S 4.1: No. 2. Find the eigenvalues and eigenvectors of the matrix \( A \) given below:

\[
A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}
\]

Solution: \( \det(A - \lambda I) = (2 - \lambda)(-1 - \lambda) = 0 \) and so eigenvalues are \( \lambda = 2, -1 \). Corresponding to \( \lambda = 2 \) an eigenvector is found by solving for \( X = [x_1, x_2]^T \) satisfying \( (A - 2I)X = 0 \) which gives rise to the augmented matrix reduction as shown:

\[
\begin{pmatrix} 0 & 1 & 0 \\ 0 & -3 & 0 \end{pmatrix} \Rightarrow R_2 + 3R_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

\( x_2 = 0, x_1 = t \), arbitrary. Therefore one eigenvector \( x = [1, 0]^T \). Corresponding to \( \lambda = -1 \), we have the linear system \( (A + I)x = 0 \) giving rise to the augmented matrix reduction as below

\[
\begin{pmatrix} 3 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \tfrac{1}{3}R_1 \begin{pmatrix} 1 & \tfrac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Therefore \( x_2 = t \) and \( x_1 = -\tfrac{1}{3} \). With choice \( t = 3 \), we have eigenvector corresponding to \( \lambda = -1 \) given by \( x = [x_1, x_2]^T = [-1, 3]^T \).

S 4.1: No. 4. Same question as No. 2, except that

\[
A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}
\]

Solution: In this case \( \det(A - \lambda I) = (1 - \lambda)(4 - \lambda) = 0 \), implies \( \lambda = 1 \) or \( \lambda = 4 \). Corresponding to \( \lambda = 1 \), corresponding eigenvector satisfies \( (A - 1)X = 0 \) whose corresponding augmented matrix row reduces as shown:

\[
\begin{pmatrix} 0 & -2 & 0 \\ 0 & 3 & 0 \end{pmatrix} \Rightarrow -\tfrac{1}{2}R_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \Rightarrow R_2 - 3R_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

Therefore \( x_2 = 0 \) and \( x_1 = t \), arbitrary and an eigenvector corresponding to \( \lambda = 1 \) is \([1, 0]^T\). Corresponding to \( \lambda = 4 \), we have the following row-reduction of the augmented matrix corresponding to \( (A - 4I)x = 0 \) give rise to

\[
\begin{pmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

which is already in RREF form and gives \( x_2 = t, x_1 = 2t \). Therefore with \( t = 1 \), an eigenvector corresponding to \( \lambda = 4 \) is \( x = [2, 1]^T \).

S 4.1: 6. Same question as No. 2, except that

\[
A = \begin{pmatrix} 3 & -1 \\ 5 & -3 \end{pmatrix}
\]
**Solution:** \( \det(A - \lambda I) = (3 - \lambda)(-3 - \lambda) + 5 = \lambda^2 - 4 = 0 \) gives rise to eigenvalues \( \lambda = \pm 2 \). Corresponding to eigenvalue \( \lambda = 2 \), to find eigenvector we solve linear system \((A - 2I)x = 0\) for which the corresponding augmented matrix row reduces as follows:

\[
\begin{pmatrix}
1 & -1 & 0 \\
5 & -5 & 0
\end{pmatrix} \Rightarrow R_2 - 5R_1 \begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Therefore, it follows that \( x_2 = t \) and \( x_1 = t \) and an eigenvector corresponding to \( \lambda = 2 \) is \((1, 1)^T\).

Corresponding to eigenvalue \( \lambda = -2 \), to find eigenvector we solve linear system \((A + 2I)x = 0\) for which the corresponding augmented matrix row reduces as follows:

\[
\begin{pmatrix}
5 & -1 & 0 \\
5 & -1 & 0
\end{pmatrix} \Rightarrow R_2 - R_1 \begin{pmatrix}
5 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix} \Rightarrow -R_1 \begin{pmatrix}
1 & -\frac{1}{5} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Therefore, it follows that \( x_2 = t \) and \( x_1 = \frac{t}{5} \) and an eigenvector corresponding to \( \lambda = -2 \) is \((1, 5)^T\).

**S 4.1: 18.** Consider the \( 2 \times 2 \) matrix \( A \) given below. Show that there are no scalars \( \lambda \) so that \( A - \lambda I \) is singular.

\[
A = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad b \neq 0
\]

**Solution:** Note that condition that \( A - \lambda I \) is singular is \( \det(A - \lambda I) = 0 = (a - \lambda)^2 + b^2 \) This does not have real roots \( \lambda \) since if \( a, b, \lambda \) are all real it is the sum of two squares, which cannot be zero since \( b \neq 0 \).

**S 4.2, No. 12** Calculate \( \det A \) for \( A \) given below. Determine whether \( A \) is singular or nonsingular.

\[
A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 4 & 2 & 10 \end{pmatrix}
\]

**Solution:** If you noticed that the third column \( A_3 = 2A_1 + A_2 \), implying that the column vectors of \( A \) are dependent, then you know right away without further calculations that \( \det A = 0 \) and \( A \) is singular. If you did not notice that, let’s calculate \( \det A \) as below:

\[
\begin{aligned}
\det A &= (1) \det \begin{pmatrix} 3 & 7 \\ 2 & 10 \end{pmatrix} + (-2) \det \begin{pmatrix} 2 & 4 \\ 2 & 10 \end{pmatrix} + 4 \det \begin{pmatrix} 2 & 4 \\ 3 & 7 \end{pmatrix} \\
&= 1(30 - 14) - 2(20 - 8) + 4(14 - 12) = 16 - 24 + 8 = 0
\end{aligned}
\]

and hence \( A \) is singular.

**S 4.2: 14.** Calculate \( \det A \) for \( A \) given below. Determine whether \( A \) is singular or nonsingular.

\[
A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}
\]
Solution: Since $A$ singular is equivalent to $\det A = 0$, we check the determinant to find

$$\det A = (1) \det \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} + (-1) \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 1 - 1(4 - 3) = 0$$

and hence $A$ is singular.

S 4.2: 18. Same question as No. 14 above with

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Solution: Expanding by the first row, we have

$$\det A = (-1)^1(1 + 2) \det \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = (-1)(-1) \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

Since $\det A \neq 0$, $A$ is nonsingular.

S 4.2: 24. Let $A$ and $B$ be $n \times n$ matrices. Use Theorems 2 and 3 on page 287 text to give an easy proof of the following

a. If either $A$ or $B$ is singular, then $AB$ is singular.

Solution: Since $\det(AB) = \det(A) \det(B) = 0$ if either $A$ or $B$ is singular as this implies either $\det A = 0$ or $\det B = 0$. However, $\det(AB) = 0$ implies $AB$ is singular.

b. If $AB$ is singular, then either $A$ or $B$ is singular.

Solution: Once again $AB$ singular implies $\det(AB) = 0$. This means $\det A \times \det B = 0$, which can only happen if either $\det A = 0$ or $\det B = 0$, implying either $A$ or $B$ is singular.

S 4.2: No. 32. Let $U$, $V$ be $n \times n$ upper-triangular matrices. Prove the following special case of Theorem 2: $\det(UV) = \det(U) \det(V)$

Solution: Note a matrix $A$ is upper-triangular when $a_{i,j} = 0$ for $i > j$, i.e. when the first subscript corresponding to row is larger than the second subscript corresponding to column. We know from matrix multiplication definition that the $(i,j)$ th entry of matrix $UV$ is given by

$$(UV)_{i,j} = \sum_{k=1}^{n} U_{i,k} V_{k,j} = \sum_{k=i}^{n} U_{i,k} V_{k,j}$$

where use used the fact that $U_{i,k} = 0$ for $i = 1, 2, \ldots, k - 1$ as it is upper-triangular. However, if $i > j$, then in the summation above $k \geq i > j$ and $V$ upper-triangular implies $V_{k,j} = 0$ as well. Thus every term in the summation above is zero when $i > j$. Therefore $T = UV$ is an upper-triangular matrix.

Now, we know $\det T = t_{11}t_{22}\cdots t_{nn}$. So, let’s calculate $t_{i,i} = (UV)_{i,i}$:

$$t_{i,i} = (UV)_{i,i} = \sum_{k=1}^{n} U_{i,k} V_{k,i} = \sum_{k=i}^{n} U_{i,k} V_{k,i} = U_{i,i}V_{i,i}$$
since $U_{i,k} = 0$ for $k < i$ and $V_{k,i} = 0$ for $k > i$. Therefore, it follows that

$$\det T = U_{11} V_{11} U_{22} V_{22} U_{33} V_{33} \cdots U_{nn} V_{nn} = (U_{11} U_{22} \cdots U_{nn}) (V_{11} V_{22} \cdots V_{nn})$$

S 4.4: No. 8. Find the characteristic polynomial and eigenvalues for the following matrix. Also, give the algebraic multiplicity of each eigenvalue.

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{pmatrix}$$

**Solution:** Characteristic polynomial is $p(\lambda) = \det(A - \lambda I)$, which in this case is

$$\det \begin{pmatrix} -2 - \lambda & -1 & 0 \\ 0 & 1 - \lambda & 1 \\ -2 & -2 & -1 - \lambda \end{pmatrix} = (-2 - \lambda) \det \begin{pmatrix} 1 - \lambda & 1 \\ -2 & -1 - \lambda \end{pmatrix} - 2 \det \begin{pmatrix} -1 & 0 \\ 1 - \lambda & 1 \end{pmatrix}$$

$$= -(2 + \lambda) ((1 - \lambda)(-1 - \lambda) + 2) - 2(-1) = -(2 + \lambda) (\lambda^2 + 1) + 2 = -2\lambda^2 - \lambda(\lambda^2 + 1)$$

$$= -\lambda(\lambda^2 + 2\lambda + 1) = -\lambda(\lambda + 1)^2$$

So, $p(\lambda) = -\lambda(\lambda + 1)^2$ and we have eigenvalues $\lambda = 0, -1$, with algebraic multiplicity 1 for $\lambda = 0$ and 2 for $\lambda = -1$ (as a factor of $(\lambda + 1)$ appears twice).

S 4.4: 10. Same question as No. 8, but with

$$A = \begin{pmatrix} -7 & 4 & -3 \\ 8 & -3 & 3 \\ 32 & -16 & 13 \end{pmatrix}$$

**Solution:** Characteristic polynomial is $p(\lambda) = \det(A - \lambda I)$, which in this case is

$$\det \begin{pmatrix} -7 - \lambda & 4 & -3 \\ 8 & -3 - \lambda & 3 \\ 32 & -16 & 13 - \lambda \end{pmatrix} = (-7 - \lambda) \det \begin{pmatrix} -3 - \lambda & 3 \\ -16 & 13 - \lambda \end{pmatrix} - 4 \det \begin{pmatrix} 8 & 3 \\ 32 & 13 - \lambda \end{pmatrix}$$

$$- 3 \det \begin{pmatrix} 8 & -3 - \lambda \\ 32 & 16 \end{pmatrix}$$

$$= -(7 + \lambda) ((13 - \lambda)(-15 - \lambda) + 48) - 4(8 - 8\lambda) - 3(32 + 32\lambda) = -(7 + \lambda) (9 - 10\lambda + \lambda^2) + 64(1 - \lambda)$$

$$= -(7 + \lambda)(9 - \lambda)(1 - \lambda) + 64(1 - \lambda) = -(1 - \lambda) (63 + 2\lambda - \lambda^2 - 64) = -(\lambda - 1)^3$$

So, $p(\lambda) = -(\lambda - 1)^3$ and we have only one distinct eigenvalue $\lambda = 1$ with algebraic multiplicity 3 as $(\lambda - 1)$ factor appears three times.

S 4.4: 14. Same question as No. 8 above, but with

$$A = \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$
Solution: \( \det(A - \lambda I) \) gives

\[
\det (A - \lambda I) = (1 - \lambda) \det \begin{pmatrix} 1 - \lambda & -1 & -1 \\ -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \end{pmatrix} + \det \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \end{pmatrix}
\]

\[
- \det \begin{pmatrix} -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \\ -1 & -1 & -1 \end{pmatrix} + \det \begin{pmatrix} -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \\ -1 & -1 & -1 \end{pmatrix}
\]

Now, it is useful to do each of the four determinants above separately. We have

\[
\det \begin{pmatrix} 1 - \lambda & -1 & -1 \\ -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \end{pmatrix} = -(\lambda - 2)^2(\lambda + 1)
\]

\[
\det \begin{pmatrix} -1 & -1 & -1 \\ -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \end{pmatrix} = -\lambda(\lambda - 2) + 2(\lambda - 2) = -(\lambda - 2)^2
\]

\[
- \det \begin{pmatrix} -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \\ -1 & -1 & -1 \end{pmatrix} = (\lambda - 2) + (1 - \lambda)(\lambda - 2) = -(\lambda - 2)^2
\]

\[
\det \begin{pmatrix} -1 & 1 - \lambda & -1 \\ -1 & -1 & 1 - \lambda \\ -1 & -1 & -1 \end{pmatrix} = -1(2 - \lambda) - (1 - \lambda)(2 - \lambda) = -(2 - \lambda)^2
\]

So, we have on adding all the determinants

\[
\det(A - \lambda I) = -(1 - \lambda)(\lambda + 1)(\lambda - 2)^2 - 3(\lambda - 2)^2 = (\lambda - 2)^2(\lambda^2 - 4) = (\lambda - 2)^3(\lambda + 2)
\]

Therefore the eigenvalues are \( \lambda = 2 \) (algebraic multiplicity 3) and \( \lambda = -2 \) (multiplicity 1).