}{8}\right) \ , \quad \text{implying } -\frac{1}{2}\left(y - \frac{17}{8}\right) = \left(x + \frac{1}{4}\right)^2 \tag{23}$$

or vertex $(x_0, y_0) = \left(-\frac{1}{4}, \frac{17}{8}\right)$ with focus at $(0, k) = \left(0, -\frac{1}{2}\right)$ and directrix given by $y = \frac{1}{2}$.

If we change the role of $x - y$, then the equation of the parabola will be

$$4kx = y^2 \ , \quad x = \frac{y^2}{4k} \tag{24}$$

in which case the vertex will still be at $(0, 0)$, but the $x = -k$ is the directrix and $(k, 0)$ is the focus location. We can accommodate translation $(x, y) \to (x - x_0, y - y_0)$ in the same manner as before. Note that rotation

$$x = \cos\theta x' + \sin\theta y' \ , \quad y = -\sin\theta x' + \cos\theta y' \tag{25}$$

Changes the equation to the form

$$Ay'^2 + Bx'^2 + Cx'y' + Dx' + Ey' = 0 \ , \quad \text{withdiscriminant } C^2 - 4AB = 0 \tag{26}$$

2.1 Geometric Description of Ellipses

Ellipses are characterized by the geometric condition that the sum of distances d_1, d_2 from two fixed points called *focii* is a constant. Suppose, the *focii* are located at F $((c,0))$ and F' $((-c,0))$. Consider an arbitrary point on the ellipse P with coordinates (x,y). Then, Suppose

$$2a = d_1 + d_2 = \sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} \tag{27}$$

This implies

$$d_2^2 = (x+c)^2 + y^2 = 4a^2 - 4ad_1 + d_1^2 = (x-c)^2 + y^2 + 4a^2 - 4ad_1 \tag{28}$$

implying that

$$4ad_1 = 4a^2 - -4cx \ , \quad \text{implying } d_1 = \sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x \tag{29}$$

In order that $d_1 + d_2 = 2a$, we also have

$$d_2 = a + \frac{c}{a}x \tag{30}$$

Now,

$$d_1^2 = (x-c)^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2 \tag{31}$$

It is convenient to define

$$b^2 = a^2 - c^2 > 0 \tag{32}$$

Then the relation (29) implies that

$$x^2\frac{b^2}{a^2} + y^2 = b^2 \ , \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{33}$$

Here $2a$ is the major axis and $2b$ is the minor axis length, while $e = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}}$ is the minor axis length. The point $(0,0)$ is called the center. The focii locations in terms of a, b are at $\left(\pm\sqrt{a^2 - b^2}, 0\right)$. Now, we can seek a more geometric interpretation of (29) and (30). We note that (29) and (30) imply that

$$d_1 = a - \frac{c}{a}x = e\left(\frac{a}{e} - x\right) \ , \quad d_2 = e\left(x + \frac{a}{e}\right) \tag{34}$$

We note that $\frac{a}{e} - x$ is the distance of the point P from a vertical line called directrix $x = \frac{a}{e}$, while $x + \frac{a}{e}$ is the distance of the point P from the directrix $x = -\frac{a}{e}$. Therefore, we notice that for an ellipse, the ratio of the distances from a point P on the ellipse to one of the focus point to the distance to a corresponding directrix is a constant $e < 1$.

More generally, if the shape is translated without any rotation so that the center is at (x_0, y_0), then the equation of the ellipse will be

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1 \tag{35}$$

and when $b < a$, the eccentricity $e = \sqrt{1 - \frac{b^2}{a^2}}$, the focii will be at $(x_0 \pm ea, y_0)$ and directrices at $x = x_0 \pm \frac{a}{e}$

If we have an equation in the above form with $b > a$, then the focii location are at at $\left(x_0, y_0 \pm \sqrt{b^2 - a^2}\right)$ with major axis vertically aligned and minor axis horizontally aligned with $e = \sqrt{1 - \frac{a^2}{b^2}}$. The directrices will then be at $y = y_0 \pm \frac{b}{e}$. Completing squares we can convert an equation of the form

$$Ax^2 + By^2 + Dx + Ey = F \tag{36}$$

with $AB > 0$ of the same sign (and therefore nonzero) in the form (35). More generally, If we had an equation in the form

$$Ax^2 + By^2 + Cxy + Ex + Ey = F \tag{37}$$

Then, a rotation of coordinates is possible to convert it into a form (35) provided we have the ellipticity condition

$$C^2 - 4AB < 0 \tag{38}$$

2.2 Hyperbola

Now now require that $|d_1 - d_2| =$ constant where d_1, d_2 are once again distances of a point P from the focii F and F'. Clearly, we notice from geometric consideration that Suppose we choose a coordinate system where coordinates of F, F' are $(c, 0)$ and $(-c, 0)$ respectively on the x-axis. Then geometric description may be written as

$$d_1 - d_2 = \pm 2a \tag{39}$$

for some $a > 0$. In order for this condition to hold for a point $(x, 0)$ on the x axis, it is clear that from geometric consideration tht $2c > 2a$, and so the *eccentricity*

$$e = \frac{c}{a} > 1 \ . \tag{40}$$

Then, (39) implies that

$$d_1^2 = (x - c)^2 + y^2 = d_2^2 + 4a^2 \pm 4ad_2 = (x + c)^2 + y^2 + 4a^2 \pm 4ad_2 \tag{41}$$

$$d_2 = \pm \frac{c}{a}\left(x + \frac{a^2}{c}\right) = \pm e\left(x + \frac{a}{e}\right) \ , \tag{42}$$

$$d_1 = \pm \frac{c}{a}\left(x - \frac{a^2}{c}\right) = \pm e\left(x + \frac{a}{e}\right) \ , \tag{43}$$

The choice of sign in (42) and (43) has to be consistent with d_1, d_2 being distances and therefore being positive by definition, and so will depend on whether or not we are to the left or right of the directrices $x = \pm ae$. Now, taking square of (42) leads to

$$d_2^2 = x^2 + 2cx + c^2 + y^2 \left(a + \frac{c}{a}x\right)^2 = a^2 + 2cx + \frac{c^2}{a^2}x^2 \tag{44}$$

or

$$-y^2 + x^2\left(\frac{c^2}{a^2} - 1\right) = c^2 - a^2 = b^2 \ , \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{45}$$

which is the equation of hyperbola with focii at $(\pm c, 0) = \left(\pm\sqrt{a^2 + b^2}, 0\right)$. As in previous discussion for the case of an ellipse or parabola, we can have a more generalized representation

$$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1 \ , \tag{46}$$

when the hyperbolas are translated without rotation so that focii is now at

$$(x_0 \pm c, y_0) \tag{47}$$

where $c = \sqrt{a^2 + b^2}$. If we rotated by hyperbola by 90^o, then this will be equivalent to interchanging the role of x and y in (45) leading to

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \tag{48}$$

with $a > b$. The focii in this case is now rotated into the y-axis. In general, when we translate and rotate a hyperbola arbitrarily, it can be shown to have the form

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0 \tag{49}$$

with the discriminant requirement

$$C^2 - 4AB > 0 \tag{50}$$